Journal of Numerical Mathematics and Stochastics, **8** (1) : 88-98, 2016 © ***JNM@S***
http://www.jnmas.org/jnmas8-7.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

Conditional Full Support for the Brownian Bridge

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Abstract. *We present some basic concepts from stochastic portfolio theory that we shall need later and We present conditions that imply the conditional full support (CFS) property, introduced by Guasoni, Résonyi, and Schachermayer , for processes* $Z := H + K * W$*, where* W *is a Brownian motion,* H *is a continuous process, and* H *and* K *are processes either progressive or independent of* W*. Moreover, in the latter case under an additional assumption that* K *is of finite variation, we present conditions under which* Z *has CFS also when* W *is replaced with a general continuous process with CFS. The result of this paper is the establishment of the condition of conditional full support for the Brownian Bridge and requirement of the absence of arbitrage opportunities without calculating the risk-neutral probability.*

Key words : Conditional Full Support, Brownian Bridge, Absence of Arbitrage Opportunities.

AMS Subject Classifications : 34A08, 34K50, 93B05, 60HXX, 60GXX, 58J65

1. Introduction

Stochastic Portfolio Theory (SPT), as we currently think of it, began in 1995 with the manuscript "On the diversity of equity markets", which eventually appeared in 1999 by Fernholz in the Journal of Mathematical Economics. Stochastic portfolio theory is a relatively new branch of mathematical finance. It was introduced and studied by Fernholz [1, 2], and then further developed by Fernholz, Karatzas and Kardaras [3]. It provides a framework for analyzing portfolio performance under an angle which is different from the usual one. The CFS property was first introduced by Guasoni, Résonyi, and Schachermayer [4], in connection with mathematical finance, via pricing models with transaction costs. Their main result asserts that if a continuous price process has CFS, then for any $\varepsilon > 0$ there exists a so called ε-consistent price system, which is a martingale (after an equivalent change of measure).

The existence of ε-consistent price systems for all $\varepsilon > 0$ implies that the price process does not admit arbitrage opportunities under arbitrary small transaction costs, since any arbitrage strategy would generate arbitrage also in the consistent price system, which is a contradiction because of the martingale property.

Consistent price systems can be seen as generalizations of equivalent martingale measures (EMM's), since if a price process admits an EMM, then the price process itself qualifies as a trivial ε-consistent price system for any $\varepsilon > 0$.
However, CFS is worth studying even when it comes to price processes that admit EMM's, since it enables the construction of specific consistent price systems that are useful in solving superreplication problems under proportional transaction costs. This is manifested by the "face-lifting" result in [4].

Aside from having these applications in mathematical finance, CFS is an interesting fundamental property from a purely mathematical point of view. In particular, research on the CFS property can be seen as a natural continuation for the classical studies of the supports of the laws of continuous Gaussian processes, by Kallianpur [6], and diffusions, initiated by Stroock and Varadhan [9] and continued by several other authors (see e.g. [7] and the references therein).

Mikko S. Pakkanen presented in 2009 [8] conditions that imply the conditional full support for the process $Z := H + K * W$, where W is a Brownian motion, H is a continuous process, and H and K are processes either progressive or independent of W.

This paper is organized as follows. In the section 2 we present some basic concepts from stochastic portfolio theory that we shall need later on. Section 3 contains the result on consistent price system. In section 4 We present conditions that imply the conditional full support (CFS) property, for processes $Z := H + K * W$. In section 5 we provide our main result on conditional full support for the process $Z := H + K * W$ where W is a Brownian bridge.

1. Markets and Portfolios

We shall place ourselves in a model M for a financial market of the form

$$dB(t) = B(t)r(t)dt, \qquad B(0) = 1$$

$$dS_i(t) = S_i(t)\left(b_i(t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)dW_\nu(t) \right), \qquad S_i(0) = s_i > 0;\ i = 1,\dots,n, \tag{1}$$

consisting of a money-market $B(.)$ and of n stocks, whose prices $S_1(.);\dots;S_n(.)$ are driven by the d_i – *dimensional* Brownian motion $W(.) = (W_1(.);\dots;W_d(.))'$ with $d \geq n$.

We shall assume that the interest-rate process $r(.)$ for the money-market, the vector-valued process $b(.) = (b_1(.);\dots;b_n(.))'$ of rates of return for the various stocks, and the $(n * d)$-matrix-valued process $\sigma(.) = (\sigma_{i\nu}(.))_{1<i<n,\ 1\leq\nu\leq d}$ of stock-price volatilities. Then let us introduce the notation

$$a_{ij}(t) = \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\sigma_{j\nu}(t) = (\sigma(t)\sigma'(t))_{ij} = \frac{d}{dt}\langle logS_i, logS_j\rangle(t) ,$$

for the non-negative definite matrix-valued covariance process $a(.) = (a_{ij}(.))_{1\leq i,\, j\leq n}$ of the stocks in the market, as well as

$$y_i(t) = b_i(t) - \tfrac{1}{2}a_{ii}(t), \qquad i = 1,\dots,n.$$

Then we may use Ito's rule to solve (1) in the form

$$dlogS_i(t) = y_i(t)dt + \sum_{v=1}^{d} \sigma_{iv}(t)dW_v(t), \qquad i = 1,\ldots,n. \tag{2}$$

Or equivalently

$$S_i(t) = s_i \exp\left\{ \int_0^t y_i(u)du + \sum_{v=1}^{d} \int_0^t \sigma_{iv}(u)dW_v(u) \right\}, \qquad 0 \le t < \infty.$$

Definition 2.1. A portfolio $\pi(.) = (\pi_1(.), \pi_2(.), \ldots\ldots, \pi_n(.))'$ is an $\mathbb{F}$-progressively measurable process, bounded uniformly in (t,w), with values in the set

$$\bigcup_{k\in\mathbb{N}} (\pi_1, \pi_2, \ldots\ldots, \pi_n) \in \mathbb{R}^n | \pi_1^2 + \pi_2^2 + \ldots\ldots + \pi_n^2 | \le k^2,\ \pi_1 + \pi_2 + \ldots\ldots + \pi_n = 1.$$

Thus a portfolio can sell one or more stocks short but is never allowed to borrow from, or invest in, the money market. The interpretation is that $\pi_i(t)$ represents the proportion of wealth $V^{w,\pi}(t)$ invested at time t in the i^{th} stock, so the quantities

$h_i(t) = \pi_i(t)V^{w,\pi}(t), \qquad i = 1,\ldots,n,$

are the dollar amounts invested at any given time t in the individual stocks. The wealth process $V^{w,\pi}$, that corresponds to a portfolio $\pi(.)$ and initial capital $w > 0$, satisfies the stochastic equation

$$\frac{dV^{w,\pi}(t)}{V^{w,\pi}(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dS_i(t)}{S_i(t)} = \pi'(t)[b(t)dt + \sigma(t)dW(t)]$$

$$= b_\pi(t)dt + \sum_{v=1}^{d} \sigma_{\pi v}(t)dW_v(t), \qquad V^{w,\pi}(0) = w, \tag{3}$$

where

$$b_\pi(t) = \sum_{i=1}^{n} \pi_i(t)b_i(t), \qquad \sigma_{\pi v}(t) = \sum_{i=1}^{n} \pi_i(t)\sigma_{iv}(t) \quad for\ v = 1,\ldots,d.$$

These quantities are, respectively, the rate-of-return and the volatility coefficients associated with the portfolio $\pi(.)$. By analogy with (2) we can write the solution of equation (3) as

$$d\log V^{w,\pi}(t) = y_\pi(t)dt + \sum_{v=1}^{d} \sigma_{\pi v}(t)dW_v(t), \qquad V^{w,\pi}(0) = w, \tag{4}$$

or equivalently as

$$V^{w,\pi}(t) = w \exp\left\{ \int_0^t y_\pi(u)du + \sum_{v=1}^{d} \int_0^t \sigma_{\pi v}(u)dW_v(u) \right\}, \qquad 0 \le t < \infty.$$

Here

$$y_\pi(t) = \sum_{i=1}^{n} \pi_i(t)y_i(t) + y_\pi^*(t)$$

is the growth rate of the portfolio $\pi(.)$, and

$$y_\pi^*(t) = \frac{1}{2}\left(\sum_{i=1}^{n}\pi_i(t)a_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i(t)a_{ij}(t)\pi_j(t)\right)$$

is the excess growth rate of the portfolio $\pi(.)$.

2.1. **The market portfolio**

The stock price $S_i(t)$ can be interpreted as the capitalization of the i^{th} company at time t, and the quantities

$$S(t) = S_1(t) + \ldots + S_n(t) \qquad and \qquad \mu_i(t) = \frac{S_i(t)}{S(t)}, \quad i = 1, \ldots, n, \tag{5}$$

as the total capitalization of the market and the relative capitalizations of the individual companies, respectively. Clearly $0 < \mu_i(t) < 1$, $\forall i = 1, \ldots, n$ and

$$\sum_{i=1}^{n}\mu_i(t) = 1.$$

In accordance with (5) and (3) the resulting wealth process $V^{w,\mu}(.)$ satisfies

$$\frac{dV^{w,\pi}(t)}{V^{w,\pi}(t)} = \sum_{i=1}^{n}\mu_i(t)\frac{dS_i(t)}{S_i(t)} = \sum_{i=1}^{n}\frac{S_i(t)}{S(t)} = \frac{dS(t)}{S(t)}.$$

In other words,

$$V^{w,\mu}(.) \equiv \frac{w}{S(0)}S(.);$$

investing in the portfolio $\mu(.)$ is tantamount to ownership of the entire market, in proportion of course to the initial investment.

For this reason, we shall call $\mu(.)$ of (5) the market portfolio, and the process μ_i the market weight processes. By analogy with (4), we have

$$d\log V^{w,\mu}(t) = y_\mu(t)dt + \sum_{\nu=1}^{d}\sigma_{\mu\nu}(t)dW_\nu(t), \qquad V^{w,\mu}(0) = w, \tag{6}$$

and comparison of this last equation (6) with (2) gives the dynamics of the market-weights

$$d\,log\,\mu_i(t) = (y_i(t) - y_\mu(t))dt + \sum_{\nu=1}^{d}(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t))dW_\nu(t).$$

To state the main result of the paper we need the following variant of the notion of conditional full support. Let $T > 0$ be a fixed time horizon. We consider a filtered probability space $(\Omega, (\mathfrak{F}_t)_{t\in[0,T]}, \mathbf{P})$, where the filtration is assumed to satisfy the usual conditions with $\mathfrak{F}_0$ being trivial and all events belong to $\mathfrak{F}_T$.

Definition 2.2. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set and $S(t)_{t\in[0,T]}$ be a continuous adapted process taking values in $\mathcal{O}$.
We say that S has conditional full support in $\mathcal{O}$, if for all $t \in [0.T]$ and open set $G \subset C([0.T], \mathcal{O})$

$$P(S \in G \mid \mathfrak{F}_t) > 0, \quad \text{a.s. on the event } S\mid_{[0,t]} \in \{\mathfrak{g}\mid_{[0,t]} : \mathfrak{g} \in G\}. \tag{7}$$

Moreover, we say that S has full support in $\mathcal{O}$, or simply full support when $\mathcal{O} = \mathbb{R}^n$, if (7) holds for $t = 0$ and for all open subsets of $C([0.T], \mathcal{O})$.

Recall also, the notion of consistent price system.

Definition 2.3. Let $\varepsilon > 0$. An ε – *consistent* price system with S is a pair $(\widetilde{S}, \mathbf{Q})$, where $\mathbf{Q}$ is a probability measure equivalent to $\mathbf{P}$ and $\widetilde{S}$ is a $\mathbf{Q}$ – *martingale* in the filtration $\mathfrak{F}$, such that
$$\frac{1}{1+\varepsilon} \leq \frac{\widetilde{S}_i(t)}{S_i(t)} \leq 1+\varepsilon, \quad \text{almost surely for all } t \in [0,T] \text{ and } i = 1,\ldots,n.$$

Note, that $\widetilde{S}$ is a martingale under $\mathbf{Q}$, hence we may assume that it is càdlàg, but it is not required in the definition that $\widetilde{S}$ is continuous.

Theorem [5] **2.1**. *Let $\mathcal{O} \subset (0,\infty)^n$ be the open set defined by*
$$\mathcal{O} = \mathcal{O}(\delta) = \left\{x \in (0,\infty)^n : \max_j \frac{x_j}{x_1+\ldots+x_n} < 1-\delta\right\}, \tag{8}$$
and assume that the price process takes values and has conditional full support in $\mathcal{O}$. Then for any $\varepsilon > 0$ there is an ε – consistent price system $(\widetilde{S}, Q)$ such that $\widetilde{S}$ takes values in $\mathcal{O}$.

To check the condition of theorem 2.1 we apply the next theorem. To compare it with existing results we mention that it seems to be new in the sense, that we do not assume that our process solves a stochastic differential equation as it is done by Stroock and Varadhan [9], and it is not only for one dimensional processes as it is in the paper by Pakkanen [8].

Theorem [5] **2.2**. *Let X be a n-dimensional Itô process on $[0,T]$, such that*
$$dX_i(t) = \mu_i(t)dt + \sum_{v=1}^{n} \sigma_{iv}(t)dW_v(t).$$
Assume also that $|\mu|$ is bounded and σ satisfies
$$\varepsilon|\xi|^2 \leq |\sigma'(t)\xi|^2 \leq M|\xi|^2, \quad a.s. \text{ for all } t \in [0,T] \text{ and } \xi \in \mathbb{R}^n \text{ and } \varepsilon, M > 0.$$
Then X has conditional full support.

3. Consistent Price System and Conditional Full Support

Theorem [5] **3.1** *Let $\mathcal{O} \subset R^n$ be an open set and $(S(t))_{t\in[0,T]}$ be an $\mathcal{O}$-valued, continuous adapted process having conditional full support in $\mathcal{O}$. Besides, let $(\varepsilon_t)_{t\in[0,T]}$ be a continuous positive process, that satisfies*
$$|\varepsilon_t - \varepsilon_s| \leq L_s \sup_{s\leq u\leq t} |S(u) - S(s)|, \quad \text{for all } 0 \leq s \leq t \leq T,$$
with some progressively measurable finite valued $(L_s)_{s\in[0,T]}$. Then S admits an ε – consistent price system in the sense that, there is an equivalent probability Q on F_t, a process $(\widetilde{S}(t))_{t\in[0,T]}$ taking values in $\mathcal{O}$, such that $\widetilde{S}$ is Q martingale, bounded in $L^2(Q)$ and finally $|S(t) - \widetilde{S}(t)| \leq \varepsilon_t$ almost surely for all $t \in [0,T]$.

Lemma [5] **3.1** *Under the assumption of theorem 3.1 there is a sequence of stopping times $(\tau_n)_{n\geq 1}$, a sequence of random variables $(X_n)_{n\geq 0}$ and an equivalently Q such that*

1. *$\tau_0 = 0$, (τ_n) is increasing and $\bigcup_n \{\tau_n = T\}$ has full probability,*
2. *$(X_n)_{n\geq 0}$ is a Q martingale in the discrete time filtration $(\mathfrak{g}_n = F_{\tau_n})_{n\geq 0}$, bounded in $L^2(Q)$,*
3. *if $\tau_n \leq t \leq \tau_{n+1}$ then $|S_t - X_{n+1}| \leq \varepsilon_t$.*

Corollary [5] **3.1** *Assume that the continuous adapted process S evolving in* $\mathcal{O}$ *has conditional full support in* $\mathcal{O}$. *Let* τ *be a stopping time and denote by* $Q_{S|\mathfrak{F}_\tau}$ *the regular version of the conditional distribution of S given* F_τ. *Then the support of the random measure* $Q_{S|\mathfrak{F}_\tau}$ *is*
$supp\, Q_{S|\mathfrak{F}_\tau} = \{\mathfrak{g} \in C([0,T],\mathcal{O}) : \mathfrak{g}|_{[0,\tau]} = S|_{[0,\tau]}\},$ *almost surely*.

4. Conditional Full Support for Stochastic Integrals

We shall establish CFS for processes of the form

$$Z_t := H_t + \int_0^t k_s\, dW_s, \qquad t \in [0,T],$$

where H is a continuous process, the integrator W is a Brownian motion, and the integrand k_s satisfies some varying assumptions (to be clarified below). We focus on the following three cases, each of which requires a separate treatment (see [8]).

4.1. Independent integrands and Brownian integrators

Theorem [8] **4.1**. Let us define

$$Z_t := H_t + \int_0^t k_s\, dW_s, \qquad t \in [0,T] .$$

Suppose that

- $(H_t)_{t\in[0,T]}$ is a continuous process,
- $(k_t)_{t\in[0,T]}$ is a measurable process s.t. $\int_0^T K_s^2 ds < \infty$,
- $(W_t)_{t\in[0,T]}$ is a standard Brownian motion independent of H and k.

If we have

$$meas(t \in [0,T] : k_t = 0) = 0 \qquad \mathbf{P} - a.s,$$

then Z has CFS.

As an application of this result, we show that several popular stochastic volatility models do have the CFS property.

4.1.1. *Application to a stochastic volatility model*

Let us consider price process $(P_t)_{t\in[0,T]}$ in $\mathbb{R}_+$ given by :

$$dP_t = P_t(f(t,V_t)dt + \rho\mathfrak{g}(t,V_t)dB_t + \sqrt{1-\rho^2}\,\mathfrak{g}(t,V_t)dW_t,$$

$P_0 = p_0 \in \mathbb{R}_+$ where
(a) $f,\mathfrak{g} \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$,
(b) (B,W) is a planar Brownian motion,
(c) $\rho \in (-1,1)$,
(d) V is a (measurable) process in $\mathbb{R}^d$ s.t. $\mathfrak{g}(t,V_t) \neq 0$ a.s. for all $t \in [0,T]$,
(e) (B,V) is independent of W.
We may then write using Itô's formula

$$logP_t = \underbrace{logP_0 + \int_0^t (f(s,V_s) - \frac{1}{2}\mathfrak{g}(s,V_s)^2)ds + \rho\int_0^t \mathfrak{g}(s,V_s)dB_s}_{=H_t}$$

$$\underbrace{+\sqrt{1-\rho^2}\int_0^t \mathfrak{g}(s,V_s)dW_s}_{=K_s} \quad .$$

W is independent of B and V. The previous theorem implies that $logP$ has CFS, and from the next remark, it follows that P has CFS.

Remark 4.1. If $I \in \mathbb{R}$ is an open interval and $f : \mathbb{R} \to I$ is a homeomorphism, then $\mathfrak{g} \mapsto f \circ \mathfrak{g}$ is a homeomorphism between $C_x([0,T])$ and $C_{f(x)}([0,T], I)$.
Hence, for $f(X)$, understood as a process in I, we have

$$f(X)\ has\ \mathbb{F}-CFS \Leftrightarrow X\ has\ \mathbb{F}-CFS \tag{9}$$

Next, we relax the assumption about independence, and consider the second case.

4.2. Progressive integrands and Brownian integrators

Remark 4.2. The assumption about independence between W and (H,k) cannot be dispensed with in general, without imposing additional conditions. Namely, if e.g.
$H_t = 1; k_t := e^{W_t - \frac{1}{2}t}; t \in [0,T]$,
then $Z = k = \xi(W)$, the Dolans exponential of W, which is strictly positive and thus does not have CFS, if the process is considered to be in $\mathbb{R}$.

Theorem [8] **4.2**. *Suppose that*
- $(X_t)_{t\in[0,T]}$ *and* $(W_t)_{t\in[0,T]}$ *are continuous process,*
- *h and k are progressive* $[0,T] * C([0,T])^2 \longrightarrow R$,
- ε *is a random variable,*
- *and* $F_t = \sigma\{\varepsilon, X_s, W_s : s \in [0,t]\}, t \in [0,T]$.

If W is an $F_{t\in[0,T]}$ *– Brownian motion and*
- $E[e^{\lambda\int_0^T k_s^{-2}ds}] < \infty$ *for all* $\lambda > 0$,
- $E[e^{2\int_0^T k_s^{-2}h_s^2 ds}] < \infty$ *, and*
- $\int_0^T k_s^2 ds \le \overline{K}$ *a.s for some constant* $\overline{K} \in (0,\infty)$,

then the process
$Z_t = \varepsilon + \int_0^t h_s ds + \int_0^t k_s dw_s, \qquad t \in [0,T]$
has CFS.

4.3. Independent integrands and general integrators

Since Brownian motion has CFS, one might wonder if the preceding results generalize to the case where the integrator is merely a continuous process with CFS. It should be noted that the proofs of these results use quite extensively methods specific to Brownian motion (martingales, time changes), in the case independent integrands of finite variation.

Theorem [8] **4.3**. *Suppose that*

- $(H_t)_{t\in[0,T]}$ *is a continuous process,*
- $(k_t)_{t\in[0,T]}$ *is a process of finite variation, and*
- $X = (X_t)_{t\in[0,T]}$ *is a continuous process independent of H and k.*

Let us define

$$Z_t := H_t + \int_0^t k_s dX_s, \qquad t \in [0,T].$$

If X has CFS *and*

$$\inf_{t\in[0,T]} |k_t| > 0 \qquad P-a.s,$$

then Z has CFS.

5. Conditional Full Support for the Brownian Bridge

5.1. Some Basics of the Brownian bridge

In a Brownian motion the state variable, i.e. the stock price, interest rate, is stochastic and evolves over a period of time in a random manner.

The randomness is tied to the volatility of the asset and the drift is deterministic. In the short run, the volatility dominates the process and the asset price path is truly stochastic, i.e. random. However, over the long period of time the drift will dominate the volatility and therefore, if there are small errors in estimation of the drift it will lead to large fluctuations in the future price distribution. This is one of the drawbacks of a Brownian motion (as applied to the pricing of financial assets). Over longer horizons the drift of the stochastic process (Brownian motion) becomes a complicating factor.

Besides, in a Brownian motion the final state is uncertain. The asset value can be anything above zero. Theoretically, the asset can go from zero to infinity. Of course, the asset returns are normally distributed (Gaussian distribution) from minus infinity to plus infinity.

Therefore, a Brownian motion may not be very suitable for modelling an asset which has a longer maturity period, say, 5 years, 10 years, etc. and where the final state of the asset is known; like a discount bond (Treasury bonds). A government bond can have maturities of 5 years, 10 years, 30 years and the final value of the bond is known, i.e. the par (face) value. And a bond will always redeem at par.

In the case of a long dated discount bond we need to simulate values of the asset over a longer period of time such that the stochastic process is conditional on reaching a given final state. For example, take the case of a discount bond such as a Treasury bond. The Treasury bond will always mature at the face (par) value. If the par value of a discount bond is 100 $ and given a certain yield to maturity, it is currently trading at 95 $ (market price), then at maturity it has to redeem at par. A bond always gets redeemed at the par value at maturity. Therefore, if we model a discount bond as a stochastic, random process then this process should be tied to the final state of the process. In other words, this stochastic process will evolve conditional on the (given) final state of the process.

A Brownian bridge (BB) is a "tied down" Brownian motion such that even though the stochastic process, the Brownian motion, evolves in a random manner it is conditional on the final state of the process. Simply speaking, in a Brownian Bridge the initial and the final states of the stochastic process are known; however, in between the initial and the final state the process is random. Also, in an integral setup, the drift of the stochastic process is given at the beginning and is simply the return of the process - natural log of the prices - given the initial and the final values of the asset (say, the bond) which is known.

Another important point is that while modelling a discount bond we see that in a Brownian bridge the drift drops out of the integral stochastic equation and it is replaced by a constant drift that is conditional only on the initial and final state of the asset (bond).

Besides treasury and discount bonds, Brownian bridges are also used to model barrier options and other exotic options where the terminal value is known in advance.

Let us start with a BM $(B_t, t \geq 0)$ and its natural filtration $\mathfrak{F}^B$. Next define a new filtration as $\mathfrak{F}_t^{(B1)} = \mathfrak{F}_t^B \vee \sigma(B_1)$. In this filtration, the process $(B_t, t \geq 0)$ is no longer a martingale. It is easy to be convinced of this by looking at the process $(E(B_1|\mathfrak{F}_t^{(B_1)}); t \leq 1)$: this process is identically equal to B_1, not to B_t.

Before giving the application of CFS we recall some facts on a Brownian bridge.

- The Brownian bridge $(b_t; 0 \leq t \leq 1)$ is defined as the conditioned process $(B_t; t \leq 1|\, B_1 = 0)$.
- Note that $B_t = (B_t - tB_1) + tB_1$ where, from the Gaussian property, the process $(B_t - tB_1; t \leq 1)$ and the random variable B_1 are independent. Hence $(b_t\,; 0 \leq t \leq 1) =^{law} (B_t - tB_1; 0 \leq t \leq 1)$.
- The Brownian bridge process is a Gaussian process, with zero mean and covariance function $s(1-t); s \leq t$. Moreover, it satisfies $b_0 = b_1 = 0$.
- We can represent the Brownian bridge between 0 and y during the time interval $[0, 1]$ as $(B_t - tB_1 + ty\,; t \leq 1)$.
- More generally, the Brownian bridge between x and y during the time interval $[0, T]$ may be expressed as $(x + B_t - \frac{t}{T}B_T + \frac{t}{T}(y - x);\ t \leq T$, where $(B_t\,; t \leq T)$ is a standard BM starting from 0.

5.2. **Application**

Let
$dS_t = S_t(\mu dt + \sigma db_t)$,
where μ and σ are constants, be the price of a risky asset. Assume that the riskless asset has a constant interest rate r.

The standard Brownian bridge $b(t)$ is a solution of the following stochastic equation.

$$\begin{aligned} db_t &= -\frac{b_t}{1-t}dt + dW_t\,; \qquad 0 \leq t < 1 \\ b_0 &= 0. \end{aligned} \tag{10}$$

The solution of the above equation is
$b_t = (1-t)\int_0^t \frac{1}{1-s}dW_s$.

We may now verify that S has CFS. By positivity of S, Itô's formula yields
$logS_t = logS_0 + \left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma(1-t)\int_0^t \frac{1}{1-s}dW_s\right\}, \qquad 0 \leq t < 1,$

which is

$$logS_t = \underbrace{logS_0 + \left(\mu - \frac{\sigma^2}{2}\right)t}_{=:H_t} + \int_0^t \underbrace{\sigma(1-t)\frac{1}{1-s}}_{=:K_s} dW_s, \qquad 0 \leq t < 1.$$

Clearly

1. (H_t) is a continuous process,
2. $K_s = \sigma(1-t)\frac{1}{1-s}$ is a measurable process s.t. $\int_0^t K_s^2 ds < \infty$,
3. (W_t) is a standard Brownian motion independent of H and K,

which clearly satisfies the assumptions of theorem 4.1 and $logS_t$ has CFS. From (9) it follows that S has CFS for $0 \leq t < 1$. Then for any $\varepsilon > 0$ there exists ε-consistent price system, which is a martingale and implies that the price process does not admit arbitrage opportunities.

Conjecture 5.1. *If* $t = 1$, *then the conditions of CFS can be established*.

We do think, however, that the assumptions we have made do not guarantee the absence of arbitrage.

Acknowledgements

The first author would like to thank Professor Paul Raynaud De Fitte (of Rouen University) for his reception at LMRS and his availability. He also thanks Professor M'hamed Eddahbi (of Marrakech University) for his support to finalize this work. Both authors wish to thank an anonymous reviewer for a number of useful remarks.

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Article history: Submitted November, 17, 2015; Accepted December, 12, 2016.