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Existence of Optimal Controls for Forward-Backward Stochastic Differential Equations of Mean-field Type

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Abstract. *In this paper, we prove the existence of optimal controls for systems governed by forward-backward stochastic differential equations of mean-field type (MF-FBSDEs), in which the coefficients depend not only on the state process, but also on the distribution of the state process, via the expectation of some function of the state. The cost functional is also of mean-field type. We prove this existence result by using the weak convergence techniques for the associated MF-FBSDEs on the space of continuous functions and on the space of cádlág functions endowed with the Jakubowski S-topology. Moreover, when the Roxin convexity condition is fulfilled, we establish that the set of strict control coincides with the set of relaxed control.*

Key words : Mean-field, Forward Backward Stochastic Differential Equations, Relaxed Control, Strict Control, Weak Convergence, Existence.

AMS Subject Classifications : 60H10, 60G55, 93E20

1. Introduction

Forward-backward stochastic differential equations (FBSDEs), which are encountered in stochastic control problems and mathematical finance were first studied by Antonelli in [3]. Xu treated in [32] a non-coupled forward-backward stochastic control systems. The maximum principle for fully coupled FBSDEs has been established by Wu in [31], where the control domain is taken to be convex. Then Peng and Wu [30], considered fully coupled FBSDEs and their applications to optimal control.

In 2009, Buckdahn, Djehiche, Li and Peng [9] introduced a new kind of backward stochastic differential equations, called mean-field BSDEs, which were derived as a limit to

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some high dimensional system of FBSDEs, corresponding to a large number of particles. Eversince, many authors studied a system of this kind named as a Mckean-Vlasov type (see [26] and [1]). In this respect we refer the reader to [8] and [2] as well. The existence of solutions to mean-field FBSDEs systems has been proved by Carmona and Dularue [11]. A maximum principle for fully coupled FBSDEs of mean-field type has been established by Li and Liu [25], where the control domain is not assumed to be convex. Singular optimal control for FBSDEs of mean-field type and applications to finance have been investigated by Hafayed [16]. He established also in [17], a maximum principle for mean-field FBSDEs with jumps with uncontrolled diffusion, where the domain of control is not assumed to be convex. In Hafayed et al. [18] a maximum principle for MF-FBSDEs with controlled diffusion have been established, where the domain of control is assumed to be convex.

In this paper, we study the existence of optimal controls for systems driven by MF-FBSDEs of the form

$$\begin{cases} dX_t = b(t,X_t,\mathbb{E}[\alpha(X_t)],v_t)dt + \sigma(t,X_t,\mathbb{E}[\beta(X_t)],v_t)dW_t \\ dY_t = -f(t,X_t,\mathbb{E}[\gamma(X_t)],Y_t,\mathbb{E}[\delta(Y_t)],v_t)dt + Z_t dW_t - dM_t \\ X_0 = x_0, Y_T = g(X_T,\mathbb{E}[\lambda(X_T)]), \quad t \in [0,T]. \end{cases} \tag{1}$$

where b, α, σ, $\beta, f, \gamma, \delta, g$ and λ are given functions, $(W_t, t \geq 0)$ is a standard Brownian motion, defined on some filtered probability space $(\Omega, \mathcal{F}, P)$, satisfying the usual conditions. X, Y, Z are square integrable adapted processes and M a square integrable martingale that is orthogonal to W. The control variable v_t, called strict control, is a measurable, $\mathcal{F}_t -$ adapted process with values in a compact metric space U.

We shall consider a functional cost to be minimized, over the set $\mathcal{U}$ of strict controls, as the following:

$$J(v.) := \mathbb{E}\Big[l(X_T,\theta(X_T)) + k(Y_0,\rho(Y_0)) + \int_0^T h(t,X_t,\mathbb{E}[\varphi(X_t)],Y_t,\mathbb{E}[\psi(Y_t)],v_t)dt\Big], \tag{2}$$

where $l, \theta, k, \rho, h, \varphi$ and ψ are appropriate functions.

The considered system and the cost functional, depend not only on the state of the system, but also on the distribution of the state process, via the expectation of some function of the state. The mean-field FBSDEs (1), called McKean-Vlasov systems, are obtained as the mean square limit of an interacting particle system of the form

$$\begin{cases} dX_t^{i,n} = b(t,X_t^{i,n},\frac{1}{n}\sum_{j=1}^{n}\alpha\left(X_t^{j,n}\right),v_t)dt + \sigma(t,X_t^{i,n},\frac{1}{n}\sum_{j=1}^{n}\beta\left(X_t^{j,n}\right),v_t)dW_t \\ dY_t^{i,n} = -f(t,X_t^{i,n},\frac{1}{n}\sum_{j=1}^{n}\gamma\left(X_t^{j,n}\right),Y_t^{i,n},\frac{1}{n}\sum_{j=1}^{n}\delta\left(Y_t^{j,n}\right),v_t)dt + Z_t^{i,n}dW_t^i - dM_t^i. \end{cases} \tag{3}$$

where $(W_.^i, i \geq 0)$ is a collection of independent Brownian motions. Our system MF-FBSDEs

(1.1) occur naturally in the probabilistic analysis of financial optimization and control problems of the McKean-Vlasov type.

In the case of forward SEDs without the mean-field part, the existence of strict optimal controls follows from the convexity of the image of the action space U by the mapping $(b(t,x,\bullet),\sigma\sigma^T(t,x,\bullet),h(t,x,\bullet))$, which is known as the Roxin-type convexity condition (see [12, 15, 24]). In the absence of this condition, a strict optimal control may fail to exist. The idea is then to introduce a new class $\mathcal{R}$ of admissible control in which, the controller chooses at time t, a probability measure $q_t(da)$ on the control set U, rather than an element $u_t \in U$. These are called relaxed controls.

Using compactification techniques, Fleming [13] derived the first existence result of an optimal relaxed control for SDEs with an uncontrolled diffusion coefficient. The case of SDEs with controlled diffusion coefficient has been solved by El-Karoui et al. [12], where the optimal relaxed control is shown to be Markovian. See also Haussmann and Lepeltier [19] and Bahlali, Djehiche and Mezerdi [4]. Bahlali et al. [7] proved the existence of optimal controls for systems governed by SDEs of mean-field type, where the diffusion coefficient is not controlled. Existence of optimal control for FBSDE has been proved by Bahlali, Gherbal and Mezerdi [5]. See also Buckdahn et al. [10]. Bahlali, Gherbal and Mezerdi in [6] proved existence of optimal control for linear BSDEs, and this result has been extended to a system of linear backward doubly SDEs by Gherbal [14].

In this work, we are interested in existence of optimal control for systems governed by FBSDEs of mean-field type (1) in which the coefficient depends not only on the state process, but also on the distribution of the state process, via the expectation of some function of the state. Moreover the cost functional is also of mean-field type. We prove the existence of optimal solutions, of control problems governed by this kind of systems. Our approach is based on tightness properties of the distributions of the processes defining the control problem and the Skorokhod's selection theorem on the space $\mathbb{D}$, endowed with the Jakubowski S-topology [22]. Our results happens to extend in particular those in [5] and [7].

The paper is organized as follows. In section 2, we formulate the strict control problems and introduce various assumptions to be used throughout the paper. In section 3, we present and prove the main result concerning the existence of optimal controls for systems of MF-FBSDEs. Section 4 concludes the paper with some remarks and hints on future work.

2. Statement of the Problems and Assumptions

2.1. **Strict control problem**

We study the existence of strict optimal controls for systems governed by the following FBSDE of mean-field type

$$\begin{cases} X_t = x + \int_0^t b(s,X_s,\mathbb{E}[\alpha(X_s)],u_s)ds + \int_0^t \sigma(s,X_s,\mathbb{E}[\beta(X_s)],u_s)dW_s \\ Y_t = g(X_T,\mathbb{E}[\lambda(X_T)]) + \int_t^T f(s,X_s,\mathbb{E}[\gamma(X_s)],Y_s,\mathbb{E}[\delta(Y_s)],u_s)ds \\ \qquad\qquad - \int_t^T Z_s dW_s - (M_T - M_t), \end{cases} \tag{4}$$

and the expected cost on the time interval $[0,T]$ is given by

$$J(u.) := \mathbb{E}\Big[l(X_T,\mathbb{E}[\theta(X_T)]) + k(Y_0,\mathbb{E}[\rho(Y_0)]) + \int_0^T h(t,X_t,\mathbb{E}[\varphi(X_t)],Y_t,\mathbb{E}[\psi(Y_t)],u_t)dt\Big], \quad (5)$$

where u_t is a strict control, $(W_t, t \geq 0)$ is a d-dimensional Brownian motion defined on some filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ and M is a square integrable martingale that is orthogonal to W.
Our objective is to minimize the cost function (5), over the set of admissible controls, which are a $\mathcal{F}_t$-measurable processes valued in a compact metric space $U \subset \mathbb{R}^k$.
It should be noted that the probability space and the Brownian motion may change with the control u. Therefore, we need to have another definition of the admissible control, given as follows.

Definition 2.1. A 6-tuple $\upsilon. = (\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P,W.,u.)$ is called ω-admissible strict control, and (X_t,Y_t,Z_t) a ω-admissible triple if:
i)- $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ is a filtered probability space satisfying the usual conditions;
ii)- W_t is an d-dimensional standard Brownian motion defined on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$;
iii)-u_t is an F_t-adapted process on $(\Omega,\mathcal{F},P)$ taking values in the action space U;
iv)-(X_t,Y_t,Z_t) is the unique solution of the MF-FBSDE (4) on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ under u_t.
The set of all ω-admissible controls is denoted by U^ω.

Our stochastic optimal control problem under the weak formulation can be stated as follows:
Minimize (5) over $\mathcal{U}^\omega$. We say that the ω-admissible control υ^* is ω-optimal control, if it satisfies

$$J(\upsilon^*_\cdot) = \inf_{\upsilon.\in\mathcal{U}^\omega} J(\upsilon). \quad (6)$$

2.2. **Notation and assumptions**

We now introduce spaces of processes to be considered later.
$\mathbb{M}^2([0,T];\mathbb{R}^m)$: the set of jointly measurable, processes $\{Y_t, t\in 0,T\}$ with values in $\mathbb{R}^m$ such that Y_t is $\mathcal{F}_t$-measurable for a.e. $t\in[0,T]$, and satisfy

$$\mathbb{E}\int_0^T |Y_t|^2 dt < \infty.$$

Let $\mathbb{S}^2([0,T];\mathbb{R}^n)$: the set of jointly measurable, processes $\{X_t, t\in 0,T\}$ with values in $\mathbb{R}^n$ such that X_t is $\mathcal{F}_t$-measurable for a.e. $t\in[0,T]$, and satisfy

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X_t|^2\Big] < \infty.$$

$\mathbb{C}([0,T];\mathbb{R}^n)$: the space of continuous functions from $[0,T]$ to $\mathbb{R}^n$, equipped with the topology of uniform convergence.
$\mathbb{D}([0,T];\mathbb{R}^m)$: the Skorokhod space of càdlàg functions from $[0,T]$ to $\mathbb{R}^m$, that is functions

which are continuous from the right with left hand limits, equipped with the S-topology of Jakubowski (see [22]).

Consider next the following conditions.

(**H1**) Assume that the functions

$$b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^n,$$

$$\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d},$$

$$f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \to \mathbb{R}^m,$$

$$g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m,$$

$$\alpha,\beta,\lambda,\gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$\delta : [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,$$

are bounded and continuous. Moreover assume that there exists $K > 0$, such that for every $(x_1,x_2,x_1',x_2') \in \mathbb{R}^{4n}, (y_1,y_2,y_1',y_2')\mathbb{R}^{4m}$,

$$|f(t,x_1,x_2,y_1,y_2,u) - f(t,x_1',x_2',y_1',y_2',u)|$$

$$\leq K\ (|x_1 - x_1'| + |x_2 - x_2'| + |y_1 - y_1'| + |y_2 - y_2'|),$$

$$|b(t,x_1,x_2,u) - b(t,x_1',x_2',u)| \leq K\ (|x_1 - x_1'| + |x_2 - x_2'|),$$

$$|\sigma(t,x_1,x_2,u) - \sigma(t,x_1',x_2',u)| \leq K\ (|x_1 - x_1'| + |x_2 - x_2'|).$$

Also, the functions α,β,γ are uniformly Lipschitz in x and δ is uniformly Lipschitz in y.

(**H2**) Assume that the functions

$$l : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

$$k : [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R},$$

$$h : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \to \mathbb{R},$$

$$\varphi,\theta : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$\psi,\rho : [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,$$

are bounded and continuous. Moreover assume that

$$|h(t,x_1,x_2,y_1,y_2,u) - h(t,x_1',x_2',y_1',y_2',u)|$$

$$\leq K\ (|x_1 - x_1'| + |x_2 - x_2'| + |y_1 - y_1'| + |y_2 - y_2'|).$$

(**H3**) (U,d) is a compact metric space.

(**H4**) (Roxin-type convexity condition): The set

$$(b,\sigma\sigma^T,f,h)(t,x,x',y,y',U) := \{b^i(t,x,x',u),(\sigma\sigma^T)^{ij}(t,x,x',u),$$

$$f^j(t,x,x',y,y',u),h(t,x,x',y,y',u)\diagdown u \in U, i = 1,\cdots,n, j = 1,\cdots,m\},$$

is convex and closed in $\mathbb{R}^{n+m+1}$.

To proof the existence of optimal solution of our strict control problem $\{(4),(5),(6)\}$, we need a certain structure of compactness. The weak formulation allows us to find the compactness of the image measure of some processes involved on a certain functional space. However, because the control u is measurable only in t and there is no convenient compactness property on the space of measurable functions, we need to embed it in a larger space with proper compactness.

We denote by $\mathbb{V}$ the space of positive Radon measures μ on $[0,T] \times U$ such that

$$\mu([0,s]\times U) = s, \forall \in [0,T]. \tag{7}$$

Equipped with the topology of stable convergence of measures, $\mathbb{V}$ is a compact metrizable space, (see Jacod and Mémin [21]). On the other hand, by (7), μ can be represented as $\mu(dt,du) = \mu(t,du)dt$, where $\mu(t,du)$ is a probability measure on U for almost all t and is determined uniquely except on a t-null set. In this context, any U-valued measurable process $u.$ may be embedded into $\mathbb{V}$ in which $u.$ corresponds to the Dirac measure $\delta_{u.}(dt,da)$ as follows: for any bounded and uniformly continuous functions ϖ we have

$$\varpi(t,x,u_t) = \int_U \varpi(t,x,u)\delta_{u.}(t,da) := \widehat{\varpi}(t,x,\delta_{u.}).$$

3. Existence of Optimal Controls

Our results in this paper extends those of [5] and [7] to a systems governed by FBSDE of mean-field type and with controlled diffusion coefficient.

Theorem 3.1. *Under conditions* (**H1**)-(**H4**)*, the strict control problem* $\{(4),(5),(6)\}$ *has an optimal solution.*

Before proving this theorem in subsection 3.1, we need some auxiliary results on the tightness of the distributions of the processes defining the control problem.

Let $\upsilon^n_{\cdot} = (\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_{t\geq 0}, P^n, W^n_{\cdot}, u^n)$ be a minimizing sequence, that is $\lim_{n\to\infty} J(\upsilon^n_{\cdot}) = \inf_{\upsilon.\in\mathcal{U}^{\omega}} J(\upsilon.)$. Let (X^n,Y^n,Z^n) be the unique solution of the following MF-FBSDE

$$\begin{cases} X^n_t = x + \int_0^t b(s,X^n_s,\mathbb{E}[\alpha(X^n_s)],u^n_s)ds \\ \qquad\qquad + \int_0^t \sigma(s,X^n_s,\mathbb{E}[\beta(X^n_s)],u^n_s)dW^n_s, \\ Y^n_t = g(X^n_T,\mathbb{E}[\lambda(X^n_T)]) \\ \qquad\qquad + \int_t^T f(s,X^n_s,\mathbb{E}[\gamma(X^n_s)],Y^n_s,\mathbb{E}[\delta(Y^n_s)],u^n_s)ds \\ \qquad\qquad - \int_t^T Z^n_s dW^n_s. \end{cases} \tag{8}$$

Lemma 3.1. *Let* (X^n,Y^n,Z^n) *be the unique solution of the system (8). There exists a positive constant C such that*

$$\sup_n \mathbb{E}\left(\sup_{0\leq t\leq T}|X^n_t|^2 + \sup_{0\leq t\leq T}|Y^n_t|^2 + \int_t^T |Z^n_s|^2 ds\right) \leq C. \tag{9}$$

Proof. Its easily to show that

$$\sup_n \mathbb{E}\left[\sup_{0\le t\le T}|X_t^n|^2\right] < +\infty.$$

By using the boundedness of b and σ and using Burkholder-Davis-Gundy's inequality.

On the other hand, applying Itô's formula to $|Y_t^n|^2$, we obtain

$$\begin{aligned}
\mathbb{E}\left[|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds\right] &= \mathbb{E}[|g(X_T^n, \mathbb{E}[\lambda(X_T^n)])|^2] \\
&\quad + 2\mathbb{E}\left[\int_t^T \langle Y_s^n, f(s, X_s^n, \mathbb{E}[\gamma(X_s^n)], Y_s^n, \mathbb{E}[\delta(Y_s^n)], u_s^n)\rangle ds\right] \\
&\le \mathbb{E}[|g(X_T^n, \mathbb{E}[\lambda(X_T^n)])|^2] + \mathbb{E}\left[\int_t^T |Y_s^n|^2 ds\right] \\
&\quad + \mathbb{E}\left[\int_t^T |f(s, X_s^n, \mathbb{E}[\gamma(X_s^n)], Y_s^n, \mathbb{E}[\delta(Y_s^n)], u_s^n)|^2 ds\right].
\end{aligned}$$

Using the boundedness of g and f and by Gronwall's lemma, it follows that

$$\sup_n \mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds\right] < +\infty. \qquad \blacksquare$$

Lemma 3.1. *The sequence of distributions of the processes* $(X^n, W^n, Y^n, \int_0^\cdot Z_s^n dW_s^n)$ *is tight on the space* $\Gamma := C([0,T];\mathbb{R}^n) \times C([0,T];\mathbb{R}^d) \times D([0,T];\mathbb{R}^m) \times D([0,T];\mathbb{R}^{m\times d})$ *endowed with the topology of uniform convergence for the first and second factor and endowed with the S-topology of Jakubowski (see[22]) for the third and fourth factor.*

Proof. According to Kolmogorov's theorem (see Ikeda and Watanabe [20], page 18), we need to verify that

$$\mathbb{E}[|X_t^n - X_s^n|^4] \le K_1|t-s|^2,$$

$$\mathbb{E}[|W_t^n - W_s^n|^4] \le K_2|t-s|^2,$$

for some constants K_1 and K_2 independent from n.

We have

$$\begin{aligned}
\mathbb{E}[|X_t^n - X_s^n|^4] &\le C\mathbb{E}\left[\left|\int_s^t b(s, X_s^n, \mathbb{E}[\alpha(X_s^n)], u_s^n) ds\right|^4\right] \\
&\quad + C\mathbb{E}\left[\left|\int_s^t \sigma(s, X_s^n, \mathbb{E}[\beta(X_s^n)], u_s^n) dW_s^n\right|^4\right].
\end{aligned}$$

Using Burkholder-Davis-Gundy's inequality to the martingale part and the boundedness of b and σ, we obtain

$$\begin{aligned}
\mathbb{E}[|X_t^n - X_s^n|^4] &\le C\mathbb{E}\left[\left(\int_s^t |b(s, X_s^n, \mathbb{E}[\alpha(X_s^n)], u_s^n)|^2 ds\right)^2\right] \\
&\quad + C\mathbb{E}\left(\int_s^t |\sigma(s, X_s^n, \mathbb{E}[\beta(X_s^n)], u_s^n)|^2 ds\right)^2 \\
&\le K_1|t-s|^2.
\end{aligned}$$

The second inequality is obvious.

Let us prove that $(Y_\cdot^n, \int_0^\cdot Z_s^n dW_s^n)$ is tight on the space $\mathbb{D}([0,T];\mathbb{R}^m) \times \mathbb{D}([0,T];\mathbb{R}^{m\times d})$.

Let $0 = t_0 < t_1 < \ldots < t_n = T$. We define the conditional variation by

$$CV(Y_\cdot^n) := \sup \mathbb{E}\left[\sum_i |\mathbb{E}(Y_{t_{i+1}}^n - Y_{t_i}^n) / \mathcal{F}_{t_i}^{W^n}| \right],$$

where the supremum is taken over all partitions of the interval $[0,T]$. By the same method given in [29], we get

$$CV(Y_\cdot^n) \leq C\mathbb{E}\left[\int_0^T |f(s, X_s^n, \mathbb{E}[\gamma(X_s^n)], Y_s^n, \mathbb{E}[\delta(Y_s^n)], u_s^n)| ds \right],$$

where C is a constant depending only on t. Hence combining conditions (**H1**) and lemma 3.1, we deduce that

$$\sup_n \left[CV(Y_\cdot^n) + \sup_{0 \leq t \leq T} \mathbb{E}[|Y_t^n|] + \sup_{0 \leq t \leq T} \mathbb{E}\left[\left| \int_0^t Z_s^n dW_s^n \right| \right] \right] < +\infty.$$

Thus the Meyer-Zheng tightness criteria is fulfilled (see [28]), then the sequences $Y_\cdot^n$ and $\int_0^\cdot Z_s^n dW_s^n$ are tight. ■

Lemma 3.3. *The family of distributions of the relaxed control* $(\delta_{u_\cdot^n})_n$ *is tight in V.*

Proof. Since $[0,T] \times U$ is compact, then by applying Prokhorov's theorem, the space $\mathbb{V}$ of probability measures on $[0,T] \times U$ is then compact. Since $(\delta_{u_\cdot^n})_n$ valued in the compact space $\mathbb{V}$, then the family of distributions associated to $(\delta_{u_\cdot^n})_n$ is tight. ■

3.1. **Proof of theorem 3.1**

Proof. Using lemmas 3.2 and 3.3, it follows that the sequence of processes $\eta^n := \left(\delta_{u^n}, X_\cdot^n, W_\cdot^n, Y_\cdot^n, \int_0^\cdot Z_s^n dW_s^n\right)$ is tight on the space $\mathbb{V} \times \Gamma$. Then by the Skorokhod representation theorem, there exists a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$, a sequences $\widetilde{\eta}^n = \left(\widetilde{\pi}^n, \widetilde{X}^n, \widetilde{W}^n, \widetilde{Y}^n, \int_0^\cdot \widetilde{Z}_s^n d\widetilde{W}_s^n\right)$ and $\widetilde{\eta} = \left(\widetilde{\pi}, \widetilde{X}, \widetilde{W}, \widetilde{Y}, \int_0^\cdot \widetilde{Z}_s d\widetilde{W}_s\right)$ defined on this space and a countable subset D of $[0,T]$ such that on D^c, we have

(i) for each $n \in \mathbb{N}$, $\text{law}(\eta^n) \equiv \text{law}(\widetilde{\eta}^n)$,

(ii) there exists a subsequence $(\widetilde{\eta}^{n_k})$ of $(\widetilde{\eta}^n)$, still denoted $(\widetilde{\eta}^n)$, which converges to $\widetilde{\eta}, \widetilde{P}$-a.s. on the space $\mathbb{V} \times \Gamma$,

(iii) $(\widetilde{Y}^n, \int_0^\cdot \widetilde{Z}_s^n d\widetilde{W}_s^n)$ converges to the càdlàg processes $(\widetilde{Y}, \int_0^\cdot \widetilde{Z}_s d\widetilde{W}_s)$, $dt \times \widetilde{P}$-a.s.

(iv) $\sup_{0 \leq t \leq T} \left| \widetilde{X}_t^n - \widetilde{X}_t \right| \to 0$, $\widetilde{P}$-a.s.

(v) $(\widetilde{\pi}^n)$ converges in the stable topology to $\widetilde{\pi}, \widetilde{P}$-a.s.

Set

$$\begin{cases} \widetilde{\mathcal{F}}_t^n := \sigma(\widetilde{W}_s^n, \widetilde{X}_s^n, \widetilde{Y}_s^n, s \leq t) \vee (\widetilde{\pi}^n)^{-1}(\mathcal{B}_t(\mathbb{V})), \\ \widetilde{\mathcal{F}}_t := \sigma(\widetilde{W}_s, \widetilde{X}_s, \widetilde{Y}_s, s \leq t) \vee (\widetilde{\pi})^{-1}(\mathcal{B}_t(\mathbb{V})), \end{cases}$$

where $\mathcal{B}_t(\mathbb{V})$ is defined by

$$\mathcal{B}_t(\mathbb{V}) := \sigma\{\pi \in \mathbb{V} \mid \pi(\phi^s) \in B\} : s \in [0,t], B \in \mathcal{B}(\mathbb{R}),$$

$\pi \in \mathbb{V}$ is a linear functional on $\mathcal{C}([0,T] \times U)$ in the way:

$\pi(\phi) := \int_0^T \int_U \phi(t,u)\pi(dt,du), \forall \phi \in \mathcal{C}([0,T] \times U),$ and $\phi^t(s,u) := \phi(s \wedge t, u).$

According to property (i), we have the following MF-FBSDE on $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}^n_t\}_{t\geq 0}, \widetilde{P}\right)$

$$\begin{cases} \widetilde{X}^n_t = x + \int_0^t \int_U b(s, \widetilde{X}^n_s, \mathbb{E}[\alpha(\widetilde{X}^n_s)], u)\widetilde{\pi}^n_s(du)ds \\ \qquad\qquad + \int_0^t \int_U \sigma(s, \widetilde{X}^n_s, \mathbb{E}[\beta(\widetilde{X}^n_s)], u)\widetilde{\pi}^n_s(du)d\widetilde{W}^n_s, \\ \widetilde{Y}^n_t = g(\widetilde{X}^n_T, \mathbb{E}[\lambda(\widetilde{X}^n_T)]) \\ \qquad\qquad + \int_t^T \int_U f(s, \widetilde{X}^n_s, \mathbb{E}[\gamma(\widetilde{X}^n_s)], \widetilde{Y}^n_s, \mathbb{E}[\delta(\widetilde{Y}^n_s)], u)\widetilde{\pi}^n_s(du)ds \\ \qquad\qquad - (\widetilde{N}^n_T - \widetilde{N}^n_t), \end{cases} \tag{10}$$

and by previos notation, we have

$$\begin{cases} \widetilde{X}^n_t = x + \int_0^t \widehat{b}(s, \widetilde{X}^n_s, \mathbb{E}[\alpha(\widetilde{X}^n_s)], \widetilde{\pi}^n_s)ds + \int_0^t \widehat{\sigma}(s, \widetilde{X}^n_s, \mathbb{E}[\beta(\widetilde{X}^n_s)], \widetilde{\pi}^n_s)d\widetilde{W}^n_s, \\ \widetilde{Y}^n_t = g(\widetilde{X}^n_T, \mathbb{E}[\lambda(\widetilde{X}^n_T)]) + \int_t^T \widehat{f}(s, \widetilde{X}^n_s, \mathbb{E}[\gamma(\widetilde{X}^n_s)], \widetilde{Y}^n_s, \mathbb{E}[\delta(\widetilde{Y}^n_s)], \widetilde{\pi}^n_s)ds \\ \qquad\qquad - (\widetilde{N}^n_T - \widetilde{N}^n_t), \end{cases} \tag{11}$$

where $\widetilde{N}^n_t = \int_0^t \widetilde{Z}^n_s d\widetilde{W}^n_s$.

Since $\widetilde{W}^n_.$ is an $\{\widetilde{\mathcal{F}}^n_t\}_{t\geq 0}$-Brownian motion, all the integrals in (10) and in (11) are well-defined.

Using properties (ii), (iii), (iv),(v), under (**H1**)-(**H4**) and passing to the limit in the MF-FBSDE (11), one can show that there exists a countable set $D \subset [0, T)$ such that

$$\begin{cases} \widetilde{X}_t = x + \widetilde{B}(t) + \widetilde{\Sigma}(t), t > 0 \\ \widetilde{Y}_t = g(\widetilde{X}_T, \mathbb{E}[\lambda(\widetilde{X}_T)]) + (\widetilde{F}(T) - \widetilde{F}(t)) - (\widetilde{N}_T - \widetilde{N}_t), t \in [0,T] \setminus D. \end{cases} \tag{12}$$

Since $\widetilde{Y}$ and $\widetilde{N}$ are càdlàg, then one can get for every $t \in [0,T]$

$\widetilde{Y}_t = g(\widetilde{X}_T, \mathbb{E}[\lambda(\widetilde{X}_T)]) + (\widetilde{F}(T) - \widetilde{F}(t)) - (\widetilde{N}_T - \widetilde{N}_t).$

Also we have

$$\begin{aligned} \inf_{u.\in\mathcal{U}^{\omega}} J(u.) &= \lim_{n\to\infty} J(\delta_{u^n_.}) = \lim_{n\to\infty} J(\widetilde{\pi}^n_.) \\ &:= \lim_{n\to\infty} \widetilde{\mathbb{E}} l(\widetilde{X}^n_T, \widetilde{\mathbb{E}}[\theta(\widetilde{X}^n_T)]) + k(\widetilde{Y}^n_0, \widetilde{\mathbb{E}}[\rho(\widetilde{Y}^n_0)]) \end{aligned} \tag{13}$$

$$+\int_0^T \widehat{h}t,\widetilde{X}_t^n,\widetilde{\mathbb{E}}[\varphi(\widetilde{X}_t^n)],\widetilde{Y}_t^n,\widetilde{\mathbb{E}}[\psi(\widetilde{Y}_t^n],\widetilde{\pi}_t^n)dt,$$
$$=\widetilde{\mathbb{E}}l(\widetilde{X}_T,\widetilde{\mathbb{E}}[\theta(\widetilde{X}_T)])+k(\widetilde{Y}_0,\widetilde{\mathbb{E}}[\rho(\widetilde{Y}_0)])+\widetilde{H}(T).$$

The rest of the proof is inspired from the work of Yong and Zhou [33]. Let us consider the sequence $a^n(s):=\widehat{\sigma}\widehat{\sigma}^T(s,\widetilde{X}_s^n,\mathbb{E}[\beta(\widetilde{X}_s^n)],\widetilde{\pi}_s^n), s\in[0,T]$. Setting

$$\begin{cases} b_n^i(s):=\widehat{b}^i(s,\widetilde{X}_s^n,\mathbb{E}[\alpha(\widetilde{X}_s^n)],\widetilde{\pi}_s^n), i=1,\cdots,n,\\ a_n^{ik}(s):=\widehat{\sigma}\widehat{\sigma}^T(s,\widetilde{X}_s^n,\mathbb{E}[\beta(\widetilde{X}_s^n)],\widetilde{\pi}_s^n), i=1,\cdots,n,k=1,\cdots,d,\\ f_n^j(s):=\widehat{f}^j(s,\widetilde{X}_s^n,\mathbb{E}[\gamma(\widetilde{X}_s^n)],\widetilde{Y}_s^n,\mathbb{E}[\delta(\widetilde{Y}_s^n)],\widetilde{\pi}_s^n),j=1,\cdots,m,\\ h_n(s):=\widehat{h}(s,\widetilde{X}_s^n,\mathbb{E}[\varphi(\widetilde{X}_s^n)],\widetilde{Y}_s^n,\mathbb{E}[\psi(\widetilde{Y}_s^n)],\widetilde{\pi}_s^n),\end{cases}$$

Since $b_n^i\to b^i, i=1,\cdots,n, f_n^j\to f^j, j=1,\cdots,m$, $h_n\to h$, $a_n^{ik}\to a^{ik}$ weakly, and

$$\begin{aligned}&b^i(s,\omega)\in b^i(s,\widetilde{X}_s(\omega),\mathbb{E}[\alpha(\widetilde{X}_s(\omega))],U),\\ &(\sigma\sigma^T)^{ik}(s,\omega)\in(\sigma\sigma^T)^{ik}(s,\widetilde{X}_s(\omega),\mathbb{E}[\beta(\widetilde{X}_s(\omega))],U),\\ &f^j(s,\omega)\in f^j(s,\widetilde{X}_s(\omega),\mathbb{E}[\gamma(\widetilde{X}_s(\omega))],\widetilde{Y}_s(\omega),\mathbb{E}[\delta(\widetilde{Y}_s(\omega))],U),\\ &h(s,\omega)\in h(s,\widetilde{X}_s(\omega),\mathbb{E}[\varphi(\widetilde{X}_s(\omega))],\widetilde{Y}_s(\omega),\mathbb{E}[\psi(\widetilde{Y}_s(\omega))],U),\\ &\forall(s,\omega)\in[0,T]\times\widetilde{\Omega}, i,k=1,\cdots,n,j=1,\cdots,m\end{aligned}\tag{14}$$

From (12), (**H4**) and a measurable selection theorem (see Li-Yong [27], p. 102, Corollary 2.26), there is a U-valued, $\tilde{\mathcal{F}}_t$-adapted process $\tilde{u}.$ such that

$$\begin{aligned}&b(s,\omega)=b(s,\widetilde{X}_s(\omega),\mathbb{E}[\alpha(\widetilde{X}_s(\omega))],\widetilde{u}_s(\omega)),\\ &\sigma\sigma^T(s,\omega)=\sigma\sigma^T(s,\widetilde{X}_s(\omega),\mathbb{E}[\beta(\widetilde{X}_s(\omega))],\widetilde{u}_s(\omega)),\\ &f(s,\omega)=f(s,\widetilde{X}_s(\omega),\mathbb{E}[\gamma(\widetilde{X}_s(\omega))],\widetilde{Y}_s(\omega),\mathbb{E}[\delta(\widetilde{Y}_s(\omega))],\widetilde{u}_s(\omega)),\\ &h(s,\omega)=h(s,\widetilde{X}_s(\omega),\mathbb{E}[\varphi(\widetilde{X}_s(\omega))],\widetilde{Y}_s(\omega),\mathbb{E}[\psi(\widetilde{Y}_s(\omega))],\widetilde{u}_s(\omega)).\end{aligned}\tag{15}$$

Since $\widetilde{\Sigma}(t)$ given in (12), is an $\widetilde{\mathcal{F}}_t$-martingale, we have

$$\langle\widetilde{\Sigma}^n\rangle(t)=\int_0^t\widehat{\sigma}\widehat{\sigma}^T(s,\widetilde{X}_s^n,\mathbb{E}[\beta(\widetilde{X}_s^n)],\widetilde{\pi}_s^n)ds\equiv\int_0^t a^n(s)ds,$$

where $\langle\widetilde{\Sigma}^n\rangle$ is the quadratic variation of $\widetilde{\Sigma}^n$. Thus $\widetilde{\Sigma}^n(\widetilde{\Sigma}^n)^T(t)-\int_0^t a^n(s)ds$ is an $\widetilde{\mathcal{F}}_t$-martingale and from the fact that

$$\int_s^t a^n(r)dr \textit{ converges weakly to } \int_s^t\sigma\sigma^T(r,\widetilde{X}_r,\mathbb{E}[\alpha(\widetilde{X}_r)],\widetilde{u}_r)dr,$$

we can show that $\widetilde{\Sigma}(\widetilde{\Sigma})^T(t)-\int_0^t\sigma\sigma^T(r,\widetilde{X}_r,\mathbb{E}[\alpha(\widetilde{X}_r)],\widetilde{u}_r)dr$ is an $\widetilde{\mathcal{F}}_t$-martingale, which implies

$$\langle\widetilde{\Sigma}\rangle(t)=\int_0^t\widehat{\sigma}\widehat{\sigma}^T(s,\widetilde{X}_s,\mathbb{E}[\beta(\widetilde{X}_s)],\widetilde{u}_s)ds.$$

By a martingale representation theorem, there is an extension space $(\overline{\Omega},\mathcal{F},\mathcal{F}_t,\overline{P})$ of

$\left(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_t,\widetilde{P}\right)$ on which there lives an d-dimensional $\mathcal{F}_t$-Brownian motion $\overline{W}_t$ such that

$$\widetilde{\Sigma}(t) = \int_0^t \sigma(s,\widetilde{X}_s,\mathbb{E}[\beta(\widetilde{X}_s)],\widetilde{u}_s)d\overline{W}_s. \tag{16}$$

Similarly, one can show that

$$\begin{aligned}\widetilde{B}(t) &= \int_0^t b(s,\widetilde{X}_s,\mathbb{E}[\alpha(\widetilde{X}_s)],\widetilde{u}_s)ds,\\ \widetilde{F}(t) &= \int_0^t f(s,\widetilde{X}_s,\mathbb{E}[\gamma(\widetilde{X}_s)],\widetilde{Y}_s,\mathbb{E}[\delta(\widetilde{Y}_s)],\widetilde{u}_s)ds, \qquad (17)\\ \widetilde{H}(t) &= \int_0^t h(s,\widetilde{X}_s,\mathbb{E}[\varphi(\widetilde{X}_s)],\widetilde{Y}_s,\mathbb{E}[\psi(\widetilde{Y}_s)],\widetilde{u}_s)ds.\end{aligned}$$

Also, since $\widetilde{N}$ is a $\widetilde{\mathcal{F}}_s$-martingale. Therefore by the martingale decomposition theorem, there exist a process $\widetilde{Z} \in \mathbb{M}^2([0,T];\mathbb{R}^{m\times d})$ such that

$$\widetilde{N}_t = \int_0^t \widetilde{Z}_s d\overline{W}_s + \widetilde{M}_t, \text{ and } \left\langle \widetilde{M},\overline{W}\right\rangle_t = 0. \tag{18}$$

Putting the values of $\widetilde{\Sigma}(t),\widetilde{B}(t),\widetilde{F}(t),\widetilde{N}_t,\widetilde{H}(t)$ (from (16),(17),(18)) into (12) and (13), we get

$$\begin{cases}\widetilde{X}_t := x + \int_0^t b(s,\widetilde{X}_s,\mathbb{E}[\alpha(\widetilde{X}_s)],\widetilde{u}_s)ds \\ \qquad\qquad + \int_0^t \sigma(s,\widetilde{X}_s,\mathbb{E}[\beta(\widetilde{X}_s)],\widetilde{u}_s)d\overline{W}_s, t \geq 0, \\ \\ \widetilde{Y}_t := g(\widetilde{X}_T,\mathbb{E}[\lambda(\widetilde{X}_T)]) + \int_t^T f(s,\widetilde{X}_s,\mathbb{E}[\gamma(\widetilde{X}_s)],\widetilde{Y}_s,\mathbb{E}[\delta(\widetilde{Y}_s)],\widetilde{u}_s)ds \\ \qquad\qquad - \int_t^T \widetilde{Z}_s d\overline{W}_s - (\widetilde{M}_T - \widetilde{M}_t), t \in [0,T],\end{cases}$$

and

$$\begin{aligned}\inf_{u.\in\mathcal{U}^\omega} J(u.) &= \widetilde{\mathbb{E}} l(\widetilde{X}_T,\widetilde{\mathbb{E}}[\theta(\widetilde{X}_T)]) + k(\widetilde{Y}_0,\widetilde{\mathbb{E}}[\rho(\widetilde{Y}_0)]) \\ &\qquad + \int_0^T \widehat{h} t,\widetilde{X}_t,\widetilde{\mathbb{E}}[\varphi(\widetilde{X}_t)],\widetilde{Y}_t,\widetilde{\mathbb{E}}[\psi(\widetilde{Y}_t],\widetilde{u}_t)dt \\ &= J(\widetilde{\upsilon}_.).\end{aligned}$$

By the definition 2.1 , we arrive that $\widetilde{\upsilon}. := (\overline{\Omega},\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\overline{P},\overline{W}.,\widetilde{u}.)$ is an ω-optimal control.■

4. Conclusion

In this paper, the existence of optimal controls for systems of forward-backward SDEs of mean-field type with controlled diffusion, is proved. The coefficients of the system depend not only on the state process, but also on the distribution of the state process. Moreover, the cost functional is also of mean-field type. The basic idea in the proof of this result is the relaxed control, which is needed in order to provide some compact structure. Moreover, under the Roxin's convexity condition, the set of strict controls coincides with that of relaxed controls. An emerging open question is to prove this result, when the generator depends on the second backward variable Z. Note that when the coefficients of the system are linear, we don't need to use the above techniques, but we can prove the existence of a strong strict optimal control (that is adapted to a fixed filtration) by using the convexity of the cost functional and Mazur's theorem. This will be our main concern in a future relevant work.

5. Appendix A :The S-Topology

The S-topology has been introduced by Jakubowski [22], as a topology defined on the Skorokhod space of càdlàg functions $\mathbb{D}([0,T];\mathbb{R}^k)$. This topology is weaker than the Skorokhod topology where the tightness criteria are easier to establish. These criteria are the same as for the Meyer and Zheng topology [28].

Let $N^{a,b}(Y)$ denotes the number of up-crossings of the function $Y \in \mathbb{D}([0;T];\mathbb{R}^m)$ in a given level $a < b$. Then let us recall some facts about the S-topology.

Proposition A.1. *(criterion for S-tightness). A sequence $(Y^n)_{n>0}$ is S-tight if and only if it is relatively compact on the S-topology.*

Proposition A.2. *Let $(Y^n)_{n>0}$ be a family of stochastic processes in $\mathbb{D}([0;T];R^m)$. Then this family is tight for the S-topology if and only if $(\|Y^n\|)_n$ and $(N^{a,b}(Y^n))_n$ are tight for each $a < b$.*

We recall (see Meyer & Zheng [28] and Jakubowski [22],[23]) that for a family $(Y^n)_n$ of quasi-martingales on the probability space $(\Omega,\{\mathcal{F}_t\}_{0\le t\le T},P)$, the following condition insures the tightness of the family $(Y^n)_n$ on the space $\mathbb{D}([0,T];\mathbb{R}^m)$ endowed with the S-topology where, for a quasi-martingale $Y^n, CV(Y^n)$ stands for the conditional variation of Y on $[0,T]$, and is defined by

$$CV(Y) = \sup \mathbb{E}\left[\sum_i |\mathbb{E}(Y^n_{t_i+1} - Y^n_{t_i}) \mid \mathcal{F}^n_{t_i}|\right],$$

where the supremum is taken over all partitions of $[0,T]$.
The process Y is called *quasimartingale* if $CV(Y) < +\infty$. When Y is a $\mathcal{F}_t$-martingale, $CV(Y) = 0$.

Proposition A.3. *(The a.s. Skorokhod representation). Let $(\mathbb{D},S)$ be a topological space on which there exists a countable family of S-continuous functions separating points in Y. Let $\{Y^n\}_{n\in\mathbb{N}}$ be a uniformly tight sequence of laws on $\mathbb{D}$. In every subsequence $\{Y^{n_k}\}$ one can find a further subsequence $\{Y^{n_{k_l}}\}$ and stochastic processes $\{Y^l\}$ defined on $([0,T],\mathcal{B}_{[0,T]},l)$ such that*

$$Y^l \sim Y^{n_{k_l}},\ l = 1,2,\dots \tag{19}$$

for each $w \in [0,T]$

$$Y^l(w) \underset{S}{\to} Y^0(w), \text{ as } l \to +\infty, \tag{20}$$

and for each $\varepsilon > 0$, there exists an S-compact subset $K_\varepsilon \subset D$ such that

$$P(\{w \in [0,T] : Y^l(w) \in K_\varepsilon, l = 1,2,\dots\}) > 1-\varepsilon. \tag{21}$$

One can say that (20) and (21) describe "the almost sure convergence in compacts" and that (19), (20) and (21) define the strong a.s. Skorokhod representation for subsequences ("strong" because of condition (21)).

Proposition A.4. *Let* (Y^n, M^n) *be a multidimensional process in* $D([0,T];R^m)$ *converging to* (Y, M) *in the S-topology. Let* $(F_t^{Y^n})_{t\geq 0}$ *(resp.* $(F_t^Y)_{t\geq 0}$*) be the minimal complete admissible filtration for* Y^n *(resp.Y). We assume that* $\sup_n E[\sup_{0\leq t\leq T}|M_t^n|^2] < C_T \; \forall T > 0$, M^n *is a* F^{Y^n}*-martingale and M is a* F^Y*-adapted. Then M is a* F^Y*-martingale.*

Proposition A.5. *Let* $(Y^n)_{n>0}$ *be a sequence of processes converging weakly in* $\mathbb{D}([0,T];R^m)$ *to Y. We assume that* $\sup_n \mathbb{E}[\sup_{0\leq t\leq T}|Y_t^n|^2] < +\infty$. *Hence, for any* $t \geq 0$, $\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2\right] < +\infty$.

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