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Mixed Bifractional Brownian Motion: Definition and Preliminary Results

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Abstract. *In this paper we, firstly, introduce a new Gaussian process as an extension of the well known bifractional Brownian motion in the form of a linear combination of a finite number of independent bifractional Brownian motions. We have opted to call this process a mixed bifractional Brownian motion. Secondly, we study some stochastic properties and characteristics of this process: the Hölder continuity, self similarity, quadratic variation, Markov property and differentiability of the trajectories, long range dependence, stationarity of the increments and behavior of the noise generated by the increments of this process. We believe that our process can be a possible candidate for models which involve self similarity, long range dependence and non stationarity of increments.*

Key words : Gaussian Process, Self Similarity, Fractional Brownian Motion, Bifractional Brownian Motion, Quadratic Variation, Differentiability, Long-Range Dependence.

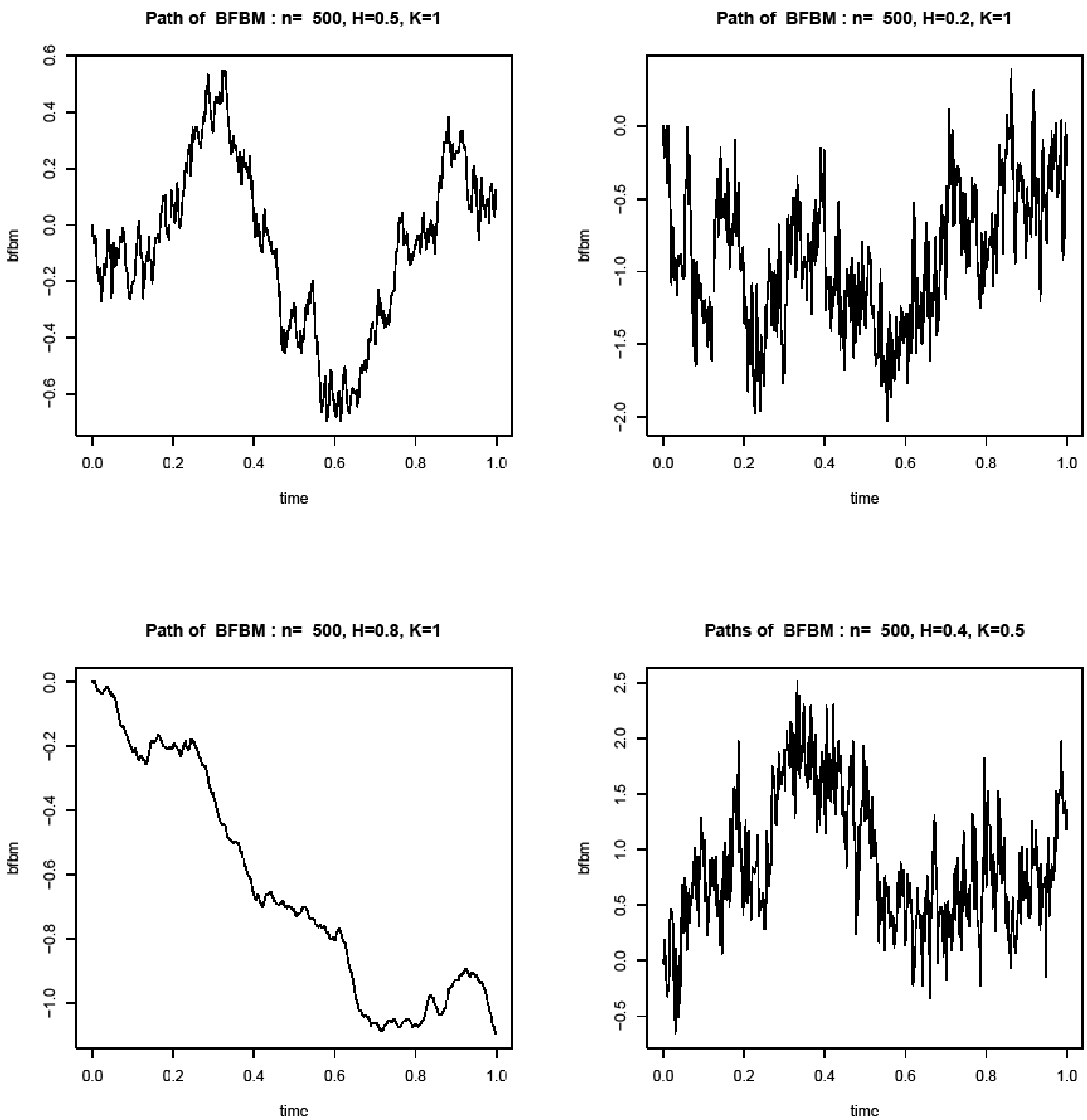
AMS Subject Classifications : 60G15, 60G17, 60H18

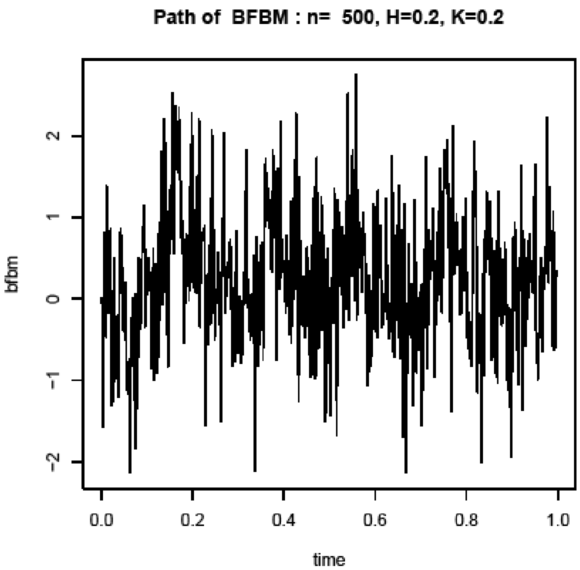
1. Introduction

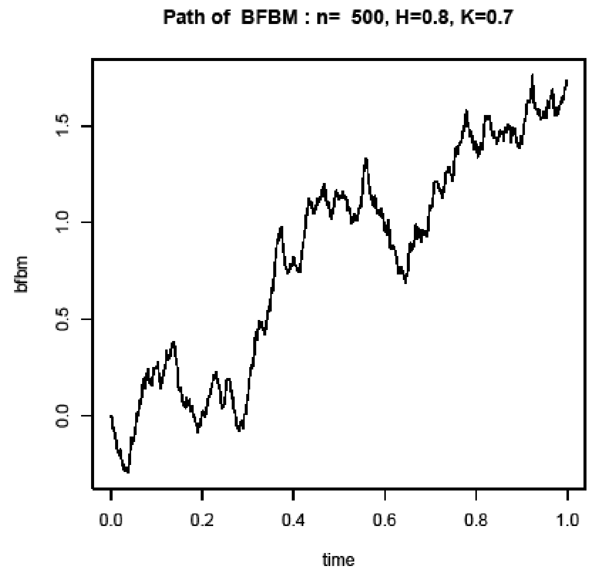
Let $B^{H,K} := \left\{B_t^{H,K}, t \geq 0\right\}$ be a one-parameter bifractional Brownian motion, (*BFBM* for short), valued in $\mathbb{R}$, with parameters $H \in (0,1)$ and $K \in (0,1]$, introduced in Houdré and Villa [8] as a centered Gaussian process, starting from zero, with the covariance function:

$$\mathsf{E}(B_t^{H,K} B_s^{H,K}) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right). \tag{1}$$

Figure 1: Paths of the *BFBM* $B^{H,K}$ by using the Cholesky method







The case $K = 1$ corresponds to the well known fractional Brownian motion $B^H := \{B_t^H, t \geq 0\}$ of the Hurst index $H \in (0,1)$, (*FBM* for short), introduced in Mandelbrot and Van Ness [14]. The *BFBM* $B^{H,K}$ is *HK*-self similar. This is an immediate consequence of the fact that the covariance function in (1) is homogeneous of order $2HK$: for any $h > 0$,

$$\left\{B_{ht}^{H,K}, t \geq 0\right\} \underline{\underline{d}} \left\{h^{HK} B_t^{H,K}, t \geq 0\right\},$$

where $\underline{\underline{d}}$ means equality in the law of all finite dimensional distributions.
In the above figures, we have used the Cholesky method for the Gaussian process and the software *R* for simulating the paths of the *BFBM* $B^{H,K}$. The integer n is the length of the desired sample.

The self similarity and the stationarity of the increments are two main properties for which the *FBM* exhibited success as a modeling tool in engineering, mathematical finance, hydrology and queueing theory, (we refer here, for example, to Addison [1], Cheridito [5], Comegna et al. [6] and Taqqu [21]). The *BFBM* is an extension of the *FBM* which preserves many properties of the *FBM*, but not the stationarity of the increments. Moreover, Russo and Tudor [18] showed that the *BFBM* $B^{H,K}$ behaves as a *FBM* B^{HK} of the Hurst index *HK*. These properties make the *BFBM* a possible candidate for models which involve self similarity, long range dependence, (*LRD* for short), and non stationarity of increments.

It turns out that the *BFBM* is related to some stochastic partial differential equations, (*SPDEs* for short), (see e.g. Lei and Nualart [12] and Swanson [20]). Moreover, suppose that $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ is the solution of the one-dimensional stochastic heat equation on $\mathbb{R}$ with the initial condition $u(0,x) = 0$ defined as follows:

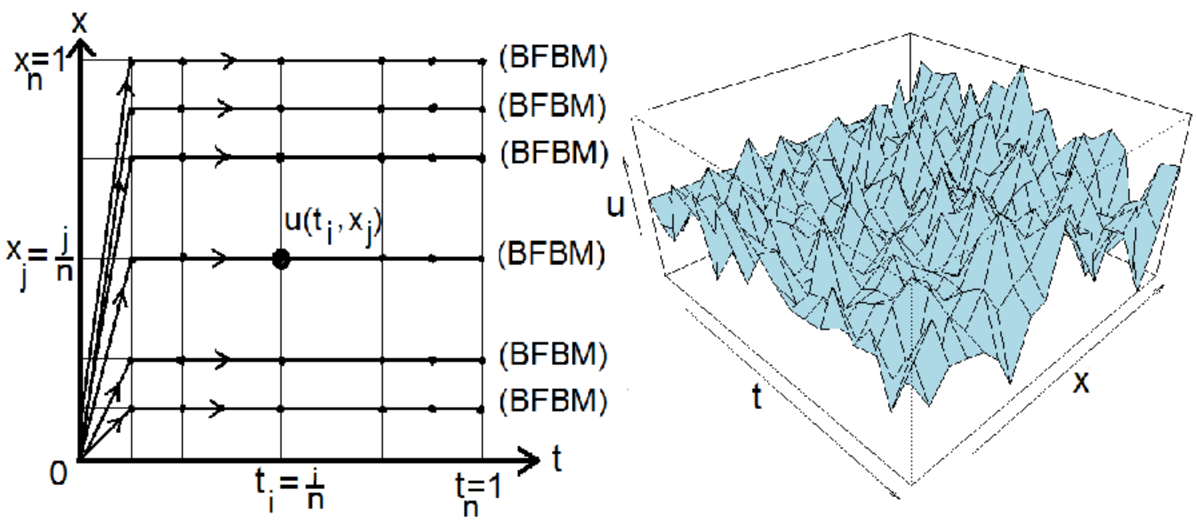
$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 W}{\partial t\, \partial x}, \qquad (*)$$

where $\{W(t,x), t \geq 0, x \in \mathbb{R}\}$ is a two-parameter Wiener process. In other words, W is a centered Gaussian process with covariance:

$\mathsf{E}(W(t,x)W(s,y)) = (s \wedge t)(|x| \wedge |y|)$.

Then, for any $x \in \mathsf{R}$, the process $\{u(t,x), t \geq 0\}$ is a *BFBM* $B^{H,K}$ with parameters $H = K = \frac{1}{4}$, multiplied by the constant $(2\pi)^{\frac{1}{4}}(2)^{-\frac{1}{8}}$. Combining this fact with the Cholesky method, we can easily simulate the solution $u(t,x)$ of the *SPDE* $(*)$, we obtain the paths exhibited in Figure 2.

Figure 2 : Paths of the solution of the SPDE (*)



Remark 1.2. 1. Because of random fluctuations, stochastic differential equations (*SDEs*), offer more realistic mathematical formulations, compared to ordinary differential equations. Moreover, in finance, for example, people extend finite dimensional systems of *SDEs* to infinite dimensional *SPDEs*, (see Benth et al. [4]). Indeed, many physical, biological and financial phenomena can be modeled by *SPDEs*. However, explicit solutions to most of the problems do not exist. Therefore it is natural to simulate a discrete version of these *SPDEs*, (we refer, e.g. to Barth and Lang [2] and Jentzen and Kloeden [9]). In case of the *SPDE* $(*)$, we have obtained a discrete version by using only the link with the trajectories of the *BFBM*, (see the left side of Figure 2), apparently for the first time.

2. Öz Bakan et al. [16] have used the *SPRK* scheme for the discretization of the stochastic control problem governed by *SPDEs* where a maximal principle for the optimal control of the harvesting problem is studied. They have formulated the control problem of *SPDEs* in terms of *SDEs* with the help of matrices and vectors. In a future outlook, we shall try to follow the work [16] for stochastic control problems governed by the *SPDE* $(*)$ by using the link with the *BFBM*. It will be also interesting to find a link between the *BFBM* and the stochastic quasi-linear heat equation used in [16].

The increments of the *BFBM* $B^{H,K}$ are only independents in the case of Brownian motion, $(H = \frac{1}{2}, K = 1)$, and they are not stationary for any $K \in (0,1)$ except the case of the *FBM*, $(K = 1)$. However, $B^{H,K}$ is a quasi-helix in the sense of Kahane [10] : for all $t,s \in [0,1]$,

$$2^{-K}|t-s|^{2HK} \leq \mathsf{E}(B_t^{H,K} - B_s^{H,K})^2 \leq 2^{1-K}|t-s|^{2HK}. \tag{2}$$

Moreover, if we put: $\sigma_\varepsilon^2(t) := \mathsf{E}\left(B_{t+\varepsilon}^{H,K} - B_t^{H,K}\right)^2$, then

$$\lim_{\varepsilon \to 0} \frac{\sigma_\varepsilon^2(t)}{\varepsilon^{2HK}} = 2^{1-K}, \quad t > 0. \tag{3}$$

Therefore, the increments of the *BFBM* $B^{H,K}$ are approximately stationary for small increments: if t is close to s, then

$$\mathsf{E}\left(B_t^{H,K} - B_s^{H,K}\right)^2 \approx 2^{1-K}|t-s|^{2HK}.$$

Since the *BFBM* $B^{H,K}$ is a Gaussian process, then for any $p > 0$, we have

$$\mathsf{E}\left|B_t^{H,K} - B_s^{H,K}\right|^p = C\left(\mathsf{E}\left|B_t^{H,K} - B_s^{H,K}\right|^2\right)^{\frac{p}{2}}. \tag{4}$$

Therefore, by virtue of (2) and the famous Kolmogorov continuity criterion, $B^{H,K}$ is Hölder continuous of order δ for any $\delta < HK$. Moreover, with probability one, the trajectories of the *BFBM* $B^{H,K}$ are not differentiable and due to unbounded variation on any finite interval (except, of course, in the degenerate case $H = K = 1$). More precisely, for every $H \in (0,1)$ and $K \in (0,1]$, we have

$$\lim_{\varepsilon \to 0} \sup_{t \in [t_0-\varepsilon, t_0+\varepsilon]} \left| \frac{B_t^{H,K} - B_{t_0}^{H,K}}{t - t_0} \right| = +\infty,$$

with probability one for every t_0. Then according to Russo and Tudor [18], if we put:

$$V_t^{\pi,\alpha}(B^{H,K}) := \sum_{i=1}^{n-1} \left|B_{t_{i+1}}^{H,K} - B_{t_i}^{H,K}\right|^\alpha,$$

where $\pi := (0 = t_0 <, \ldots, < t_n = t)$ denoting a partition of $[0,t]$ and $|\pi| := \max_i |t_{i+1} - t_i|$, then

$$L^1(\Omega) - \lim_{|\pi| \to 0} V_t^{\pi,\alpha}(B^{H,K}) = 0, \ if\ \alpha > \tfrac{1}{HK}\ ; \ = +\infty, \ if\ \alpha < \tfrac{1}{HK}\ ; = 2^{\frac{1-K}{HK}} \rho_{HK} t, \ if\ \alpha = \tfrac{1}{HK}.$$

Remark 1.2. An interesting property of the *BFBM* $B^{H,K}$ is the expression of its quadratic variation ($< B^{H,K} >_t, \alpha = 2$). The following properties hold true:

- If $\frac{1}{2} < HK < 1$, then the quadratic variation of $B^{H,K}$ is zero.
- If $0 < HK < \frac{1}{2}$, then the quadratic variation of $B^{H,K}$ does not exist.
- If $HK = \frac{1}{2}$, then the quadratic variation of $B^{H,K}$ at time t is equal to $2^{1-K}t$.

The last property is remarkable. Indeed, for $HK = \frac{1}{2}$ we have a Gaussian process which has the same quadratic variation as a Brownian motion. Moreover, $B^{H,K}$ is not a semimartingale if $HK \neq \frac{1}{2}$. Nevertheless, a stochastic integral with respect to the *BFBM* was developed, (see, for examples, Es-Sebaiy and Tudor [7] and Russo and Tudor [18]).

To end this account on facts about *BFBM*, we present a useful decomposition in the law of $B^{H,K}$. This was given and used by Lei and Nualart [12] to yield the $\frac{1}{HK}$-variation of $B^{H,K}$, and also used by Maejima and Tudor [13] to prove that the incremental process of $B^{H,K}$ is approximately stationary for large increments. In fact: let $W := \{W_\theta, \theta \geq 0\}$ be a Brownian motion independent of $B^{H,K}$, then for any $K \in (0,1)$, let $X^K := \{X_t^K, t \geq 0\}$ be the process defined by:

$$X_t^K := \int_0^\infty (1 - e^{-\theta t})\theta^{-\frac{(1+K)}{2}} dW_\theta.$$

Then X^K is a centered Gaussian process with the covariance function

$$\mathsf{E}(X_t^K X_s^K) = \frac{\Gamma(1-K)}{K}(t^K + s^K - (t+s)^K).$$

This process is characterized by trajectories which are infinitely differentiable on $(0, +\infty)$ and absolutely continuous on $[0, +\infty)$. This fact will be useful in the sequel, particularly in section 6.

The authors of [12] showed the following decomposition in the law of the *BFBM* $B^{H,K}$:

$$(C_1(K)X_t^{H,K} + B_t^{H,K}\ ;\ t \geq 0)\ \underline{\underline{d}}\ (C_2(K)B_t^{HK}\ ;\ t \geq 0), \tag{5}$$

where $X_t^{H,K} := X_{t^{2H}}^K$, $C_1(K) = \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}$ and $C_2(K) = 2^{\frac{(1-K)}{2}}$.

The rest of this paper is organized as follows. In section 2, we define our new process and we called in short *MBBM*, and calculate its covariance function. As an application, we study the Hölder continuity and self similarity of its trajectories. In section 3, we study the quadratic variation of the *MBBM* by using that of the *BFBM* and the second moment of increments of the *BFBM*. In section 4, we deal with the Markov property of the *MBBM* by using the famous result of Revuz and Yor [19] concerning the case of Gaussian processes. In section 5, we prove the non-differentiability of the *MBBM*, and follow a classical argument used by Houdré and Villa in [8]. However, in section 6, we study the α-differentiability of the *MBBM* via a local Hölderian behavior. Finally, in the last section, we study the *LRD* of the *MBBM* and the limiting process of the increments process of the *MBBM*. This last fact is to be explained from the perspective of the noise generated by the *MBBM*. The paper ends with a conclusion and an outlook on future studies, on further research questions and applications. We believe that our process can be a possible candidate for models which involve self similarity, *LRD* and non stationarity of increments.

2. Definition and Preliminary Properties

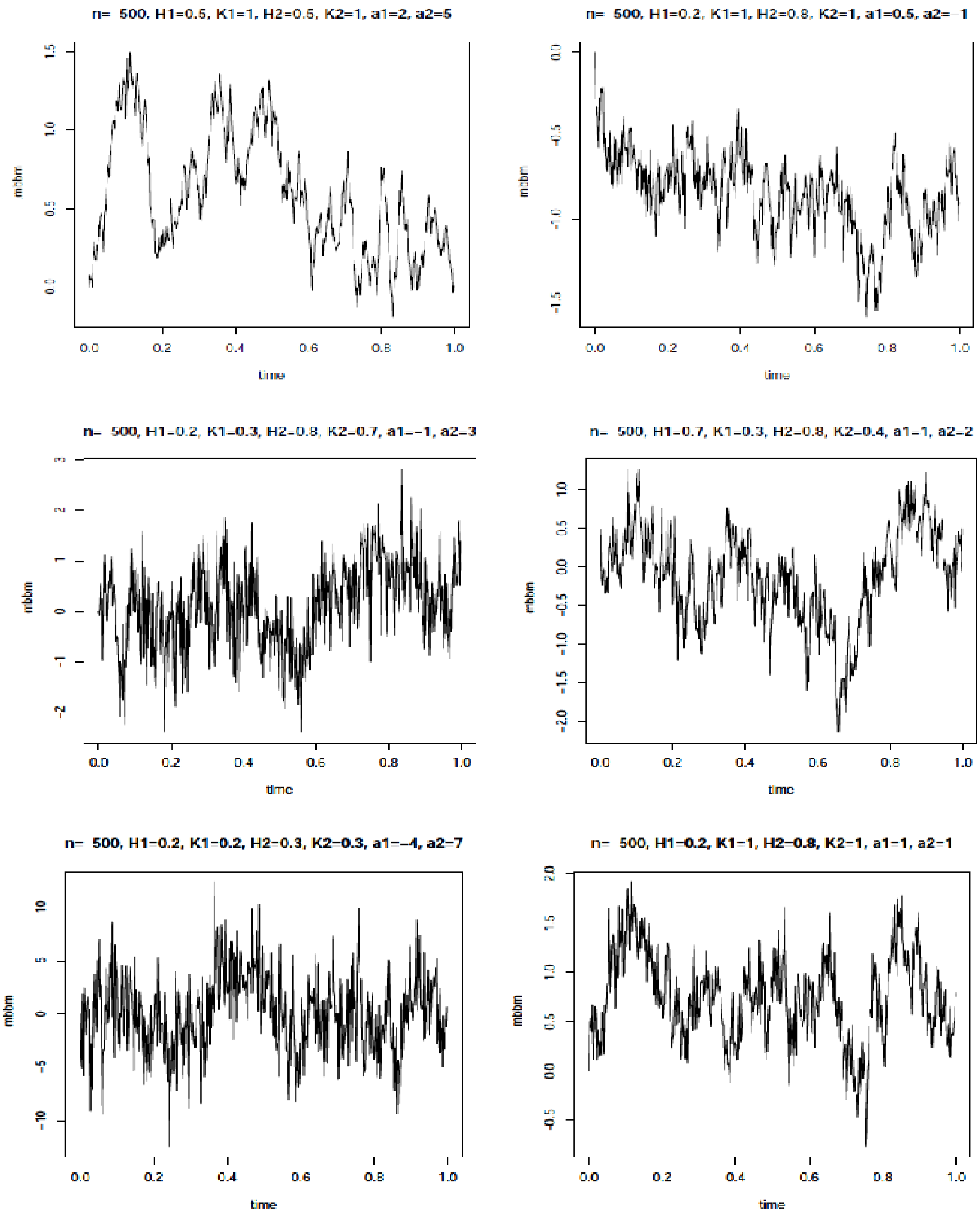
Definition 2.1. Let $m \geq 2$ an integer, $a = (a_1, \ldots, a_m) \in \mathsf{R}^m \backslash \{(0, \ldots, 0)\}$, $H = (H_1, \ldots, H_m) \in (0,1)^m$ and $K = (K_1, \ldots, K_m) \in (0,1]^m$.
We call mixed bifractional Brownian motion, (*MBBM*), of parameters a, H and K, the process denoted by $N^{H,K} := \left\{N_t^{H,K}(a), t \geq 0\right\}$, and defined on the probability space (Ω, F, P) as follows:

$$N_t^{H,K}(a) := \sum_{i=1}^m a_i B_t^{H_i,K_i}, \quad \forall t \geq 0,$$

where $\left\{B_t^{H_i,K_i}, t \geq 0\right\}$, $i = 1, \ldots, m$, are m independent *BFBM* of parameters H_i and K_i defined on (Ω, F, P).

Example 2.1. In the case: $K = (1, \ldots, 1)$, we find the process introduced in Thäle [22].

Figure 3: Paths of the *MBBM* $N^{H,K}, m = 2,$ by using the Cholesky method



In particular, for $m = 2$, if $H_1 = \frac{1}{2}$, $H_2 \in (0,1)$ and $K = (1,1)$, we find the mixed fractional Brownian motion introduced in Cheridito [5] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. This last process was studied later by Zili in [25]. Now we are ready to study our process. We begin by the Hölder continuity and self similarity of its trajectories via the calculation of its covariance function.

Lemma 2.1.
(i) The MBBM $N^{H,K}$ is a centered Gaussian process starting from zero.
(ii) For all $s,t \geq 0$, the covariance function is given by:

$$Cov(N_t^{H,K}(a), N_s^{H,K}(a)) = \sum_{i=1}^{m} \frac{a_i^2}{2^{K_i}} \left[(t^{2H_i} + s^{2H_i})^{K_i} - |t-s|^{2H_iK_i} \right], \tag{6}$$

in particular, for any $t \geq 0$, $\mathsf{E}\left(N_t^{H,K}(a)\right)^2 = \sum_{i=1}^{m} a_i^2 t^{2H_i K_i}$.

(iii) Increments. For any $0 \leq s \leq t$,

$$\mathsf{E}\left(N_t^{H,K}(a) - N_s^{H,K}(a)\right)^2$$
$$= \sum_{i=1}^{m} a_i^2 \left(t^{2H_i K_i} + s^{2H_i K_i} - 2^{1-K_i}\left[\left(t^{2H_i} + s^{2H_i}\right)^{K_i} - |t-s|^{2H_iK_i} \right] \right).$$

(iv) Hölder continuity. Let $H_{i_0}K_{i_0} = \min\left\{H_iK_i \ ; \ i = 1, \ldots, m\right\}$. For all $T > 0$, the MBBM $N^{H,K}$ admits a version whose sample paths are Hölder continuous of order $\delta < H_{i_0}K_{i_0}$ on the interval $[0,T]$.
(v) The increments process of the MBBM $N^{H,K}$ are not stationary except the case where $K = (1, \ldots, 1)$.
(vi) A mixed self similarity property. For any $h > 0$, the MBBM $N^{H,K}$ satisfy:
$\left\{N_{ht}^{H,K}(a), t \geq 0\right\} \underline{\underline{d}} \left\{N_t^{H,K}\left(a_1 h^{H_1K_1}, \ldots, a_m h^{H_mK_m}\right), t \geq 0\right\}$.

Proof. The proofs of $(i), (ii)$ are simple consequences of the definition of the *MBBM*. (iii) can easily be deduced from (ii). The point (iv) is a consequence of (2) and (4). In fact, we have

$$\sum_{i=1}^{m} \frac{a_i^2}{2^{K_i}} |t-s|^{2H_iK_i} \leq \mathsf{E}\left(N_t^{H,K} - N_s^{H,K}\right)^2 \leq \sum_{i=1}^{m} \frac{a_i^2}{2^{K_i-1}} |t-s|^{2H_iK_i}.$$

For (v), it suffices to see that if $K_i \neq 1$ holds, then $\mathsf{E}\left(N_{2t}^{H,K} - N_t^{H,K}\right)^2 \neq \mathsf{E}\left(N_t^{H,K}\right)^2$ for any $t \geq 0$, with

$$\mathsf{E}\left(N_{2t}^{H,K} - N_t^{H,K}\right)^2 = \sum_{i=1}^{m} a_i^2 t^{2H_iK_i} \left(4^{H_iK_i} + 1 - 2^{1-K_i}\left[(4^{H_i} + 1)^{K_i} - 1 \right] \right).$$

Now we deal with (vi). Since the *MBBM* $N^{H,K}$ is a centered Gaussian process, then we have only to prove that the two processes : $\left\{N_{ht}^{H,K}(a), t \geq 0\right\}$ and $\left\{N_t^{H,K}\left(a_1 h^{H_1K_1}, \ldots, a_m h^{H_mK_m}\right), t \geq 0\right\}$ have the same covariance function.

By virtue of (6), we have

$$Cov\left(N_t^{H,K}\left(a_1h^{H_1K_1},\ldots,a_mh^{H_mK_m}\right),N_s^{H,K}\left(a_1h^{H_1K_1},\ldots,a_mh^{H_mK_m}\right)\right)$$
$$=\sum_{i=1}^{m}\frac{(a_ih^{H_iK_i})^2}{2^{K_i}}\left(\left(t^{2H_i}+s^{2H_i}\right)^{K_i}-|t-s|^{2H_iK_i}\right)$$
$$=\sum_{i=1}^{m}\frac{a_i^2}{2^{K_i}}\left([(ht)^{2H_i}+(hs)^{2H_i}]^{K_i}-|ht-hs|^{2H_iK_i}\right)$$
$$=\sum_{i=1}^{m}a_i^2\mathsf{E}\left(N_{ht}^{H_i,K_i}(a)N_{hs}^{H_i,K_i}(a)\right)$$
$$=Cov\left(N_{ht}^{H,K}(a)N_{hs}^{H,K}(a)\right),$$

which is our desired result. ■

3. Quadratic Variation

In this section, we study the quadratic variation of the *MBBM* by using that of the *BFBM* and the second moment of increments of the *BFBM*. The following lemma is a consequence of remark 1.2.

Lemma 3.1.
1. *If for all* $i=1,\ldots,m$, *we have:* $\frac{1}{2}<H_iK_i<1$, *then* $<N^{H,K}>_t=0$.
2. *If it exists* $i=1,\ldots,m$ *such that:* $0<H_iK_i<\frac{1}{2}$, *then* $<N^{H,K}>_t=+\infty$.
3. *If for all* $i=1,\ldots,m$, *we have:* $H_iK_i=\frac{1}{2}$. *Then* $<N^{H,K}>_t=\sum_{i=1}^{m}a_i^2 2^{1-K_i}t$.

Proof. 1. Invoke

$$V_t^{\pi,2}(N^{H,K}(a))=\sum_{i=1}^{n-1}\left(N_{t_{i+1}}^{H,K}-N_{t_i}^{H,K}\right)^2=\sum_{i=1}^{n-1}\left(\sum_{j=1}^{m}a_j\left(B_{t_{i+1}}^{H_jK_j}-B_{t_i}^{H_jK_j}\right)\right)^2$$
$$=\sum_{i=1}^{n-1}\sum_{l,l'=1}^{m}a_la_{l'}\left(B_{t_{i+1}}^{H_lK_l}-B_{t_i}^{H_lK_l}\right)\left(B_{t_{i+1}}^{H_{l'}K_{l'}}-B_{t_i}^{H_{l'}K_{l'}}\right)$$
$$=\sum_{i=1}^{n-1}a_l^2V_t^{\pi,2}(B^{H_l,K_l})+\sum_{l\neq l'}^{m}a_la_{l'}U(l,l'),$$

where

$$U(l,l'):=\sum_{i=1}^{n-1}\left(B_{t_{i+1}}^{H_lK_l}-B_{t_i}^{H_lK_l}\right)\left(B_{t_{i+1}}^{H_{l'}K_{l'}}-B_{t_i}^{H_{l'}K_{l'}}\right).$$

Clearly $\mathsf{E}(U(l,l^{'})) = 0$. Then to reach the required conclusion it suffices to show that: $\lim_{n\to\infty} \mathsf{E}|U(l,l^{'})| = 0$.

Using the Hölder inequality and (2), we get

$$\mathsf{E}|U(l,l^{'})| \leq \sum_{i=1}^{n-1} \mathsf{E}\left|\left(B_{t_{i+1}}^{H_l K_l} - B_{t_i}^{H_l K_l}\right)\left(B_{t_{i+1}}^{H_{l'} K_{l'}} - B_{t_i}^{H_{l'} K_{l'}}\right)\right|$$

$$\leq \sum_{i=1}^{n-1}\left(\mathsf{E}\left(B_{t_{i+1}}^{H_l K_l} - B_{t_i}^{H_l K_l}\right)^2\right)^{\frac{1}{2}}\left(\mathsf{E}\left(B_{t_{i+1}}^{H_{l'} K_{l'}} - B_{t_i}^{H_{l'} K_{l'}}\right)^2\right)^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{n-1} 2^{1-\frac{K_l+K_l^{'}}{2}} |t_{i+1} - t_i|^{H_l K_l + H_{l'} K_{l'}}$$

$$\leq 2^{1-\frac{K_l+K_l^{'}}{2}}\left(|\pi|^{H_l K_l + H_{l'} K_{l'} - 1}\right) t \longrightarrow 0 \quad as \quad n \to +\infty,$$

where

$\sum_{i=1}^{n-1} |t_{i+1} - t_i| = t$ and $H_l K_l + H_{l'} K_{l'} - 1 > 0$.

2. Is evident.

3. In this case, we need to prove that: $\lim_{n\to+\infty} \mathsf{E}|U(l,l^{'})|^2 = 0$. So, we consider the following function, which appeared in Es-Sebaiy and Tudor [7],

$$\theta_n^l(i,j) := \mathsf{E}\left(\left(B_{t_{i+1}}^{H_l K_l} - B_{t_i}^{H_l K_l}\right)\left(B_{t_{j+1}}^{H_l K_l} - B_{tj}^{H_l K_l}\right)\right).$$

We then have,

$$\mathsf{E}|U(l,l^{'})|^2 = \sum_{i,j=1}^{n-1} \theta_n^l(i,j)\theta_n^{l'}(i,j)$$

$$= \sum_{i=1}^{n-1} (\theta_n^l(i,i))^2 \left(\theta_n^{l'}(i,i)\right)^2 + 2\sum_{i<j}^{n-1} \theta_n^l(i,j)\theta_n^{l'}(i,j).$$

Finally by using the properties of the functions $\theta_n^l(i,i)$ and $\theta_n^l(i,j)$ for $i < j$, obtained in [7] to prove Lemma 2, we arrive at the desired result. ■

4. The Markov Property

Here, we deal with the Markov property of the *MBBM* by using the famous result of Revuz and Yor [19] concerning the case of Gaussian processes.

Theorem 4.1. *For every* $a = (a_1, .., a_m) \in R^m \backslash \{(0, \ldots, 0)\}$, $H = (H_1, .., H_m) \in (0,1)^m$ *and* $K = (K_1, \ldots, K_m) \in (0,1)^m$ *such that* $\exists\ i \in \{1, \ldots, m\}$ *such that* $H_i K_i \neq \frac{1}{2}$, *the MBBM* $\left\{N_t^{H,K}(a), t \geq 0\right\}$ *is not a Markovian process.*

Proof. By lemma 2.1, $N^{H,K}$ is a centered Gaussian process and for all $t > 0$,

$$CovN_t^{H,K}(a), N_t^{H,K}(a) = \mathsf{E}N_t^{H,K}(a)^2 = \sum_{i=1}^{m} a_i^2 t^{2H_iK_i} > 0.$$

When $N^{H,K}$ is a Markovian process, according to Revuz and Yor [19], for all $s < t < u$, we should have,

$$CovN_s^{H,K}, N_u^{H,K} CovN_t^{H,K}, N_t^{H,K} = CovN_s^{H,K}, N_t^{H,K} CovN_t^{H,K}, N_u^{H,K}. \tag{7}$$

Let us consider the two numbers:

$$H_{i_0}K_{i_0} = \min H_iK_i \ ; \ i \in \{1,..,m\}, \quad H_{i_1}K_{i_1} = \max H_iK_i \ ; \ i \in \{1,..,m\},$$

for two distinct cases.
First case: If $H_iK_i > \frac{1}{2}$, with $H_{i_1}K_{i_1} > \frac{1}{2}$. By virtue of (6) and (7) with
$1 < s = \sqrt{t} < t < u = t^2$,
we would have,

$$\left(\sum_{i=1}^{m} a_i^2 t^{2H_iK_i}\right) \sum_{i=1}^{m} \frac{a_i^2}{2^{K_i}} t^{4H_i} + t^{H_iK_i} - t^{4H_iK_i} 1 - t^{-3/2\,2H_iK_i}$$

$$= \sum_{i=1}^{m} \frac{a_i^2}{2^{K_i}} t^{H_i} + t^{2H_iK_i} - t^{2H_iK_i} 1 - t^{-1/2\,2H_iK_i}$$

$$\times \sum_{i=1}^{m} \frac{a_i^2}{2^{K_i}} t^{2H_i} + t^{4H_iK_i} - t^{4H_iK_i} 1 - t^{-1\,2H_iK_i}.$$

Therefore, when $t \to +\infty$, we can write

$$\sum_{i=1}^{m} a_i^2 t^{2H_iK_i} \times \sum_{i=1}^{m} \frac{a_i^2}{2^{K_i}} K_i t^{4H_iK_i-3H_i} + \frac{1}{2} K_i(K_i-1) t^{4H_iK_i-6H_i} + o(t^{4H_iK_i-6H_i})$$

$$- 2H_iK_i t^{4H_iK_i-3/2} + H_iK_i(2H_iK_i-1) t^{4H_iK_i-3} + o(t^{4H_iK_i-3})$$

$$- \sum_{i=1}^{m} \frac{a_i^2}{2^{K_i}} K_i t^{2H_iK_i-H_i} + \frac{1}{2} K_i(K_i-1) t^{2H_iK_i-2H_i} + o(t^{2H_iK_i-2H_i})$$

$$- 2H_iK_i t^{2H_iK_i-1/2} + H_iK_i(2H_iK_i-1) t^{2H_iK_i-1} + o(t^{2H_iK_i-1})$$

$$- \sum_{i=1}^{m} \frac{a_i^2}{2^{K_i}} K_i t^{4H_iK_i-2H_i} + \frac{1}{2} K_i(K_i-1) t^{4H_iK_i-4H_i} + o(t^{4H_iK_i-4H_i})$$

The left member of the last equation would tend to zero as t goes to infinity. Consequently

$$\lim_{t\to\infty}\frac{a_{i_1}^4}{2^{K_{i_1}}}H_{i_1}K_{i_1}(2H_{i_1}K_{i_1}-1)(1-2^{-K_{i_1}}H_{i_1}K_{i_1}(2H_{i_1}K_{i_1}-1))t^{6H_{i_1}K_{i_1}-3}=0,$$

which is true if, and only if $H_{i_1}K_{i_1}=\frac{1}{2}$. So $N^{H,K}$ is not a Markovian process.

Second case: If $H_iK_i<\frac{1}{2}$, with $H_{i_0}K_{i_0}<\frac{1}{2}$. In the same way, as explained in the first case, with

$$0<s=t^2<t<u=\sqrt{t}<1 \qquad t\to 0,$$

we get

$$\lim_{t\to 0}\frac{a_{i_0}^4}{2^{K_{i_0}}}H_{i_0}K_{i_0}(2H_{i_0}K_{i_0}-1)(1-2^{-K_{i_0}}H_{i_0}K_{i_0}(2H_{i_0}K_{i_0}-1))t^{3H_{i_0}K_{i_0}+3}=0,$$

which is true if, and only if $H_{i_0}K_{i_0}=\frac{1}{2}$. So in this case too, $N^{H,K}$ is not a Markovian process.■

Remark 4.1. The little Landau "o" means that: "as $t\to t_0$, $f(t)=o(g(t))$" means $f(t)/g(t)\to 0$ as $t\to t_0$.

5. The Non-differentiability of the *MBBM*

This section reports on the following result concerning non-differentiability of the *MBBM*. Here we use the classical arguments used by Houdré and Villa in [8].

Proposition 5.1. *For every $H=(H_1,\ldots,H_m)\in]0,1[^m$ and $K=(K_1,\ldots,K_m)\in]0,1]^m$ there holds*

$$\lim_{\epsilon\to 0^+}\sup_{t\in[t_0-\epsilon,\,t_0+\epsilon]}\left|\frac{N_t^{H,K}-N_{t_0}^{H,K}}{t-t_0}\right|=+\infty,$$

with probability one for every $t_0\in R$.

Proof. For $l,n\in\mathbb{N}$, let

$$A_n^{(l)}=\left\{w\in\Omega:\sup_{t\in[t_0-\frac{1}{n},\,t_0+\frac{1}{n}]}\left|\frac{N_t^{H,K}-N_{t_0}^{H,K}}{t-t_0}\right|>l\right\},$$

Clearly $A_n^{(l)}\supseteq A_{n+1}^{(l)}$ when $A^{(l)}=\bigcap_{n=1}^{\infty}A_n^{(l)}$, also $A^{(l)}\supseteq A^{(l+1)}$. So to prove the result, it is enough to show that,

$$P\bigcap_{l=1}^{\infty}A^{(l)}=\lim_{l\to\infty}P(A^{(l)})=\lim_{l\to\infty}\lim_{n\to\infty}P(A_n^{(l)})=1.$$

But,

$$P(A_n^{(l)})\geq PN_{t_0+\frac{1}{n}}^{H,K}-N_{t_0}^{H,K}>\frac{l}{n},$$

and we will show that,

$$\lim_{n\to+\infty}PN_{t_0+\frac{1}{n}}^{H,K}-N_{t_0}^{H,K}\leq\frac{l}{n}=0.$$

$N^{H,K}_{t_0+\frac{1}{n}} - N^{H,K}_{t_0}$ is a centered Gaussian random variable with variance $\sigma_n^2(t_0)$ such that,

$$\sigma_n^2(t_0) = \mathsf{E}\left(N^{H,K}_{t_0+\frac{1}{n}} - N^{H,K}_{t_0}\right)^2 = \sum_{i=1}^{m} a_i^2 \mathsf{E}\left(B^{H_i,K_i}_{t_0+\frac{1}{n}} - B^{H_i,K_i}_{t_0}\right)^2.$$

Clearly,

$$P\left(\left|N^{H,K}_{t_0+\frac{1}{n}} - N^{H,K}_{t_0}\right| \leq \frac{l}{n}\right)$$

$$= \frac{1}{\sigma_n(t_0)\sqrt{2\pi}} \int_{-\frac{l}{n}}^{\frac{l}{n}} \exp\left(-\frac{x^2}{2\sigma_n^2(t_0)}\right) dx \leq \frac{l}{n}\sqrt{\frac{2}{\pi}} \times \frac{1}{\sigma_n(t_0)},$$

and by (3) with $\varepsilon = \frac{l}{n}$, we have: $\lim_{n\to+\infty} n^2\sigma_n^2(t_0) = +\infty$, with

$$\lim_{n\to+\infty} P\left(\left|N^{H,K}_{t_0+\frac{1}{n}} - N^{H,K}_{t_0}\right| \leq \frac{l}{n}\right) = 0,$$

and the required result follows. ■

6. The α-differentiability of the *MBBM*

The following notion of α-derivative of a function have been introduced by Kowankar and Gangal, [11], and studied by Ben Adda and Cresson in [3]. A geometrical meaning of this derivative is that it gives the local Hölderian behavior of the function.

Definition 6.1. Let f be a continuous function on $[a,b]$, and let $\alpha \in (0,1)$.
1. Call a right, (respectively left), local fractional α-derivative of f at $t_0 \in [a,b]$ the following quantity:

$$d_\sigma^\alpha f(t_0) = \Gamma(1+\alpha) \lim_{t\to t_0^\sigma} \frac{\sigma(f(t)-f(t_0))}{|t-t_0|^\alpha},$$

for $\sigma = +$ (respectively $\sigma = -$), where Γ is the Euler function.
2. The function f is α-derivative at $t_0 \in [a,b]$ if and only if $d_+^\alpha f(t_0)$ and $d_-^\alpha f(t_0)$ exist and are equal. In this case, we denote by $d^\alpha f(t_0)$ the α-derivative at t_0.

Theorem 6.1. *Let* $H_{i_0}K_{i_0} = \min\{H_iK_i\ ;\ i = 1,\ldots,m\}$.
1. *For all* $\alpha \in (0, H_{i_0}K_{i_0})$, *the sample paths of the MBBM* $N^{H,K}$ *are almost surely* α*-differentiable at every* $t_0 \geq 0$, *and*

$$\forall t_0 \geq 0, \quad P\left(d^\alpha N^{H,K}_{t_0} = 0\right) = 1.$$

2. *For all* $\alpha \in (H_{i_0}K_{i_0}, 1)$, *the sample paths of the MBBM* $N^{H,K}$ *are nowhere* α*-differentiable, almost surely*.

Proof. 1. Clearly the probability is the same if we use the decomposition in the law (5). Here we only give the proof for the case $\sigma = +$ and the proof for $\sigma = -$ is similar. For $0 \leq t_0 < t$, we have

$$P\left(d^\alpha N^{H,K}_{t_0} = 0\right) = P\left(\lim_{t\to t_0^+} \frac{N^{H,K}_t - N^{H,K}_{t_0}}{(t-t_0)^\alpha} = 0\right)$$

$$= P\left(\lim_{t\to t_0^+}\sum_{i=1}^m a_i \frac{B_t^{H_i,K_i} - B_{t_0}^{H_i,K_i}}{(t-t_0)^\alpha} = 0\right)$$

$$= P\left(\lim_{t\to t_0^+}\sum_{i=1}^m a_i \frac{C_2 B_t^{H_iK_i} - B_{t_0}^{H_iK_i} - C_1 X_t^{H_i,K_i} - X_{t_0}^{H_i,K_i}}{(t-t_0)^\alpha} = 0\right)$$

$$= P\left(\lim_{t\to t_0^+}\sum_{i=1}^m a_i \left(\frac{C_2 B_{t-t_0}^{H_iK_i}}{(t-t_0)^\alpha} - C_1(t-t_0)^{1-\alpha}\frac{X_t^{H_i,K_i} - X_{t_0}^{H_i,K_i}}{t-t_0}\right) = 0\right)$$

$$= P\left(\lim_{t\to t_0^+}\sum_{i=1}^m a_i C_2 (t-t_0)^{H_iK_i-\alpha} B_1^{H_iK_i} - C_1(t-t_0)^{1-\alpha} X_{t_0}^{H_i,K_i\,\prime} = 0\right)$$

$$= P(0=0) = 1,$$

where we have used in the last expression the self similarity and the stationarity of increments of the *FBMs* $B^{H_i\,K_i}$, and the fact that the trajectories of $X^{H_i,\,K_i}$ are infinitely differentiable on $(0,+\infty)$.

2. For any $d > 0$, we may define the events

$$A(t) = \sup_{0\le s\le t} \frac{N_s^{H,K}(a)}{s^\alpha} > d.$$

Then for any decreasing sequence $t_n \to 0$, we have $A(t_{n+1}) \subset A(t_n)$. Thus

$$P\lim_{n\to\infty} A(t_n) = \lim_{n\to\infty} P(A(t_n)),$$

and by using the mixed self similarity of $N^{H,\,K}$,

$$P(A(t_n)) \ \ge P\frac{N_{t_n}^{H,\,K}(a)}{t_n^\alpha} > d = P\sum_{i=1}^m a_i t_n^{H_i\,K_i-\alpha} B_1^{H_i,\,K_i} > d.$$

Since $\alpha > H_{i_0}K_{i_0}$, then

$$P(A(t_n)) \ \ge P a_{i_0} B_1^{H_{i_0},\,K_{i_0}} + \sum_{i\ne i_0}^m a_i t_n^{H_iK_i - H_{i_0}K_{i_0}} B_1^{H_i,\,K_i} > d t_n^{\alpha - H_{i_0}\,K_{i_0}},$$

and

$$\lim_{n\to\infty} P(A(t_n)) \ \ge P a_{i_0} B_1^{H_{i_0},\,K_{i_0}} \ge 0 = 1.$$

Here the proof ends. ■

Remark 6.1. Clearly a similar way as above gives information about the α-differentiability of the *BFBM* $B^{H,\,K}$.

7. On the Increments Process of the *MBBM*

The first result in this section concerns the limiting process of the increments process of the *MBBM*. The limit is a process with stationary increments. Here we combine Theorem 2.1 of [13] with the independence of the *n BFBMs* in the definition of the *MBBM*.

Theorem 7.1. *The increments process of the MBBM* $N^{H,K}$ *is not stationary except for the case of* $K = (1,\ldots,1)$. *It is approximately stationary, however, for large increments in the sense that, when* $h \to +\infty$, *the increments process*

$$\left(N^{H,K}_{t+h} - N^{H,K}_{h}, t \geq 0\right),$$

converges to the process

$$M^{H,K}_t := \sum_{i=1}^{m} a_i 2^{\frac{1-K_i}{2}} B^{H_i K_i}_t.$$

Remark 7.1. For some stochastic properties and characteristics of the process $M^{H,K}_t$, we refer the reader to Thäle, [22].

The result of theorem 7.1 will be explained from the perspective of the noise generated by the *MBBM* $N^{H,K}$, defined by

$$Z_n = N^{H,K}_{n+1} - N^{H,K}_{n}.$$

Recall then that in the case: $K = (1,\ldots,1)$, we have for every $h \in \mathbb{N}$ and for every $n \geq 0$, $\mathbb{E}(Z_h Z_{h+n}) = \mathbb{E}(Z_0 Z_n)$. But when $K \neq (1,\ldots,1)$, we may denote

$$R(0,n) := \mathbb{E}(Z_0 Z_n) = \mathbb{E}N^{H,K}_1 N^{H,K}_{n+1} - N^{H,K}_{n},$$

and

$$R(h,h+n) := \mathbb{E}(Z_h Z_{h+n}) = \mathbb{E}N^{H,K}_{h+1} - N^{H,K}_{h} N^{H,K}_{h+n+1} - N^{H,K}_{h+n}.$$

Furthermore, let us compute the term $R(h,h+n)$ and estimate how different it is from $R(0,n)$ in the case of the *MBBM*. For every $n \geq 1$, we have

$$R(h,h+n) = \sum_{i=1}^{m} \frac{a_i^2}{2^{K_i}} f_i^h(n) + g_i(n),$$

where

$$f_i^h(n) = (h+1)^{2H_i} + (h+n+1)^{2H_i K_i} - (h+1)^{2H_i} + (h+n)^{2H_i K_i}$$

$$- h^{2H_i} + (h+n+1)^{2H_i K_i} + h^{2H_i} + (h+n)^{2H_i K_i},$$

and

$$g_i(n) = (n+1)^{2H_i K_i} + (n-1)^{2H_i K_i} - 2n^{2H_i K_i}, \; i = 1,\ldots,m.$$

Remark 7.2. For $i = 1,\ldots,m$, the function g_i is, modulo a constant, the covariance function of the fractional Brownian noise with the Hurst index $H_i K_i$. Indeed, for $n \geq 1$,

$g_i(n) = 2\mathrm{E}B_1^{H_iK_i}B_{n+1}^{H_iK_i} - B_n^{H_iK_i}$.

The second result shows, by means of Theorem 3.3 of [13], how far the mixed bifractional Brownian noise is from the stationary case.

Theorem 7.2. *For each $i = 1,\dots,m$ and n it holds that as $h \to +\infty$,*

$f_i^h(n) = 2H_i^2K_i(K_i - 1)h^{2(H_iK_i-1)}(1 + o(1))$.

Therefore $\lim_{h\to+\infty} f_i^h(n) = 0$, *for each n.*

Remark 7.3. The mixed bifractional Brownian noise is not stationary. However, the meaning of the theorem above is that it converges to a stationary sequence.

We may now use Theorem 4.1 and Remark 4.2 in [13], and Proposition 7 and Remark 7 in [18], to arrive at the following result that concerns the long range dependence of the *MBBM*.

Lemma 7.1. *For every integer $h \geq 1$ and $n \geq 0$, we have*
1. *Long memory: If it exists $i = 1,\dots,m$ such that $H_iK_i > \frac{1}{2}$, then*

$\sum_{n\geq 0} R(h,h+n) = \infty$.

2. *Short memory: If for all $i = 1,\dots,m$ we have $H_iK_i < \frac{1}{2}$, then*

$\sum_{n\geq 0} R(h,h+n) = \infty$.

Remark 7.4. The *LRD* and long memory are synonymous notions. *LRD* measures long-term correlated processes. *LRD* exists when past events influence the present and possibly future events. *LRD* is a characteristic of phenomena whose autocorrelation functions decay rather slowly. The presence and the extent of *LRD* is usually measured by the parameters of the process. The "specialness" of *LRD* indicates that most stationary stochastic processes do not have it. This also makes it intuitive that non-stationary processes can provide an alternative explanation to the empirical phenomena that the notion of *LRD* is designed to address. This connection between long memory and lack of stationarity is very important, (see for example Maejima and Tudor [13] in case of the *BFBM*, and C-René Dominique and Luis Eduardo Solis Rivera-Solis [17] in case of the mixed *FBM*).

In the end of this section, we consider the asymptotic behavior of correlation for large increments of the *MBBM*. In fact, by using theorem 7.1, we have

$$\lim_{h\to+\infty} C_{t,s,h} := Cov(N_{t+h}^{H,K} - N_t^{H,K}, N_{s+h}^{H,K} - N_s^{H,K}) = Cov(M_t^{H,K}, M_s^{H,K}).$$

Next, we may state the following result.

Lemma 7.2. *The large increments of the MBBM $N^{H,K}$ are asymptotically:*
1. *Persistent (positively correlated): if for all $i = 1,\dots,m$: $H_iK_i > \frac{1}{2}$.*

2. *Uncorrelated: if for all* $i = 1, \ldots, m$*:* $H_i K_i = \frac{1}{2}$.
3. *Anti-persistent (negatively correlated): if for all* $i = 1, \ldots, m$: $H_i K_i > \frac{1}{2}$.

This lemma can be recast as follows.

Corollary 7.1. *If b and c are two numbers such that* $|b| \leq |c|$, *then for h large enough, we have*

$$C_{t,s,h}(a_1, \ldots, b, \ldots a_m) > C_{t,s,h}(a_1, \ldots, c, \ldots a_m) \qquad H_i K_i < \frac{1}{2}, i = 1, \ldots, m.$$

$$C_{t,s,h}(a_1, \ldots, b, \ldots a_m) < C_{t,s,h}(a_1, \ldots, c, \ldots a_m) \qquad H_i K_i > \frac{1}{2}, i = 1, \ldots, m.$$

Consequently, if $H_i K_i < \frac{1}{2}$ *(respectively* $H_i K_i > \frac{1}{2}$*),* $i = 1, \ldots, m$*, then*

- *The smaller (larger) b is, the more correlated the increments of* $N_t^{H,K}(a_1, \ldots, b, \ldots a_m)$ *are*.
- *The larger (smaller) b is, the less correlated the increments of* $N_t^{H,K}(a_1, \ldots, b, \ldots a_m)$ *are*.

Remark 7.5. The result of corollary 7.1 will be useful in modeling of a particular phenomenon. Indeed, we can choose H, K and a suitably, in such a manner that the *MBBM* $N^{H,K}(a)$ permits taking the sign and the level of correlation between the large increments of this phenomenon into account.

8. Conclusion and Outlook

We should point out that in this paper, we have only presented the *MBBM* and studied some of its stochastic properties and characteristics. This is enough for the purpose of this paper. We believe that our reported process can be a possible candidate for models which involve self similarity, long range dependence and non stationarity of increments. It can be used to generalize the work of C-René Dominique and Luis Eduardo Solis Rivera-Solis [17] in case of the mixed *FBM* where the market alternated between anti-persistence and persistence. In future outlook, firstly, we will try to study some *SDEs* driven by the *MBBM*, (for the case of *SDEs* driven by a sum of independent *FBMs*, we refer to Mishura [15] and Zähle [24]). We will develop a stochastic calculus for the *MBBM* from the work of Es-Sebaiy and Tudor [7] in case of the *BFBM*. Secondly, since it is highly important to identify the values of the parameters H and K in order to understand the structure of the process and its applications. We will try to estimate the parameters H and K of the *MBBM* $N^{H,K}(a)$ by using the new theory of estimating the Hurst parameter using the conic multivariate adaptive regression splines method developed in Yerlikaya et al. [23], in case of the FBM.

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