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Maximum Principle for the Optimal Control Problem of a Forward Backward SDE With Jumps in the Mean-field Model

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Abstract. *We prove a necessary and sufficient condition for optimal control of a controlled forward-backward stochastic differential equation with random jumps of the mean-field type. The coefficients of our system decide the law of the solution, the control variable is allowed to enter both the diffusion and jumps coefficients.*

Key words : Forward Backward SDE, Poisson Random Measure, Maximum Principle, Optimal Control, Control Variable, Mean-field Model.

AMS Subject Classifications : 60H10, 60H05

1. Introduction

We study a stochastic control problem in which the controlled state process is driven by both a Brownian motion and a compensated Poisson random measure where the state process is governed by a forward-backward stochastic differential equation with jumps of the mean-field type, which is called a Mckean-Vlasov type equation, in the sense that the coefficients of the FBSDE with jumps are allowed to depend on the state of the process as well as on its expected value. More precisely, the controlled FBSDE is defined as

$$\begin{aligned} dx_t \; &= \mathbb{E}'\left[f(t,(x_t)',x_t,v_t)\right]dt + \mathbb{E}'\left[\sigma\left(t,(x_t)',x_t,v_t\right)\right]dB_t \\ &+ \mathbb{E}'\left[\int_E c\left(t,(x_{-t})',x_{-t},v_t,e\right)\right]\widetilde{N}(de,dt), \\ -dy_t \; &= \mathbb{E}'\left[\int_E g(t,(x_t)',(y_t)',(z_t)',(r_t(e))',x_t,y_t,z_t,r_t(e),v_t)\pi(de)\right]dt \end{aligned}$$

$$
\begin{aligned}
& - z_t dB_t - \int_E r_t(e)\widetilde{N}(de,dt), \\
x_0 &= x, \\
y_T &= \mathbb{E}'\left[\Phi\left((x_T)', x_T\right)\right], \qquad (1)
\end{aligned}
$$

for some functions f, σ, c, g, Φ, Brownian motion B and a compensated Poisson random measure $\tilde{N}$. The control v_t is allowed to take values in some control state space U and in a cost functional

$$
\begin{aligned}
J(v(.)) &= \mathbb{E}\left[\int_0^T \mathbb{E}'\left[\int_E l\left(t,(x_t^v)',(y_t^v)',(z_t^v)',(r_t^v(e))',\ x_t^v, y_t^v, z_t^v, r_t^v(e), v_t\right)\pi(de)\right]dt \right. \\
&\quad \left. + \mathbb{E}'\left[h((x_T^v)', x_T^v) + \gamma\left((y_0^v)', y_0^v\right)\right]\right]. \qquad (2)
\end{aligned}
$$

The state equation (1) and the cost functional (2) have other forms such as:

$$
\begin{aligned}
dx_t &= \int_\Omega f(t, x_t(\omega'), x_t, v_t(\omega',))P(d\omega')dt + \int_\Omega \sigma(t, x_t(\omega'), x_t, v_t(\omega',))P(d\omega')dB_t \\
&\quad + \int_\Omega \int_E c(t, x_{-t}(\omega'), x_{-t}, v_t(\omega',), e)P(d\omega')\widetilde{N}(de,dt), \\
-dy_t &= \int_\Omega \int_E g(t, x_t(\omega'), y_t(\omega'), z_t(\omega'), r_t(\omega', e), x_t, y_t, z_t, r_t(e), v_t(\omega',))P(d\omega')\pi(de)dt \\
&\quad - z_t dB_t - \int_E r_t(e)\widetilde{N}(de,dt), \qquad (3) \\
x_0 &= x, \\
y_T &= \int_\Omega \Phi(x_T(\omega'), x_T)P(d\omega'),
\end{aligned}
$$

and

$$
\begin{aligned}
J(v(.)) &= \mathbb{E}\left[\int_0^T \int_\Omega \int_E l(t, x_t^v(\omega'), y_t^v(\omega'), z_t^v(\omega'), (r_t^v(\omega', e)),\ x_t^v, y_t^v, z_t^v, r_t^v(e), v_t)P(d\omega')\pi(de)dt \right. \\
&\quad \left. + \int_\Omega (h(x_T^v(\omega'), x_T^v) + \gamma(y_0^v(\omega'), y_0^v))P(d\omega')\right]. \qquad (4)
\end{aligned}
$$

The mean-field models were initially suggested to study the aggregate behavior of a large number of mutually interacting particles in diverse areas of statistical mechanics (e.g., in the derivation of Boltzmann or Vlasov equation in the kinetic gas theory), quantum mechanics and quantum chemistry (e.g., the density functional models or also Hartree and Hartree-Fock type models), economics, finance and game theory. For N players of stochastic differential games and the related problem of the existence of Nash equilibrium points, one can let n tend to infinity to derive, in a periodic setting, the mean filed limit equation. In Buckdahn et al. [4], the authors studied the mean field backward stochastic differential equation by using a purely stochastic approach. Recently Carmona et al. [6, 7, 8] proved an existence result for the solution of a fully coupled forward-backward stochastic differential equation of the mean-field model by applying the Wasserstein's distance. In that case the law of the solution is present in the drift and volatility. The stochastic maximum principle for controlled systems has been investigated by many authors. To list some of them, we mention Kushner [12], Bismut [2, 3], Haussmann [10, 11], Peng [16], Situ [19], Øksendal and Sulem [15], Elliot [9], Tang and Li [20], and Shi and Wu [18]. The stochastic maximum principle (SMP) in the mean-field model was obtained by Buckdahn et al. [5], Anderson and Djehiche [1], Meyer-Brandis et al. [14], and Li [13]. The first version of the SMP for the mean-field model with jumps was proved by Shen and Siu in [17] with an application to the mean variance problem. Our contribution to this subject is to extend the result of Juan Li in forward backward stochastic control with jumps in the mean-field type controls, to prove the maximum principle in the case where the system is a decoupled controlled forward-backward stochastic differential equation governed by Brownian motion and Poisson random measure. Assuming that the state variable control is convex, we use a convex classical perturbation; and since the domain control is convex, we use only one adjoint process. This will lead to a necessary condition for optimality; and with some additional convexity conditions we prove the sufficient conditions.

Our paper is organized as follows. In the second section, we will explain the problem and give its necessary preliminaries and notations and provide some results about the mean-field FBSDEs. Section 3 contains a formulation of the SMP. The associated variational equations and variational inequality are presented in section 4, while section 5 is devoted to the pertaining adjoint equation. Section 6 reports on the necessary and sufficient conditions for optimality, which are our main result. An illustrative application in finally provided in section 7.

2. Preliminaries and Notation

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space where $\{\mathcal{F}_t : t \geq 0\}$ is a filtration satisfying the usual conditions, $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the following two mutually independent stochastic processes:

(i) a d-dimensional standard Brownian motion $\{B_t\}_{t\geq 0}$.

(ii) a Poisson random measure N on $\mathbb{R}_+ \times E$, where $E = \mathbb{R}^l - \{0\}$ is equipped with its Borel field $\mathcal{B}(E)$. The compensator of N is $\hat{N}(dt, de) = dt\pi(de)$ which makes $\left\{\widetilde{N}((0,t] \times A) = \left(N - \hat{N}\right)((0,t] \times A)\right\}_{t\geq 0}$ a martingale. For all $A \in \mathcal{B}(E)$ satisfying $\pi(A) < \infty$. Here π is an arbitrarily given $\sigma-$finite Lévy measure on $(E, \mathcal{B}(E))$, $i.e$ a measure on $(E, \mathcal{B}(E))$ with the property that $\int_E (1 \wedge |e|^2)\pi(de) < \infty$. We set T to be an arbitrarily prescribed positive number and we call $[0, T]$ the time duration. We assume

$$\mathcal{F}_t = \sigma\left[\iint_{(0,s]\times A} N(ds, dz) : s \leq t, A \in \mathcal{B}(E)\right] \vee \sigma[B_s : s \leq t] \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $P-$null sets. We also introduce the following spaces of processes which are used in what follows. Let $\mathcal{S}^2([0,T], \mathbb{R})$ denote the set of real valued and $\mathcal{F}_t-$adapted càdlàg process $\left(\Psi(t)\right)_{t\in[0,T]}$ which satisfy $\mathbb{E}\left[\sup_{t\in[0,T]}|\Psi(t)|^2 < +\infty\right]$. Then $\mathcal{M}^2([0,T], \mathbb{R}^n)$ denote the set of $n-$dimensional, $\mathcal{F}_t-$progressively measurable process $\left(\Psi(t)\right)_{t\in[0,T]}$, such that $\|\Psi\|_2^2 = \mathbb{E}\left[\int_0^T |\Psi(t)|^2 dt < +\infty\right]$. We also denote by $F_p^2([0,T], \mathbb{R}^m)$, the set of function $m-$dimensional $f(.,t,e)$, define in $\Omega \times [0,T] \times E$, and $\mathcal{P} \otimes \mathcal{B}(E)$ $-$measurable mapping such that : $\|f\|^2 = \mathbb{E}\int_0^T \int_E |f(t,e)|^2 \pi(de)dt < +\infty$.

Let us now consider a function $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \pi; \mathbb{R}) \to \mathbb{R}$ with the property that $(g(t,y,z,r))_{t\in[0,T]}$ is $\mathcal{P}$ measurable for each (y,z,r) in $\mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \pi; \mathbb{R})$, and we also make the following assumption on g.

$(H1)$: There exists a constant $C > 0$ such that for all $t \in [0,T]$ $y_1,\ y_2 \in \mathbb{R},\ z_1,\ z_2 \in \mathbb{R}^d,\ r_1,$ $r_2 \in L^2(E,\mathcal{B}(E),\pi;\mathbb{R}),$
$|g(y_1,z_1,r_1) - g(y_2,z_2,r_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \|r_1 - r_2\|),\ a.s.$

$(H2)$: $g(.,0,0,0) \in \mathcal{M}^2([0,T],\mathbb{R}).$

The following results on BSDE with jumps is by now well known, for its proof, see Lemma 2.4 in Tang and Li [20].

Lemma 2.1. *Under assumptions* $(H1)$ *and* $(H2)$, *for any random variable* $\xi \in L^2(\Omega,\mathcal{F}_T,P)$, *the BSDE with jumps*

$$y_t = \xi + \int_t^T g(t,y_s,z_s,r_s)ds - \int_t^T z_s dB_s - \int_t^T \int_E r_s(e)\widetilde{N}(de,ds),\ \ 0 \leq t \leq T,$$

has a unique adapted solution $(y_t,z_t,r_t) \in S^2([0,T],\mathbb{R}) \times M^2([0,T],\mathbb{R}^d) \times F_p^2([0,T],\mathbb{R}).$

2.1. **Mean-Field BSDEs with jumps**

Let $(\bar{\Omega},\bar{\mathcal{F}},\bar{P}) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P)$ be a non completed product of $(\Omega,\mathcal{F},P)$ with itself. We endow this product space with the filtration $\bar{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{F}_t$. Any random variable $\xi \in L^0(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$ originally defined on Ω is extended canonically to $\bar{\Omega}$: $\xi'(\omega',\omega) = \xi(\omega'),\ \ (\omega',\omega) \in \bar{\Omega}$. For any $\theta \in L^1(\bar{\Omega},\bar{\mathcal{F}},\bar{P})$ the variable $\theta(.,\omega) \to \mathbb{R}$ belongs to $L^1(\Omega,\mathcal{F},P),\ P(d\omega) - a.s.$ We denote its expectation by

$$\mathbb{E}'[\theta(.,\omega)] = \int_\Omega \theta(\omega',\omega)P(d\omega').$$

We note that $\mathbb{E}'[\theta] = \mathbb{E}'[\theta(.,\omega)] \in L^1(\Omega,\mathcal{F},P)$, and

$$\bar{\mathbb{E}}[\theta] = \int_{\bar{\Omega}} \theta d\bar{P} = \int_\Omega \mathbb{E}'[\theta(.,\omega)]P(d\omega) = \mathbb{E}[\mathbb{E}'[\theta]].$$

For all $(\widetilde{y},\widetilde{z},\widetilde{r},y,z,r) \in (\mathbb{R} \times \mathbb{R}^d \times L^2(E,\mathcal{B}(E),\pi;\mathbb{R}))^2$,
$g : \bar{\Omega} \times [0,T] \times (\mathbb{R} \times \mathbb{R}^d \times L^2(E,\mathcal{B}(E),\pi;\mathbb{R}))^2 \to \mathbb{R}$ is a measurable process such that $g(.,\widetilde{y},\widetilde{z},\widetilde{r},y,z,r)$ is a $\bar{\mathcal{F}}_t$ adapted for all $(\widetilde{y},\widetilde{z},\widetilde{r},y,z,r)$, and which satisfies the following assumptions.

$(H3)$ There exists a constant $C > 0$ such that $\bar{P} - a.s,\ \forall t \in [0,T],$
$((\widetilde{y}_1,\widetilde{z}_1,\widetilde{r}_1,y_1,z_1,r_1),(\widetilde{y}_2,\widetilde{z}_2,\widetilde{r}_2,y_2,z_2,r_2)) \in (\mathbb{R} \times \mathbb{R}^d \times L^2(E,\mathcal{B}(E),\pi;\mathbb{R}))^2.$

$$|g(\widetilde{y}_1,\widetilde{z}_1,\widetilde{r}_1,y_1,z_1,r_1) - g(\widetilde{y}_2,\widetilde{z}_2,\widetilde{r}_2,y_2,z_2,r_2)| \leq C(|\widetilde{y}_1 - \widetilde{y}_2| + |\widetilde{z}_1 - \widetilde{z}_2| + \|\widetilde{r}_1 - \widetilde{r}_2\| + |y_1 - y_2| + |z_1 - z_2| + \|r_1 - r_2$$

$(H4)$ $g(.,0,0,0,0,0,0) \in \mathcal{M}^2([0,T],\mathbb{R}).$

We now recall a result of Shen and Siu [17].

Lemma 2.1. *Under the assumption* $(H3)$ *and* $(H4)$, *for any random variable* $\xi \in L^2(\Omega,\mathcal{F}_T,P)$, *the mean-field BSDEs with jumps*

$$y_t = \xi + \int_t^T E'[g(t,y'_s,z'_s,y_s,z_s)]ds - \int_t^T z_s dB_s - \int_t^T \int_E r_s(e)\widetilde{N}(de,ds) \tag{5}$$

has a unique adapted solution $(y_t,z_t,r_t(.)) \in S^2([0,T],\mathbb{R}) \times M^2([0,T],\mathbb{R}^d) \times F_p^2([0,T],\mathbb{R}).$

Remark 2.1. We emphasize that, due to our notations, the driving coefficient of (5) has to be interpreted as follows

$$\begin{aligned}\mathbb{E}'[g(s,y'_s,z'_s,r'_s,y_s,z_s,r_s)](\omega) &= \mathbb{E}'[g(s,y'_s,z'_s,r'_s,y_s(\omega),z_s(\omega),r_s(\omega))]\\ &= \int_\Omega g(\omega',\omega,s,y_s(\omega'),z_s(\omega'),r_s(\omega'),y_s(\omega),z_s(\omega),r_s(\omega))P(d\omega').\end{aligned}$$

2.2. **McKean-Vlasov SDEs with jumps**

We also introduce the McKean-Vlasov type SDEs with jumps [17]. Let $f:\bar{\Omega}\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $\sigma:\bar{\Omega}\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times d}$ and $c:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times E\to\mathbb{R}^n$ be measurable functions which are supposed to be satisfied by the following conditions.

$(H5)$: $f(.,\tilde{x},x)$, $\sigma(.,\tilde{x},x)$ and $c(.,\tilde{x},x,e)$ are $\bar{\mathcal{F}}$- progressively measurable continuous processes, for all $\tilde{x}$, $x\in\mathbb{R}^n$, and there exists some constant $C>0$ such that:

$$|f(t,\tilde{x},x)|^2+|\sigma(t,\tilde{x},x)|^2+\int_E|c(t,\tilde{x},x,e)|^2\pi(de)\leq C(1+|\tilde{x}|^2+|x|^2),$$

$a.s,$ for all $0\leq t\leq T,\tilde{x},x\in\mathbb{R}^n$.

$(H6)$: There exists some constant $C>0$ such that $a.s,$ for all $0\leq t\leq T$, $\tilde{x}$, $x\in\mathbb{R}^n$:

$$\begin{cases} |f(t,\tilde{x}_1,x_1)-f(t,\tilde{x}_2,x_2)|^2 & \leq C(|\tilde{x}_1-\tilde{x}_2|^2+|x_1-x_2|^2),\\ |\sigma(t,\tilde{x}_1,x_1)-\sigma(t,\tilde{x}_2,x_2)|^2 & \leq C(|\tilde{x}_1-\tilde{x}_2|^2+|x_1-x_2|^2),\\ \int_E|c(t,\tilde{x}_1,x_1,e)-c(t,\tilde{x}_2,x_2,e)|^2\pi(de) & \leq C(|\tilde{x}_1-\tilde{x}_2|^2+|x_1-x_2|^2).\end{cases}$$

Theorem 2.2. *Under assumptions* $(H5)$ *and* $(H6)$, *the mean-field SDE*

$$\begin{cases} dx_t &= \mathbb{E}'\left[f\left(t,(x_t)',x_t\right)\right]dt+\mathbb{E}'\left[\sigma(t,(x_t)',x_t)\right]dB_t\\ &\quad+\mathbb{E}'\left[\int_E c\left(t,(x_{-t})',x_{-t},e\right)\right]\widetilde{N}(de,dt),\\ x_0 &= x,\end{cases}\tag{6}$$

has a unique solution $x_t\in S^2([0,T],\mathbb{R}^n)$.

We remark that due to our notational convention,

$$\mathbb{E}'\left[f(t,(x_t)',x_t)\right](\omega)=\int_\Omega f(\omega',\omega,x_t(\omega'),x_t(\omega))P(d\omega').$$

2.3. **Decoupled mean-field forward-backward SDE with jumps**

Given two real- valued functions g, Φ which will be satisfied by the following conditions.

$(H7)$ $\Phi:\bar{\Omega}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is a $\bar{\mathcal{F}}_T\otimes\mathcal{B}(\mathbb{R}^n)\times\mathcal{B}(\mathbb{R}^n)$ measurable random variable.
$g:\bar{\Omega}\times[0,T]\times(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathcal{B}(E),\pi;\mathbb{R}))^2\to\mathbb{R}$ is a measurable process such that

$g(.,\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{r},x,y,z,r)$ is a $\bar{\mathcal{F}}_t$ adapted for all
$(\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{r},x,y,z,r) \in \left(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times L^2(E,\mathcal{B}(E),\pi;\mathbb{R})\right)^2$.

$(H8)$ There exists a constant $C > 0$ such that for all $\widetilde{x}_1,x_1 \in \mathbb{R}^n$, $\widetilde{y}_1$, $y_1 \in \mathbb{R}$, $\widetilde{z}_1$, $\widetilde{z}_2 \in \mathbb{R}^d$, $\widetilde{r}_1$, $r_2 \in L^2(E,\mathcal{B}(E),\pi;\mathbb{R})$:
$|\Phi(\widetilde{x}_1,x_1) - \Phi(\widetilde{x}_2,x_2)| \leq C(|\widetilde{x_1} - \widetilde{x}_2| + |x_1 - x_2|)$.

$$\begin{aligned}
&|g(\widetilde{x}_1,\widetilde{y}_1,\widetilde{z}_1,\widetilde{r}_1,x_1,y_1,z_1,r_1) - g(\widetilde{x}_2,\widetilde{y}_2,\widetilde{z}_2,\widetilde{r}_2,x_2,y_2,z_2,r_2)| \\
&\leq C(|\widetilde{x_1} - \widetilde{x}_2| + |\widetilde{y}_1 - \widetilde{y}_2| + |\widetilde{z}_1 - \widetilde{z}_2| \\
&+\|\widetilde{r}_1 - \widetilde{r}_2\| + |x_1 - x_2| + |y_1 - y_2| \\
&+|z_1 - z_2| + \|r_1 - r_2\|).
\end{aligned}$$

Theorem 2.3. *Under the assumptions* $(H5)$, $(H6)$, $(H7)$ *and* $(H8)$ *the mean-field FBSDEs*

$$\begin{cases}
dx_t &= \mathbb{E}'\left[f(t,(x_t)',x_t)\right]dt + \mathbb{E}'\left[\sigma(t,(x_t)',x_t)\right]dB_t \\
&+\mathbb{E}'\left[\int_E c\left(t,(x_{-t})',x_{-t},e\right)\right]\widetilde{N}(de,dt), \\
-dy_t &= \mathbb{E}'\left[\int_E g(t,(x_t)',(y_t)',(z_t)',(r_t(e))',x_t,y_t,z_t,r_t(e))\pi(de)\right]dt \\
&-z_t dB_t - \int_E r_t(e)\widetilde{N}(de,dt), \\
x_0 &= x, \\
y_T &= \mathbb{E}'\left[\Phi\left((x_T)',x_T\right)\right],
\end{cases} \tag{7}$$

have a unique adapted solution
$(x_t,y_t,z_t,r_t(.)) \in \mathcal{S}^2([0,T],\mathbb{R}^n) \times \mathcal{S}^2([0,T],\mathbb{R}) \times \mathcal{M}^2([0,T],\mathbb{R}^d) \times F_p^2([0,T],\mathbb{R}^m)$.

3. Formulation of the Problem

In this section we study the stochastic maximum principle where the system is described by Forward-Backward SDEs with jumps of the mean-field type. Our goal is to give a necessary and sufficient condition for optimality. So we consider the following mean-field control problem

$$\begin{cases}
dx_t^v &= \mathbb{E}'\left[f(t,(x_t^v)',x_t^v,v_t)\right]dt + \mathbb{E}'\left[\sigma(t,(x_t^v)',x_t^v,v_t)\right]dB_t \\
&+\mathbb{E}'\left[\int_E c\left(t,(x_{-t}^v)',x_{-t}^v,v_t,e\right)\right]\widetilde{N}(de,dt), \\
-dy_t^v &= \mathbb{E}'\left[\int_E g(t,(x_t^v)',(y_t^v)',(z_t^v)',(r_t^v(e))',x_t^v,y_t^v,z_t^v,r_t^v(e),v_t)\pi(de)\right]dt \\
&-z_t^v dB_t - \int_E r_t^v(e)\widetilde{N}(de,dt), \\
x_0^v &= x, \\
y_T^v &= \mathbb{E}'\left[\Phi\left((x_T^v)',x_T^v\right)\right].
\end{cases} \tag{8}$$

The cost functional $J,$ to be minimized, is given by

$$J(v(.)) \ = \mathbb{E}\Big[\int_0^T \mathbb{E}'\Big[\int_E l\big(t,(x_t^v)',(y_t^v)',(z_t^v)',(r_t^v(e))',x_t^v,y_t^v,z_t^v,r_t^v(e),v_t\big)\pi(de)\Big]dt$$

$$+\mathbb{E}'h((x_T^v)',x_T^v)+\gamma\big((y_0^v)',y_0^v\big)\Big]. \tag{9}$$

We define the admissible control set as fellows.

$U_{ad} = \big\{v(.) \in L^2_{\mathcal{F},p}([0,T],\mathbb{R}^k); v(t) \in U,\ a.e. t \in [0,T], \mathbb{P}-a.s.\big\},$

U be a non-empty convex subset of $\mathbb{R}^k$, where $f,\ \sigma,\ c,\ g,\ \Phi,\ l,\ h$ and γ are a mapping such that:

$f : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U \to \mathbb{R}^n,\ \sigma : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U \to \mathbb{R}^{n\times d},$

$c : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U\times E \to \mathbb{R}^{n\times k},$

$g : [0,T]\times(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(E,\mathcal{B}(E),\pi;\mathbb{R}^m))^2\times U \to \mathbb{R}^m,\ \Phi(\widetilde{x},x) : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^m,$

$l : [0,T]\times(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(E,\mathcal{B}(E),\pi;\mathbb{R}^m))^2\times U \to \mathbb{R},\ h : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R},$

$\gamma : \mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}.$

The optimal control is to minimize the cost function $J(v(.))$ over the space U_{ad}. A control $u \in U_{ad}$ is said to be optimal if

$$J(u) = \min_{v\in U_{ad}} J(v).$$

Throughout this paper, we assume (H) to mean the following conditions:

i) $f,\ \sigma$ and c are Lipschitz in $(\tilde{x},x,v)$ and g is Lipschitz in $(\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{r},x,y,z,r,v)$.

ii) $f,\ \sigma,\ c,\ g,\ l,\ h$ and γ are continuously differentiable in their variables including $(\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{r},x,y,z,r,v)$.

iii) $f_{\widetilde{x}},\ \sigma_{\widetilde{x}},\ \int_E |c_{\widetilde{x}}(.,.,.,.,e)|^2\pi(de),\ g_{\widetilde{x}},\ g_{\widetilde{y}},\ g_{\widetilde{z}},\ g_{\widetilde{r}},\ f_x,\ \sigma_x,\ \int_E |c_x(.,.,.,.,e)|^2\pi(de),\ g_x,\ g_y,\ g_z,\ g_r,$ $\int_E |c_v(.,.,.,.,e)|^2\pi(de)$ and g_v are bounded.

iv) $l_{\widetilde{x}},\ l_{\widetilde{y}},\ l_{\widetilde{z}},\ l_{\widetilde{r}},\ l_x,\ l_y,\ l_z,\ l_r$ and l_v are bounded by $C(1+|\widetilde{x}|+|\widetilde{y}|+|\widetilde{z}|+|\widetilde{r}|+|x|+|y|+|z|+|r|+|v|)$, the derivatives $h_{\widetilde{x}}$ and h_x are bounded by $C(1+|\widetilde{x}|)$ and $C(1+|x|)$ respectively, also the derivatives $\gamma_{\widetilde{y}}$ and γ_y are bounded by $C(1+|\widetilde{y}|)$ and $C(1+|y|)$ respectively.

v) $\forall(\widetilde{x},x) \in (\mathbb{R}^n)^2,\ \ \Phi(\widetilde{x},x) \in L^2(\bar{\Omega},\bar{\mathcal{F}}_T)$, and $\Phi(.,.)$ is continuously differentiable in $(\widetilde{x},x)$ and $\Phi_{\widetilde{x}},\ \Phi_x$ are bounded.

vi) For all $t \in [0,T],\ g(t,0,0,0,0,0,0,0,0) \in L^2_{\mathcal{F}}([0,T],\mathbb{R}^m)$.

We may deduce here that if the appropriate H conditions are satisfied, then equation (8) has a unique solution

$(x(.),y(.),z(.),r(.,.)) \in \mathcal{S}^2(\mathbb{R}^n)\times\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{M}^2(\mathbb{R}^{d\times m})\times F_P^2([0,T],\mathbb{R}^m).$

4. Variational Equations and Variational Inequality

Let u be an optimal control and let $(x_t,y_t,z_t,r_t(.))$ be the corresponding trajectory. Let v be such that $u+v \in U_{ad}$. Since U_{ad} is convex, then for any $0 \le \theta \le 1$,

$u_t^\theta \equiv u_t + \theta v_t$ is also in U_{ad}.

Also we denote by $(x_t^\theta,y_t^\theta,z_t^\theta,r_t^\theta)$ the a trajectory corresponding to u^θ.

Lemma 4.1. *Under the above assumptions on the coefficients, we have*

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|x_t^\theta - x_t\right|^2\right] = 0,$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|y_t^\theta - y_t|^2\right] = 0,$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\int_0^T\left|z_t^\theta - z_t\right|^2 dt\right] = 0,$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\int_0^T\int_E|r_t^\theta(e) - r_t(e)|^2\pi(de)dt\right] = 0.$$

Proof. Let u be an optimal control and let $(x_t, y_t, z_t, r_t(.))$ be the corresponding trajectory, and we denote by $(x_t^\theta, y_t^\theta, z_t^\theta, r_t^\theta)$ the a trajectory corresponding to the perturbed control u^θ, to get

$$\begin{aligned}x_t^\theta - x_t = &\int_0^t \mathbb{E}'\left[f(s, \left(x_s^\theta\right)', x_s^\theta, u_s^\theta) - f(s, (x_s^u)', x_s^u, u_s)\right]ds\\ &+\int_0^t \mathbb{E}'\left[\sigma(s, \left(x_s^\theta\right)', x_s^\theta, u_s^\theta) - \sigma(\left(x_s^\theta\right)', x_s^\theta, u_s^\theta)\right]dB_s\\ &+\int_0^t \mathbb{E}'\left[\int_E c\left(s, (x_{s-}^\theta)', x_{s-}^\theta, u_s^\theta, e\right) - c\left(s, (x_{s-}^u)',\ x_{s-}^u, u_s,\ e\right)\right]\widetilde{N}(de, ds),\end{aligned}$$

$$\begin{aligned}y_t^\theta - y_t = &\int_t^T \mathbb{E}'\int_E\Big[g(s, (x_s^\theta)', (y_s^\theta)', (z_s^\theta)', (r_s^\theta(e))', x_s^\theta, y_s^\theta, z_s^\theta, r_s^\theta(e), u_s^\theta)\\ &- g(s, (x_s)', (y_s)', (z_s)', (r_s(e))', x_s, y_s, z_s, r_s(e), u_s)\Big]\pi(de)ds\\ &-\int_t^T[z_s^\theta - z_s]dB_s - \int_t^T\int_E[r_s^\theta(e) - r_s(e)]\widetilde{N}(de, dt) + \mathbb{E}'\left[\Phi\left((x_T^\theta)', x_T^\theta\right) - \Phi\left((x_T)', x_T\right)\right].\end{aligned}$$

Since $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, and by the Lipschitz condition on coefficients of our system, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}|x_s^\theta - x_s|^2\right] \le\ &3T\mathbb{E}\left[\int_0^t\left|\mathbb{E}'\left[f(s, (x_s^\theta)', x_s^\theta, u_s^\theta)\right.\right.\right.\\ &\left.\left.\left.-f(s, (x_s^u)', x_s^u, u_s)\right]\right|^2 ds\right]\\ &+12\mathbb{E}\left[\int_0^t\left|\mathbb{E}'\left[\sigma(s, (x_s^\theta)', x_s^\theta, u_s^\theta)\right.\right.\right.\\ &\left.\left.\left.-\sigma(s, (x_s^u)', x_s^u, u_s)\right]\right|^2 ds\right]\\ &+12\mathbb{E}\left[\int_0^t\int_E\left|\mathbb{E}'\left[c\left(s, (x_s^\theta)', x_s^\theta, u_s^\theta, e\right)\right.\right.\right.\\ &\left.\left.\left.-c\left(s, (x_s^u)', x_s^u, u_s, e\right)\right]\right|^2\right]\pi(de)ds\\ &\le C_T\mathbb{E}\int_0^t|x_s^\theta - x_s|^2 ds + \theta^2 C_T\overline{\mathbb{E}}\int_0^t|v_s|^2 ds.\end{aligned}$$

With the same arguments, we have:

$$\mathbb{E}\left[\sup_{t\le s\le T}|y_s^\theta-y_s|^2\right]+\mathbb{E}\int_t^T|z_s^\theta-z_s|^2ds+\mathbb{E}\int_t^T\int_E|r_s^\theta(e)-r_s(e)|^2\pi(de)ds$$
$$\le C_T\mathbb{E}\left[\int_t^T|x_s^\theta-x_s|^2ds+\mathbb{E}\int_t^T|z_s^\theta-z_s|^2ds+\int_t^T\int_E|r_s^\theta(e)-r_s(e)|^2\pi(de)ds\right]+\theta^2C_T\overline{\mathbb{E}}\int_t^T|v_s|^2ds.$$

From Gronwall's lemma we have the desired result. ■

Let $(\xi(.),\eta(.),\zeta(.),\varrho(.,.))$ be the solution of

$$\left.\begin{aligned}
d\xi_t &= \mathbb{E}'[f_{\tilde{x}}(t,(x_t)',x_t,u_t)(\xi_t)' \\
&\quad +f_x(t,(x_t)',x_t,u_t)\xi_t+f_v(t,(x_t)',x_t,u_t)v_t]dt \\
&\quad +\mathbb{E}'\left[\ \sigma_{\tilde{x}}\left(t,(x_t)',x_t,u_t\right)(\xi_t)'\right. \\
&\quad \left.\sigma_x(t,(x_t)',x_t,u_t)\xi_t+\sigma_v(t,(x_t)',x_t,u_t)v_t\ \right]dB_t \\
&\quad \mathbb{E}'\left[\int_E c_{\tilde{x}}\left(t,(x_{-t})',x_{-t},u_t,e\right)(\xi_{-t})'\right. \\
&\quad \left.c_x\left(t,(x_{-t})',x_{-t},u_t,e\right)(\xi_{-t})+c_v\left(t,(x_{-t})',x_{-t},u_t,e\right)v_t\right]\widetilde{N}(de,dt), \\
\xi_0 &= 0.
\end{aligned}\right\}\qquad(10)$$

$$\left.\begin{aligned}
-d\eta_t &= \mathbb{E}'\Big[(\int_E g_{\tilde{x}}(t,e)(\xi_t)'+g_x(t,e)\xi_t+g_{\tilde{y}}(t,e)(\eta_t)'+g_y(t,e)\eta_t \\
&\quad +g_{\tilde{z}}(t,e)(\zeta_t)'+g_z(t,e)\zeta_t+g_{\tilde{r}}(t,e)(\varrho_t(e))'+g_r(t,e)\varrho_t(e) \\
&\quad +g_v(t,e)v_t]\pi(de)dt-\zeta_t dB(t)-\int_E\varrho_t(e)\widetilde{N}(de,dt), \\
\eta_T &= \mathbb{E}'\left[\Phi_{\tilde{x}}\left((x_T)',x_T\right)(\xi_T)'+\Phi_x\left((x_T)',x_T\right)\xi_T\right].
\end{aligned}\right\}\qquad(11)$$

The equations (10) and (11) are called variational equations, and under the (H) assumptions, the system (10) - (11) has a unique solution
$(\xi(.),\eta(.),\zeta(.),\varrho(.,.))\in\mathcal{S}^2(\mathbb{R}^n)\times\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{M}^2(\mathbb{R}^{m\times d})\times F_P^2([0,T],\mathbb{R}^m)$.

To simplify notation, we shall assume
$\hat{x}_t^\theta=\theta^{-1}(x_t^\theta-x_t^\theta)-\xi_t,$
$\hat{y}_t^\theta=\theta^{-1}(y_t^\theta-y_t)-\eta_t,$
$\hat{z}_t^\theta=\theta^{-1}(z_t^\theta-z_t)-\zeta_t,$
$\hat{r}_t^\theta(e)=\theta^{-1}(r_t^\theta(e)-r_t(e))\ -\varrho_t(e),$
in the following convergence results.

Lemma 4.1. *Let assumptions (H) hold, then*

$$\lim_{\theta\to0}\sup_{0\le t\le T}\mathbb{E}\left[|\hat{x}_t^\theta|^2\right]=0,$$

$$\lim_{\theta\to 0}\sup_{0\le t\le T}\mathbb{E}\Big[|\hat{y}_t^\theta|^2\Big] = 0,$$

$$\lim_{\theta\to 0}\mathbb{E}\Big[\int_0^T|\hat{z}_t^\theta|^2dt\Big] = 0,$$

$$\lim_{\theta\to 0}\mathbb{E}\Big[\int_0^T\int_E|\hat{r}_t^\theta(e)|^2\pi(de)dt\Big] = 0. \tag{12}$$

Proof. First, we have

$$\begin{aligned}
\hat{x}_t^\theta &= \tfrac{1}{\theta}\int_0^t\mathbb{E}'\Big[f(s,(x_s^\theta)',x_s^\theta,u_s^\theta)-f(s,(x_s^u)',x_s^u,u_s)\Big]ds\\
&+\tfrac{1}{\theta}\int_0^t\mathbb{E}'\Big[\sigma(s,(x_s^\theta)',x_s^\theta,u_s^\theta)-\sigma(s,(x_s^u)',x_s^u,u_s)\Big]dB_s\\
&+\tfrac{1}{\theta}\int_0^t\mathbb{E}'\Big[\int_E c\Big(s,(x_{-s}^\theta)',x_{-s}^\theta,u_s^\theta,e\Big)-c\Big(s,(x_{-s}^u)',x_{-s}^u,u_s,e\Big)\Big]\widetilde{N}(de,ds)\\
&-\int_0^t\mathbb{E}'\Big[f_{\tilde{x}}(s,(x_s)',x_s,u_s)(\xi_s)'+f_x(s,(x_s)',x_s,u_s)\xi_s\\
&+f_v(s,(x_s)',x_s,u_s)v_s\Big)\Big]ds\\
&-\int_0^t\mathbb{E}'\Big[\sigma_{\tilde{x}}(s,(x_s)',x_s,u_s)(\xi_s)'+\sigma_x(s,(x_s)',x_s,u_s)\xi(s)\\
&+\sigma_v(s,(x_s)',x_s,u_s)v_s\Big)\Big]dB_s\\
&-\int_0^t\mathbb{E}'\Big[\int_E c_{\tilde{x}}\Big(s,(x_{-s}^\theta)',x_{-s}^\theta,u_s^\theta,e\Big)(\xi_s)'+c_x\Big(s,(x_{-s}^\theta)',x_{-s}^\theta,u_s^\theta,e\Big)\xi_s\\
&+c_v\Big(s,(x_{-s})',x_{-s},u_s,e\Big))v_s\Big]\widetilde{N}(de,ds).
\end{aligned} \tag{13}$$

By applying the Taylor's development with all the above coefficients, we can rewrite (13) as follows

$$\begin{aligned}
\hat{x}_t^\theta &= \int_0^t\int_0^1\mathbb{E}'\Big[f_{\tilde{x}}\Big(s,(x_s^u)'+\lambda\theta(\hat{x}_s^\theta+\xi_s)',x_s^\theta,u_s^\theta\Big)(\hat{x}_\theta(s))'\Big]d\lambda ds\\
&+\int_0^t\int_0^1\mathbb{E}'\Big[f_x\Big(s,(x_s^u)',x_s^u+\lambda\theta(\hat{x}_s^\theta+\xi_s),u_s^\theta\Big)(\hat{x}_s^\theta)\Big]d\lambda ds\\
&+\int_0^t\int_0^1\mathbb{E}'\Big[\sigma_{\tilde{x}}\Big(s,(x_s^u)'+\lambda\theta(\hat{x}_s^\theta+\xi_s)',x_s^\theta,u_s^\theta\Big)(\hat{x}_s^\theta)'\Big]d\lambda dB_s\\
&+\int_0^t\int_0^1\mathbb{E}'\Big[\sigma_x\Big(s,(x_s^u)',x_s^u+\lambda\theta(\hat{x}_s^\theta+\xi_s),u_s^\theta\Big)(\hat{x}_s^\theta)\Big]d\lambda dB_s\\
&+\int_0^t\int_0^1\mathbb{E}'\Big[c_{\tilde{x}}\Big(s,(x_{-s}^u)'+\lambda\theta(\hat{x}_{-s}^\theta+\xi_{-s})',x_{-s}^\theta,u_s^\theta,e\Big)(\hat{x}_{-s}^\theta)'\Big]d\lambda\widetilde{N}(de,ds)\\
&+\int_0^t\int_0^1\int_E\mathbb{E}'\Big[c_x\Big(s,(x_{-s}^u)',x_{-s}^u+\lambda\theta(\hat{x}_{-s}^\theta+\xi_{-s}),u_s^\theta,e\Big)(\hat{x}_{-s}^\theta)\Big]d\lambda\widetilde{N}(de,ds)\\
&+A_t^{1,\theta}+A_t^{2,\theta}+A_t^{3,\theta},
\end{aligned}$$

where

$$
\begin{aligned}
A_t^{1,\theta} &= \int_0^t \int_0^1 \mathbb{E}'\Big[f_{\tilde{x}}\Big(s,(x_s^u)' + \lambda\theta(\hat{x}_s^\theta + \xi_s)', x_s^\theta, u_s^\theta\Big) \\
&\quad - (f_{\tilde{x}}(s,(x_s)', x_s, u_s)(\xi_s)'\Big] d\lambda ds \\
&\quad + \int_0^t \int_0^1 \mathbb{E}'\Big[f_x\Big(s,(x_s^u)', x_s^u + \lambda\theta(\hat{x}_s^\theta + \xi_s), u_s^\theta\Big) \\
&\quad - f_x(s,(x_s)', x_s, u_s)(\xi_s)\Big] d\lambda ds \\
&\quad + \int_0^t \int_0^1 \mathbb{E}'\Big[f_v\Big(s,(x_s^u)', x_s^u, \lambda\theta v_s\Big) - f_v(s,(x_s)', x_s, u_s)\Big] v_s d\lambda ds,
\end{aligned}
$$

$$
\begin{aligned}
A_t^{2,\theta} &= \int_0^t \int_0^1 \mathbb{E}'\Big[\sigma_{\tilde{x}}\Big(s,(x_s^u)' + \lambda\theta(\hat{x}_s^\theta + \xi_s)', x_s^\theta, u_s^\theta\Big) \\
&\quad - (\sigma_{\tilde{x}}(s,(x_s)', x_s, u_s)(\xi_s)'\Big] d\lambda dB_s \\
&\quad + \int_0^t \int_0^1 \mathbb{E}'\Big[\sigma_x\Big(s,(x_s^u)', x_s^u + \lambda\theta(\hat{x}_s^\theta + \xi_s), u_s^\theta\Big) \\
&\quad - \sigma_x\Big(s,(x_s^u)', x_s^u + \lambda\theta(\hat{x}_s^\theta + \xi_s), u_s^\theta\Big)\xi_s\Big] d\lambda dB_s \\
&\quad \int_0^t \int_0^1 \mathbb{E}'\Big[\sigma_v\Big(s,(x_s^u)', x_s^u, \lambda\theta v_s\Big) - \sigma_v(s,(x_s)', x_s, u_s)\Big] v_s d\lambda dB_s,
\end{aligned}
$$

$$
\begin{aligned}
A_t^{3,\theta} &= \int_0^t \int_E \int_0^1 \mathbb{E}'\Big(c_{\tilde{x}}\Big(s,(x_{-s}^u)' + \lambda\theta(\hat{x}_{-s}^\theta + \xi_{-s})', x_{-s}^\theta, u_s^\theta, e\Big) \\
&\quad - (c_{\tilde{x}}(s,(x_{-s})', x_{-s}, u_s, e)(\xi_{-s})'\Big) d\lambda \widetilde{N}(de, ds) \\
&\quad + \int_0^t \int_E \int_0^1 \mathbb{E}'\Big[c_x\Big(s,(x_{-s}^u)', x_{-s}^u + \lambda\theta(\hat{x}_{-s}^\theta + \xi_{-s}), u_s^\theta, e\Big) \\
&\quad - c_x(s,(x_{-s})', x_{-s}, u_{-s}, e)\xi_{-s}\Big] d\lambda \widetilde{N}(de, ds) \\
&\quad + \int_0^t \int_E \int_0^1 \mathbb{E}'\Big[c_v\Big(s,(x_{-s}^u)', x_{-s}^u, \lambda\theta v_s, e\Big) \\
&\quad - c_v(s,(x_{-s})', x_{-s}, u_s, e)\Big] v_s d\lambda \widetilde{N}(de, ds).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\Bigg[\sup_{s\in[0,t]} |\hat{x}_s^\theta|^2\Bigg] \leq \\
&C\mathbb{E}\int_0^t \int_0^1 \mathbb{E}'\Big[\Big|f_{\tilde{x}}\Big(s,(x_s^u)' + \lambda\theta(\hat{x}_s^\theta + \xi_s)', x_s^\theta, u_s^\theta\Big)(\hat{x}_\theta(s))'\Big|^2\Big] d\lambda ds \\
&+C\mathbb{E}\int_0^t \int_0^1 \mathbb{E}'\Big[\Big|\sigma_{\tilde{x}}\Big(s,(x_s^u)' + \lambda\theta(\hat{x}_s^\theta + \xi_s)', x_s^\theta, u_s^\theta\Big)(\hat{x}_s^\theta)'\Big|^2\Big] d\lambda ds
\end{aligned}
$$

$$+C\mathbb{E}\int_0^t\int_0^1\mathbb{E}'\Big[\big|\sigma_x\big(s,(x_s^u)',x_s^u+\lambda\theta(\hat{x}_s^\theta+\xi_s),u_s^\theta\big)(\hat{x}_s^\theta)\big|^2\Big]d\lambda ds$$
$$+C\mathbb{E}\int_0^t\int_0^1\mathbb{E}'\Big[\big|\sigma_x\big(s,(x_s^u)',x_s^u+\lambda\theta(\hat{x}_s^\theta+\xi_s),u_s^\theta\big)(\hat{x}_s^\theta)\big|^2\Big]d\lambda ds$$
$$+C\mathbb{E}\int_0^t\int_E\int_0^1\mathbb{E}'\Big[\big|c_{\tilde{x}}\big(s,(x_{-s}^u)'+\lambda\theta(\hat{x}_{-s}^\theta+\xi_{-s})',x_{-s}^\theta,u_s^\theta,e\big)(\hat{x}_{-s}^\theta)'\big|^2$$
$$+\big|c_x\big(s,(x_{-s}^u)',x_{-s}^u+\lambda\theta(\hat{x}_{-s}^\theta+\xi_{-s}),u_s^\theta,e\big)(\hat{x}_{-s}^\theta)\big|^2\Big]d\lambda\pi(de)ds$$
$$+C\mathbb{E}\left[\sup_{s\in[0,t]}\big|A_s^{1,\theta}\big|^2\right]+C\mathbb{E}\left[\sup_{s\in[0,t]}\big|A_s^{2,\theta}\big|^2\right]$$
$$+C\mathbb{E}\left[\sup_{s\in[0,t]}\big|A_s^{3,\theta}\big|^2\right].$$

Then from lemma 4.1 and the uniform Lipschitz continuity of $f_{\tilde{x}}(x,\tilde{x},v)$,$f_x(x,\tilde{x},v)$, $f_v(x,\tilde{x},v)$, $\sigma_{\tilde{x}}(x,\tilde{x},v)$, $\sigma_x(x,\tilde{x},v)$, $\sigma_v(x,\tilde{x},v)$, $\int_E c_{\tilde{x}}(x,\tilde{x},v,e)\pi(de)$, $\int_E c_x(x,\tilde{x},v,e)\pi(de)$ and $\int_E c_v(x,\tilde{x},v,e)\pi(de)$ with respect to $\tilde{x},\ x,\ v,$ we have

$$\lim_{\theta\to0}\mathbb{E}\left[\sup_{s\in[0,T]}\big|A_s^{1,\theta}\big|^2\right]+\lim_{\theta\to0}\mathbb{E}\left[\sup_{s\in[0,T]}\big|A_s^{2,\theta}\big|^2\right]+\lim_{\theta\to0}\mathbb{E}\left[\sup_{s\in[0,T]}\big|A_s^{3,\theta}\big|^2\right]=0.$$

On the other hand, since of all the derivatives of $f,\ \sigma$ and c are bounded, we obtain

$$\mathbb{E}\left[\sup_{s\in[0,t]}|\hat{x}_s^\theta|^2\right]\le C\int_0^t|\hat{x}_s^\theta|^2ds+C\mathbb{E}\Bigg(\left[\sup_{s\in[0,t]}\big|A_s^{1,\theta}\big|^2\right]$$
$$+\left[\sup_{s\in[0,t]}\big|A_s^{2,\theta}\big|^2\right]+\left[\sup_{s\in[0,t]}\big|A_s^{3,\theta}\big|^2\right]\Bigg).$$

Finally, applying Gronwall's lemma allows to complete the proof. Hence

$$\left.\begin{aligned}
-d\hat{y}_t^\theta &= \tfrac{1}{\theta}\Big(\mathbb{E}'\Big[\int_E(g(t,u_t^\theta)-g(t,u_t)\pi(de)\Big]dt\\
&\quad-(z_t^\theta-z_t^u)dB_t-\int_E(r_t^\theta(e)-r_t^u(e))\widetilde{N}(de,dt)\Big)\\
&\quad-\mathbb{E}'\Big[\int_E g_{\tilde{x}}(t,e)(\xi_t)'+g_x(t,e)\xi_t+g_{\tilde{y}}(t,e)(\eta_t)'+g_y(t,e)\eta_t\\
&\quad+g_{\tilde{z}}(t,e)(\zeta_t)'+g_z(t,e)\zeta_t+g_{\tilde{r}}(t,e)(\varrho_t(e))'+g_r(t,e)\varrho_t(e)\\
&\quad+g_v(t,e)v_t\pi(de)]dt-\zeta_t dB_t-\int_E\varrho_t(e)\widetilde{N}(de,dt),
\end{aligned}\right\}\tag{14}$$

and the use of the Taylor's development with all the above coefficients, allows for rewriting (14) as follows

$$
\begin{aligned}
\hat{y}_\theta(t) = & \int_t^T \mathbb{E}' \int_E \int_0^1 g_{\tilde{x}}\Big(s,(x_s)' + \lambda\theta(\hat{x}_s^\theta + \xi_s)', (y_s^\theta)', (z_s^\theta)' \\
& ,(r_s^\theta(e))', x_s^\theta, y_s^\theta, z_s^\theta, r_s^\theta(e), u_s^\theta\Big)(\hat{x}_s^\theta)' d\lambda \pi(de) ds \\
& + \int_t^T \mathbb{E}' \int_E \int_0^1 g_{\tilde{y}}\Big(s,(x_s)', (y_s)' + \lambda\theta(\hat{y}_s^\theta + \eta_s)', (z_s^\theta)' \\
& ,(r_s^\theta(e))', x_s^\theta, y_s^\theta, z_s^\theta, r_s^\theta(e), u_s^\theta\Big)(\hat{y}_s^\theta)' d\lambda \pi(de) ds \\
& + \int_t^T \int_0^1 \mathbb{E}' \int_E g_{\tilde{z}}\Big(s,(x_s)', (y_s)', (z_s)' + \lambda\theta(\hat{z}_s + \zeta_s)' \\
& ,(r_s^\theta(e))', x_s^\theta, y_s^\theta, z_s^\theta, r_s^\theta(e), u_s^\theta\Big)(\hat{z}_s)' d\lambda \pi(de) ds \\
& + \int_t^T \int_0^1 \mathbb{E}' \int_E g_{\tilde{r}}\Big(s,(x_s)', (y_s)', (z_s)', (r_s(e))' + \lambda\theta(\hat{r}_s^\theta(e) + \varrho_s(e))' \\
& ,x_s^\theta, y_s^\theta, z_s^\theta, r_s^\theta(e), u_s^\theta)(\hat{r}_s(e))' d\lambda \pi(de) ds \\
& + \int_t^T \mathbb{E}' \int_E \int_0^1 \Big(g_x(s,(x_s)', (y_s)', (z_s)', (r_s(e))', x_s + \lambda\theta(\hat{x}_s^\theta + \xi_s) \\
& ,y_s^\theta, z_s^\theta, r_s^\theta(e), u_s^\theta)(\hat{x}_s^\theta) d\lambda \pi(de) ds \\
& + \int_t^T \int_0^1 \mathbb{E}' \int_E \Big(g_y(s,(x_s)', (y_s)', (z_s)', (r_s(e))', x_s \\
& ,y_s + (\hat{y}_s^\theta + \eta_s), z_s^\theta, r_s^\theta(e), u_s^\theta)(\hat{y}_s^\theta) d\lambda \pi(de) ds \\
& + \int_t^T \int_0^1 \mathbb{E}' \int_E \Big(g_z(s,(x_s)', (y_s)', (z_s)', (r_s(e))', x_s \\
& ,y_s, z_s + (\hat{z}_t^\theta + \zeta_t), r_s^\theta(e), u_s^\theta, s))(\hat{z}_t^\theta) d\lambda \pi(de) ds \\
& + \int_t^T \int_0^1 \mathbb{E}' \int_E \Big(g_r(s,(x_s)', (y_s)', (z_s)', (r_s(e))', x_s \\
& ,y_s, z_s, r_s(e) + \lambda\theta(\hat{r}_s^\theta(e) + \varrho_s(e)), u_s^\theta)(\hat{r}_s^\theta(e)) d\lambda \pi(de) ds \\
& + \mathbb{E}' \int_0^1 \Big[\Phi_{\tilde{x}}\Big((x_T)' + \lambda\theta(\hat{x}_T^\theta + \xi_T)', x_T^\theta\Big)(\hat{x}_T^\theta)'\Big] d\lambda \\
& + \mathbb{E}' \int_0^1 \Big[\Phi_x\Big((x_T)', x_T + \lambda\theta(\hat{x}_T^\theta + \xi_T)\Big)\Big](\hat{x}_T^\theta) d\lambda \\
& - \int_t^T \hat{z}_t^\theta dB_s - \int_t^T \int_E \hat{r}_t(e) \widetilde{N}(de,dt) \\
& + B_\theta^{1,t} + B_\theta^{2,t} + B_\theta^{3,t} + B_\theta^{4,t} + B_\theta^{5,t} + B_\theta^{6,t} + B_\theta^{7,t} + B_\theta^{8,t} + B_\theta^{9,t} + B_\theta^{10,t} + B_\theta^{11,t},
\end{aligned}
$$

where $B_\theta^{1,t}$, $B_\theta^{2,t}$, $B_\theta^{3,t}$, $B_\theta^{4,t}$, $B_\theta^{5,t}$, $B_\theta^{6,t}$, $B_\theta^{7,t}$, $B_\theta^{8,t}$, $B_\theta^{9,t}$, $B_\theta^{10,t}$ and $B_\theta^{11,t}$ are given by

$$
\begin{aligned}
B_\theta^{5,t} = & \int_t^T \mathbb{E}' \int_E \int_0^1 \Big(g_x(s,(x_s)', (y_s)', (z_s)', (r_s(e))', x_s + \lambda\theta(\hat{x}_s^\theta) \\
& + \xi_s)\ , y_s^\theta, z_s^\theta, r_s^\theta(e), u_s^\theta) - g_x(s,e)(\xi_s) d\lambda \pi(de) ds,
\end{aligned}
$$

$$B_\theta^{6,t} = \int_t^T \mathbb{E}' \int_E \int_0^1 \Big[g_y(s, (x_s)', (y_s)', (z_s)', (r_s(e))', x_s$$
$$, y_s + (\hat{y}_s^\theta + \eta_s), z_s^\theta, r_s^\theta(e), u_s^\theta)$$
$$- g_y(s,e) \Big] (\eta_s) d\lambda \pi(de) ds,$$

$$B_\theta^{7,t} = \int_t^T \mathbb{E}' \int_E \int_0^1 \Big[g_z(s, (x_s)', (y_s)', (z_s)', (r_s(e))', x_s$$
$$, y_s, z_s + (\hat{z}_s + \zeta_s), r_s^\theta(e), u_s^\theta)$$
$$- g_z(s,e) \Big] \zeta_s d\lambda \pi(de) ds,$$

$$B_\theta^{8,t} = \int_t^T \mathbb{E}' \int_E \int_0^1 \Big[g_r(s, (x_s)', (y_s)', (z_s)', (r_s(e))', x_s$$
$$, y_s, z_s, r_s(e) + \lambda\theta(\hat{r}_s(e) + \varrho_s(e)), u_s^\theta)$$
$$- g_r(s,e \Big] \varrho_s(e) d\lambda \pi(de) ds,$$

$$B_\theta^{9,t} = \int_t^T \mathbb{E}' \int_E \int_0^1 \Big[g_v(s, (x_s)', (y_s)', (z_s)', (r_s(e))', x_s$$
$$, y_s, z_s, r_s(e), u_s + \lambda\theta v_s) - g_v(t,e) v_s) \Big] d\lambda \pi(de) ds,$$

$$B_\theta^{10,t} = \int_0^1 \mathbb{E}' \Big[\Phi_{\tilde{x}}\Big((x_T)' + \lambda\theta(\hat{x}_T^\theta + \xi_T)', x_T^\theta \Big) - \Phi_{\tilde{x}}\Big((x_T)', x_T \Big) \Big] (\xi_T)' d\lambda,$$

$$B_\theta^{11,t} = \mathbb{E}' \int_0^1 \Big[\Phi_x\Big((x_T)', x_T + \lambda\theta(\hat{x}_T^\theta + \xi_T) \Big) - \Phi_x\Big((x_T)', x_T \Big) \Big] (\xi_T) d\lambda.$$

By applying Itô's formula to $|\hat{y}_s^\theta|^2$, we have

$$\mathbb{E}\left[\sup_{s\in[0,t]} |\hat{y}_s^\theta|^2 \right] + \mathbb{E}\left[\int_t^T |\hat{z}_s^\theta|^2 ds \right] + \mathbb{E}\left[\int_t^T \int_E |\hat{r}_s^\theta(e)|^2 \pi(de) dt \right]$$
$$\leq C\mathbb{E} \int_t^T \int_0^1 \mathbb{E}' \int_E \Big| g_{\tilde{x}}\Big(s, (x_s)' + \lambda\theta(\hat{x}_s^\theta + \xi_s)', (y_s^\theta)', (z_s^\theta)'$$
$$(r_s^\theta(e))', x_s^\theta, y_s^\theta, z_s^\theta, r_s^\theta(e), u_s^\theta \Big) (\hat{x}_s^\theta)' \Big|^2 d\lambda \pi(de) ds$$
$$+ C\mathbb{E} \int_t^T \int_0^1 \mathbb{E}' \Big| g_{\tilde{y}}\Big(s, (x_s)', (y_s)' + \lambda\theta(\hat{y}_s^\theta + \eta_s)', (z_s^\theta)'$$
$$, (r_s^\theta(e))', x_s^\theta, y_s^\theta, z_s^\theta, r_s^\theta(e), u_s^\theta \Big) (\hat{y}_s^\theta)' \Big|^2 d\lambda \pi(de) ds$$
$$+ C\mathbb{E} \int_t^T \mathbb{E}' \int_E \int_0^1 \Big| g_x\Big(s, (x_s)', (y_s)', (z_s)', (r_s(e))', x_s + \lambda\theta(\hat{x}_s^\theta + \xi_s)$$

$$,y_s^{\theta},z_s^{\theta},r_s^{\theta}(e),u_s^{\theta})(\hat{x}_s^{\theta})|^2 d\lambda\pi(de)ds$$
$$+C\mathbb{E}\int_t^T\int_0^1\mathbb{E}'\int_E\left|g_y(s,(x_s)',(y_s)',(z_s)',(r_s(e))',x_s\right.$$
$$,y_s+(\hat{y}_s^{\theta}+\eta_s),z_s^{\theta},r_s^{\theta}(e),u_s^{\theta})(\hat{y}_s^{\theta})|^2 d\lambda\pi(de)ds$$
$$+\int_t^T\int_0^1\mathbb{E}'\int_E\left|g_z(s,(x_s)',(y_s)',(z_s)',(r_s(e))',x_s\right.$$
$$,y_s,z_s+(\hat{z}_t^{\theta}+\zeta_t),r_s^{\theta}(e),u_s^{\theta},s))(\hat{z}_t^{\theta})|^2 d\lambda\pi(de)ds$$
$$+\int_t^T\int_0^1\mathbb{E}'\int_E\left|g_r(s,(x_s)',(y_s)',(z_s)',(r_s(e))',x_s\right.$$
$$,y_s,z_s,r_s(e)+\lambda\theta(\hat{r}_s^{\theta}(e)+\varrho_s(e)),u_s^{\theta})(\hat{r}_s^{\theta}(e))|^2 d\lambda\pi(de)ds$$
$$+C\mathbb{E}\left[\mathbb{E}'\int_0^1\left[\left|\Phi_{\tilde{x}}\left((x_T)'+\lambda\theta(\hat{x}_T^{\theta}+\xi_T)',x_T^{\theta}\right)(\hat{x}_T^{\theta})'\right|^2\right]d\lambda\right]$$
$$+C\mathbb{E}\left[\mathbb{E}'\int_0^1\left[\left[\Phi_x\left((x_T)',x_T+\lambda\theta(\hat{x}_T^{\theta}+\xi_T)\right)\right](\hat{x}_T^{\theta})\right|^2 d\lambda\right]$$
$$+CE\sum_{i=1}^{11}\left[\sup_{s\in[0,t]}\left|B_s^{i,\theta}\right|^2\right].$$

Since all the derivatives of g are bounded and Lipschitz, then from lemma 4.1 and by application of Gronwall's lemma we obtain the last convergence relations. ■

Lemma 4.3. *Let u be an optimal control and x_t^u be the corresponding optimal trajectory. Then, for any $v\in U_{ad}$, we have*

$$0\le\mathbb{E}\left[\mathbb{E}'\left[h_{\tilde{x}}\left((x_T)',x_T\right)(\xi_T)'\right]+\left[h_x\left((x_T)',x_T\right)(\xi_T)\right]\right]$$
$$+\mathbb{E}\left[\mathbb{E}'\left[\gamma_{\tilde{y}}\left((y_0^v)',y_0^v\right)(\eta_0)'+\gamma_y\left((y_0^v)',y_0^v\right)\eta_0\right]\right]$$
$$+\mathbb{E}\int_0^T\mathbb{E}'\left[\int_E l_{\tilde{x}}(t)(\xi_t)'+l_{\tilde{y}}(t)(\eta_t)'+l_{\tilde{z}}(t)(\zeta_t)'+l_{\tilde{r}}(t)(\varrho_t(e))'\right.$$
$$\left.+l_x(t)\xi_t+l_y(t)\eta_t+l_z(t)\zeta_t+l_r(t)\varrho_t(e)+l_v(t)v_t\right]\pi(de)dt. \tag{15}$$

Proof. Since u is an optimal control, we have

$$0\le\left[J(\ u(.)\ +\theta v(.)\ )-J(u(.))\right]$$
$$=\mathbb{E}\int_0^T\left[\mathbb{E}'\left[\int_E l(t,u_t^{\theta},e)-\int_E l(t,u_t,e)\right]\pi(de)\right]dt$$
$$+\mathbb{E}\left[\mathbb{E}'\left[h\left((x_T^{\theta})',x_T^{\theta}\right)-h\left((x_T)',x_T\right)\right]\right]+\mathbb{E}\left[\mathbb{E}'\left[\gamma\left((y_0^{\theta})',y_0^{\theta}\right)-\gamma\left((y_0)',y_0\right)\right]\right]. \tag{16}$$

By employing Taylor's development with all the above coefficients, we can rewrite (16) as follows

$$0 \leq \mathbb{E}\Big[\, \mathbb{E}' \int_0^1 \Big[h_{\tilde{x}}\Big((x_T)' + \lambda\theta(\hat{x}_T^\theta + \xi_T)', x_T^\theta \Big) (\xi_T)' \Big] d\lambda \,\Big]$$
$$+ \mathbb{E}\Big[\, \mathbb{E}' \int_0^1 \Big[h_x\Big((x_T)', x_T + \lambda\theta\hat{x}_T^\theta + \xi_T \Big) \Big] (\xi_T) d\lambda \,\Big]$$
$$+ \mathbb{E}\Big[\, \mathbb{E}' \int_0^1 \Big[\gamma_{\tilde{x}}\Big((y_0)' + \lambda\theta(\hat{y}_0^\theta + \eta_0)', y_0^\theta \Big) (\eta_0)' \Big] d\lambda \,\Big]$$
$$+ \mathbb{E}\Big[\, \mathbb{E}' \int_0^1 \Big[\gamma_x\Big((y_0)', y_0 + \lambda\theta\hat{y}_0^\theta + \eta_0 \Big) \eta_0 \Big] d\lambda \,\Big]$$
$$+ \int_0^T \mathbb{E}' \int_E \int_0^1 l_{\tilde{x}}\Big(t, (x_t)' + \lambda\theta(\hat{x}_t^\theta + \xi_t)', (y_t^\theta)', (z_t^\theta)'$$
$$(r_t^\theta(e))', x_t^\theta, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta \Big) (\xi_t)' d\lambda \pi(de) dt$$
$$+ \int_0^T \mathbb{E}' \int_E \int_0^1 l_{\tilde{y}}\Big(t, (x_t)', (y_t)' + \lambda\theta(\hat{y}_t^\theta + \eta_t)', (z_t^\theta)'$$
$$(r_t^\theta(e))', x_t^\theta, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta \Big) (\eta_t)' d\lambda \pi(de) dt$$
$$+ \int_0^T \mathbb{E}' \int_E \int_0^1 l_{\tilde{z}}\Big(t, (x_t)', (y_t)', (z_t)' + \lambda\theta(\hat{z}_t^\theta + \zeta_t)'$$
$$(r_t^\theta(e))', x_t^\theta, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta \Big) (\zeta_t)' d\lambda \pi(de) dt$$
$$+ \int_0^T \mathbb{E}' \int_E \int_0^1 l_{\tilde{r}}\Big(t, (x_t)', (y_t)', (z_t)'$$
$$+ (r_t(e))' + \lambda\theta(\hat{r}_t^\theta(e) + \varrho_t(e))', x_t^\theta, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta \Big) (\varrho_t(e))' d\lambda \pi(de) dt$$
$$+ \int_0^T \mathbb{E}' \int_E \int_0^1 \Big(l_x(t, (x_t)', (y_t)', (z_t)', (r_t(e))', x_t + \lambda\theta(\hat{x}_t^\theta + \xi_t)$$
$$, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta)(\xi_t) d\lambda \pi(de) dt$$
$$+ \int_0^T \mathbb{E}' \int_E \int_0^1 \Big(l_y(t, (x_t)', (y_t)', (z_t)', (r_t(e))', x_t$$
$$, y_t + (\hat{y}_t^\theta + \eta_t), z_t^\theta, r_t^\theta(e), u_t^\theta))(\eta_t) d\lambda \pi(de) dt$$
$$+ \int_0^T \mathbb{E}' \int_E \int_0^1 \Big(l_z(t, (x_t)', (y_t)', (z_t)', (r_t(e))', x_t$$
$$, y_t, z_t + (\hat{z}_t^\theta + \zeta_t), r_t^\theta(e), u_t^\theta, t))(\zeta_t) d\lambda \pi(de) dt$$
$$+ \int_0^T \mathbb{E}' \int_E \int_0^1 \Big(l_r(t, (x_t)', (y_t)', (z_t)', (r_t(e))', x_t$$
$$, y_t, z_t, r_t(e) + \lambda\theta(\hat{r}_t^\theta(e) + \varrho_t(e)), u_t^\theta))(\varrho_t(e)) d\lambda \pi(de) dt$$
$$+ \int_0^T \mathbb{E}' \int_E \int_0^1 \Big(l_v(t, (x_t)', (y_t)', (z_t)', (r_t(e))', x_t$$
$$, y_t, z_t, r_t(e), u_t + \lambda\theta v_t, t))(v_t) d\lambda \pi(de) dt + I_\theta(t),$$

where $I_\theta(t)$ is given by

$$
\begin{aligned}
I_\theta(t) &= \mathbb{E}\Big[\mathbb{E}'\int_0^1\Big[h_{\tilde{x}}\Big((x_T)' + \lambda\theta(\hat{x}_T^\theta + \xi_T)', x_T^\theta\Big)(\hat{x}_T^\theta)'\Big]d\lambda\Big] \\
&+ \mathbb{E}\Big[\mathbb{E}'\int_0^1\Big[h_x\Big((x_T)', x_T + \lambda\theta\hat{x}_T^\theta + \xi_T\Big)\Big](\hat{x}_T^\theta)d\lambda\Big] \\
&+ \mathbb{E}\Big[\mathbb{E}'\int_0^1\Big[\gamma_{\tilde{x}}\Big((y_0)' + \lambda\theta(\hat{y}_0^\theta + \eta_0)', y_0^\theta\Big)(\hat{y}_0^\theta)'\Big]d\lambda\Big] \\
&+ \mathbb{E}\Big[\mathbb{E}'\int_0^1\Big[\gamma_x\Big((y_0)', y_0 + \lambda\theta(\hat{y}_0^\theta + \eta_0)\Big)(\hat{y}_0^\theta)\Big]d\lambda\Big] \\
&+ \int_0^T\mathbb{E}'\int_E\int_0^1 l_{\tilde{x}}\Big(t, (x_t)' + \lambda\theta(\hat{x}_t^\theta + \xi_t)', (y_t^\theta)', (z_t^\theta)' \\
&(r_t^\theta(e))', x_t^\theta, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta\Big)(\hat{x}_t^\theta)' d\lambda\pi(de)dt \\
&+ \int_0^T\mathbb{E}'\int_E\int_0^1 l_{\tilde{y}}\Big(t, (x_t)', (y_t)' + \lambda\theta(\hat{y}_t^\theta + \eta_t)', (z_t^\theta)' \\
&(r_t^\theta(e))', x_t^\theta, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta\Big)(\hat{y}_t^\theta)' d\lambda\pi(de)dt \\
&+ \int_0^T\mathbb{E}'\int_E\int_0^1 l_{\tilde{z}}\Big(t, (x_t)', (y_t)', (z_t)' + \lambda\theta(\hat{z}_t^\theta + \zeta_t)' \\
&(r_t^\theta(e))', x_t^\theta, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta\Big)(\hat{z}_t^\theta)' d\lambda\pi(de)dt \\
&+ \int_0^T\mathbb{E}'\int_E\int_0^1 l_{\tilde{r}}\Big(t, (x_t)', (y_t)', (z_t)' \\
&(r_t(e))' + \lambda\theta(\hat{r}_t^\theta(e) + \varrho_t(e))', x_t^\theta, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta\Big)(\hat{r}_t^\theta(e))' d\lambda\pi(de)dt \\
&+ \int_0^T\mathbb{E}'\int_E\int_0^1\Big(l_x(t, (x_t)', (y_t)', (z_t)', (r_t(e))', x_t + \lambda\theta(\hat{x}_t^\theta + \xi_t) \\
&, y_t^\theta, z_t^\theta, r_t^\theta(e), u_t^\theta))(\hat{x}_t^\theta)d\lambda\pi(de)dt \\
&+ \int_0^T\mathbb{E}'\int_E\int_0^1\Big(l_y(t, (x_t)', (y_t)', (z_t)', (r_t(e))', x_t \\
&, y_t + (\hat{y}_t^\theta + \eta_t), z_t^\theta, r_t^\theta(e), u_t^\theta))(\hat{y}_t^\theta)d\lambda\pi(de)dt \\
&+ \int_0^T\mathbb{E}'\int_E\int_0^1\Big(l_z(t, (x_t)', (y_t)', (z_t)', (r_t(e))', x_t \\
&, y_t, z_t + (\hat{z}_t^\theta + \zeta_t), r_t^\theta(e), u_t^\theta))(\hat{z}_t^\theta)d\lambda\pi(de)dt \\
&+ \int_0^T\mathbb{E}'\int_E\int_0^1\Big(l_r(t, (x_t)', (y_t)', (z_t)', (r_t(e))', x_t \\
&, y_t, z_t, r_t(e) + \lambda\theta(\hat{r}_t^\theta(e) + \varrho_t(e)), u_t^\theta))(\hat{r}_t^\theta(e))d\lambda\pi(de)dt.
\end{aligned}
$$

From the fact that (12) are proved in this lemma, and since all the derivatives of h, γ, l are bounded, we have $\lim_{\theta\to 0} I_\theta(t) = 0$, and $\lim_{\theta\to 0} u_\theta(t) = u(t)$. Then from the continuity of all the derivative of Φ, γ and l, we obtain the required result. ■

5. Adjoint Equation

In this subsection we introduce the adjoint process, essential for deriving the maximum principle of optimality. Consider the following adjoint equations

$$\left\{\begin{aligned}
dp_t = {} & \mathbb{E}'\int_E\big[g_y(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)p_t \\
& -l_y(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)\big]\pi(de)dt \\
& +\mathbb{E}'\int_E\big[g_z(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)p_t \\
& -l_z(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)\big]\pi(de)dB_t \\
& +\mathbb{E}'\int_E\big[g_r\big(t,(x_{-t}^u)',(y_{-t}^u)',(z_{-t}^u)',(r_{-t}^u(e))',x_{-t}^u,y_{-t}^u,z_{-t}^u,r_{-t}^u(e),u_t\big)p_{-t} \\
& -l_r\big(t,(x_{-t}^u)',(y_{-t}^u)',(z_{-t}^u)',r_{-t}^u(e)',x_{-t}^u,y_{-t}^u,z_{-t}^u,r_{-t}^u(e),u_t\big)\big]\widetilde{N}(de,dt) \\
& +\mathbb{E}'\int_E\big[g_{\tilde{y}}\big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',u_t\big)(p_t)' \\
& -l_{\tilde{y}}\big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',u_t\big)\big]\pi(de)dt \\
& +\mathbb{E}'\int_E\big[g_{\tilde{z}}\big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x^u(t))',(y^u(t))',(z^u(t))',(r_t^u(e))',u_t\big)(p_t)' \\
& +\mathbb{E}'\int_E\big[g_{\tilde{r}}\big(t,x_{-t}^u,y_{-t}^u,z_{-t}^u,r_{-t}^u(e),(x_{-t}^u)',(y_{-t}^u)',(z_{-t}^u)',(r_{-t}^u(e))',u_t\big)(p_{-t})' \\
& -l_{\tilde{r}}\big(t,x_{-t}^u,y_{-t}^u,z_{-t}^u,r_{-t}^u(e),(x_{-t}^u)',(y_{-t}^u)',(z_{-t}^u)',(r_{-t}^u(e))',u_t\big)\big]\widetilde{N}(de,dt), \\
p_0 = {} & -\mathbb{E}'\big[\gamma_{\tilde{y}}\big(y_0^u,(y_0^u)'\big)+\gamma_y\big((y_0^u)',y_0^u\big)\big],
\end{aligned}\right. \tag{17}$$

$$\left\{\begin{aligned}
-dq(t) = {} & \mathbb{E}'\big[f_x\big(t,(x_t^u)',x_t^u,u_t\big)q_t+\sigma_x\big(t,(x_t^u)',x_t^u,u_t\big)k_t \\
& +\int_E c_x\big(t,(x_{-t})',x_{-t},u_t,e\big)R_t(e)\pi(de) \\
& -\int_E g_x\big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t\big)p_t\pi(de) \\
& +\int_E l_x\big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t\big)\pi(de) \\
& +f_{\tilde{x}}\big(t,x_t,(x_t)',u_t\big)(q_t)'+\sigma_{\tilde{x}}\big(t,x_t,(x_t)',u_t\big)(k_t)' \\
& -\int_E g_{\tilde{x}}\big(t,x_t,y_t,z_t,r_t(e),(x_t)',(y_t)',(z_t)',(r_t(e))',u_t\big)(p_t)'\pi(de) \\
& +\int_E l_{\tilde{x}}\big(t,x_t,y_t,z_t,r_t(e),(x_t)',(y_t)',(z_t)',(r_t(e))',u_t\big)\pi(de) \\
& +\int_E c_{\tilde{x}}\big(t,(x_{-t})',x_{-t},u_t,e\big)(R_t(e))'\pi(de)\big]dt \\
& -k_t dB_t-\int_E R_t(e)\widetilde{N}(de,dt),
\end{aligned}\right. \tag{18}$$

$$\begin{aligned} q_T &= \mathbb{E}'\big[h_{\tilde{x}}((x_T)', x_T) + h_x(x_T, (x_T)')\big] \\ &-\Big(\mathbb{E}'\big[\Phi_{\tilde{x}}\big((x_T)', x_T\big)\big]p_T + \big[\Phi_x\big(x_T, (x_T)'\big)\big](p_T)'\Big). \end{aligned}$$

We define the Hamiltonian function H as follows

$$\begin{aligned} H(t,\tilde{x},\tilde{y},\tilde{z},\tilde{r}(.),x,y,z,r(.),p,q,k,R(.),v) &= qf(t,\tilde{x},x,v) + k\sigma(t,\tilde{x},x,v) \\ &- \int_E [pg(t,\tilde{x},\tilde{y},\tilde{z},\tilde{r}(.),x,y,z,r(.),v) \\ &+ R(e)c(t,\tilde{x},x,v,e) + l(t,\tilde{x},\tilde{y},\tilde{z},\tilde{r},x,y,z,r,v)\big]\pi(de), \end{aligned}$$

to state the main result of this paper.

Theorem 5.1. *(Stochastic maximum principle) Assume that (H) holds. Let u be an optimal control and* $(x(.),y(.),z(.),r(.,.))$ *be the corresponding optimal trajectory. Then we have*

$$\mathbb{E}\int_0^T \mathbb{E}'[H_v(t)v(t)]dt \geq 0. \tag{19}$$

Proof. By applying Itô's formula to $\xi_t q_t$ and $\eta_t p_t,$ and by taking the expectation, we obtain

$$\begin{aligned} \mathbb{E}[(\xi_T q_T)] &= \mathbb{E}\big[\xi_T\big[\mathbb{E}'\big[h_{\tilde{x}}((x_T)', x_T) + h_x(x_T, (x_T)')\big] \\ &+\Big(-\mathbb{E}'\big[\Phi_{\tilde{x}}\big((x_T)', x_T\big)\big]p_T + \big[\Phi_x\big(x_T, (x_T)'\big)\big](p_T)'\Big)\big]\big] \\ &= \mathbb{E}\int_0^T E[\xi_t d(q_t) + q_t d(\xi_t) + d(\xi_t)d(q_t)] \\ &= \mathbb{E}\int_0^T -\xi_t \mathbb{E}'\big[f_x\big(t,(x_t^u)', x_t^u, u_t\big)q_t + \sigma_x\big(t,(x_t^u)', x_t^u, u_t\big)k_t \\ &-\int_E g_x\big(t,(x_t^u)', (y_t^u)', (z_t^u)', (r_t^u(e))', x_t^u, y_t^u, z_t^u, r_t^u(e), u_t\big)p_t \\ &+\int_E l_x\big(t,(x_t^u)', (y_t^u)', (z_t^u)', (r_t^u(e))', x_t^u, y_t^u, z_t^u, r_t^u(e), u_t\big) \\ &-\int_E g_{\tilde{x}}(t, x_t^u, y_t^u, z_t^u, r_t^u(e), u_t)(p_t)' \\ &+\int_E l_{\tilde{x}}\big(x_t^u, y_t^u, z_t^u, r_t^u(e), (x_t^u)', (y_t^u)', (z_t^u)', (r_t^u(e))', u_t\big)\Big]\pi(de)dt \\ &+\mathbb{E}\int_0^T q_t \mathbb{E}'\big[(f_{\tilde{x}}(t,(x_t)', x_t, u_t)(\xi_t)' \\ &+ f_x(t,(x_t)', x_t, u_t)\xi_t + f_v(t,(x_t)', x_t, u_t)v_t)\big]dt \end{aligned}$$

$$\begin{aligned} &+ \mathbb{E}\int_0^T \mathbb{E}'\big[(\sigma_{\tilde{x}}(t,(x_t)', x_t, u_t)(\xi_t)' \\ &\qquad + \sigma_x(t,(x_t)', x_t, u_t)\xi_t + \sigma_v(t,(x_t)', x_t, u_t)v_t)\big]k(t)dt \end{aligned}$$

$$+\mathbb{E}\int_0^T \mathbb{E}'\Big[\int_E c_{\tilde{x}}\big(t,(x_{-t})',x_{-t},u_t,e\big)(\xi_{-t})' +$$
$$c_x\big(t,(x_{-t})',x_{-t},u_t,e\big)\xi_{-t} + c_v\big(t,(x_{-t})',x_{-t},u_t,e\big)v_t\Big]\pi(de)dt,$$

and

$$\begin{aligned}
\mathbb{E}[\eta_T p_T] &= \mathbb{E}[\eta_0 p_0] + \mathbb{E}\int_0^T \eta_t dp_t + \mathbb{E}\int_0^T p_t d\eta_t + \mathbb{E}\int_0^T dp_t d\eta_t \\
&= \mathbb{E}\Big[-\eta_0 \mathbb{E}'\Big[\gamma_{\tilde{y}}\big((y_0^v)',y_0^v\big) + \gamma_y\big((y_0^v)',y_0^v\big)\Big]\Big] \\
&+\mathbb{E}\int_0^T \eta_t \mathbb{E}'\Big[\int_E g_y\big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))' \\
&,x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)p_t - l_y\big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))' \\
&,x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)]\pi(de)dt \\
&+\mathbb{E}\int_0^T \eta_t \mathbb{E}'\Big[\int_E g_{\tilde{y}}\big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',u_t\big)(p_t)' \\
&-l_{\tilde{y}}\big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',u_t\big)\Big]\pi(de)dt \\
&+\mathbb{E}\int_0^T -p_t\Big[\mathbb{E}'\Big[(\int_E g_{\tilde{x}}(t,e)(\xi_t)' + g_x(t,e)\xi_t + g_{\tilde{y}}(t,e)(\eta_t)' + g_y(t,e)\eta_t \\
&+g_{\tilde{z}}(t,e)(\zeta_t)' + g_z(t,e)\zeta_t + g_{\tilde{r}}(t,e)(\varrho_t(e))' + g_r(t,e)\varrho_t(e) \\
&+g_v(t,e)v_t)]\pi(de)dt \\
&+\mathbb{E}\int_0^T \zeta_t \mathbb{E}'\Big[\int_E g_z\big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u \\
&,y_t^u,z_t^u,r_t^u(e),u_t)(p_t) - l_y\big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))' \\
&,x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)]\pi(de)dt \\
&+\mathbb{E}\int_0^T \zeta_t \mathbb{E}'\Big[\int_E g_{\tilde{z}}\big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x_t^u)' \\
&,(y_t^u)',(z_t^u)',(r_t^u(e))'\big)(p_t)' - l_{\tilde{z}}(t,x_t^u,y_t^u,z_t^u,r_t^u(e), \\
&(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',u_t\big)\Big]\pi(de)dt \\
&+\mathbb{E}\int_0^T \varrho_t(e)\mathbb{E}'\Big[\int_E g_r\big(t,(x_t^u)',(y_{-t}^u)',(z_{-t}^u)',(r_{-t}^u(e))', \\
&x_t^u,y_t^u,z_t^u,r_t^u(e),u_t))p_t - l_y\big(t,(x_t^u)',(y_t^u)',(z_t^u)', \\
&r_t^u(e)',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t\big)\Big]\pi(de)dt \\
&+\mathbb{E}\int_0^T \varrho_t(e)\mathbb{E}'\Big[\int_E g_{\tilde{r}}\big(t,(x_t^u),(y_t^u),(z_t^u),(r_t^u(e)),(x_t^u)' \\
&,(y_t^u)',(z_t^u)',(r_t^u(e))',u_t)(p_t)' - l_{\tilde{r}}(t,(x_t^u),(y_t^u),(z_t^u), \\
&(r_t^u(e)),(x_t^u)',(y_{-t}^u)',(z_{-t}^u)',(r_{-t}^u(e))',u_t\big)\Big]\pi(de)dt.
\end{aligned}$$

Replacement of the above expectations with the variational inequality (15) implies that $\mathbb{E}\int_0^T \mathbb{E}'[H_v(t)v_t]dt \geq 0.$ ■

Theorem 5.2. *(Necessary conditions for the optimality of the control) Let (H) holds and H is convex with respect to v. Then, the following infinitum is necessary condition for optimality of the control* $u \in U_{ad}$:

$$\mathbb{E}'\left[H\left(t,(x_t)',(y_t)',(z_t)',(r_t(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)\right]$$
$$= \inf_{v\in U} \mathbb{E}'\left[H\left(t,(x_t)',(y_t)',(z_t)',(r_t(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),v\right)\right], \quad (20)$$

$dtdp - a.e$ *on* $[0,T]\times\Omega$, *where* $(x^u_\cdot, y^u_\cdot, z^u_\cdot, r^u_\cdot(e))$ *denotes the corresponding optimal trajectory of the control u and* $(p_t,q_t,k_t,R_t(e))$ *is the solution to mean-field FBSDE (17) and (18).*

Proof. For any $v \in U$, we have

$$\mathbb{E}'\left[H\left(t,(x^u_t)',(y^u_t)',(z^u_t)',(r^u_t(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),v\right)\right]$$
$$-\mathbb{E}'\left[H\left(t,(x^u_t)',(y^u_t)',(z^u_t)',(r^u_t(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)\right]$$
$$= \int_0^1 \mathbb{E}'\left[H_v\left(t,(x^u_t)',(y^u_t)',(z^u_t)',(r^u_t(e))'\right.\right.$$
$$\left.\left.,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t+\lambda(v-u_t)\right)(v-u_t)\right]d\lambda$$
$$= \int_0^1 \mathbb{E}'\left[H_v\left(t,(x^u_t)',(y^u_t)',(z^u_t)',(r^u_t(e))'\right.\right.$$
$$\left.,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t+\lambda(v-u_t)\right)$$
$$-H_v\left(t,(x^u_t)',(y^u_t)',(z^u_t)',(r^u_t(e))'\right.$$
$$\left.\left.,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)\right](v-u_t)d\lambda$$
$$+\int_0^1 \mathbb{E}'\left[H_v\left(t,(x^u_t)',(y^u_t)',(z^u_t)',(r^u_t(e))'\right.\right.$$
$$\left.\left.,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)\right](v-u_t)d\lambda.$$

Since H is convex with respect to v and by (19)

$$\mathbb{E}'\left[H\left(t,(x^u_t)',(y^u_t)',(z^u_t)',(r^u_t(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),v\right)\right]$$
$$-\mathbb{E}'\left[H\left(t,(x^u_t)',(y^u_t)',(z^u_t)',(r^u_t(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)\right]$$
$$\geq 0 \; dtdp - a.e. on\; [0,T]\times\Omega.$$

Then we have the desired result. ■

6. Sufficient Conditions for Optimality of the Control

Theorem 6.1. *Let* (H) *holds and suppose that the control u satisfies (20) and* $(x^u_\cdot, y^u_\cdot, z^u_\cdot, r^u_\cdot(e))$ *denotes the corresponding trajectory with* $y_T = E'\left[L_T(x_T)' + M_T(x_T)\right]$, L_T,

$M_T \in R^{m\times n}$. Let $(p_t, q_t, k_t, R_t(e))$ be the solution to mean-field FBSDE (17) and (18), assume that H, h and γ are convex with respect to $(\tilde{x},\tilde{y},\tilde{z},\tilde{r}(.),x,y,z,r(.),v)$, $(\tilde{x},x)$ and $(y,\tilde{y})$ respectively. Then u is an optimal control of the problem (8) and (9).

Proof. Let us suppose that the control process u satisfies the condition (20), then for any control $v \in U_{ad}$, we have

$$\begin{aligned} J(v)-J(u) &= \mathbb{E}\Big[\int_0^T \mathbb{E}'\int_E (l(t,v(t))-l(t,u(t)))\pi(de)dt \\ &+\mathbb{E}'\big[h((x^v(T))',x^v(t)) - h((x^u(T))',x^u(T))\big] \\ &+\mathbb{E}'\big[\gamma\big((y^v(0))',y^v(0)\big)-\gamma\big((y^u(0))',y^u(0)\big)\big]\Big]. \end{aligned} \tag{21}$$

Since h is convex with respect to $(\tilde{x},x)$, then

$$\begin{aligned} h((x_T^v)',x_t^v)-h((x_T^u)',x_T^u) &\geq h_{\tilde{x}}\big((x_T^u)',x_T^u\big)(x_T^v-x_T^u)' \\ &+h_x\big((x_T^u)',x_T^u\big)(x_T^v-x_T^u). \end{aligned} \tag{22}$$

Now as γ is convex with respect to $(\tilde{y},y)$, then

$$\begin{aligned} \gamma\big((y_0^v)',y_0^v\big)-\gamma\big((y_0^u)',y_0^u\big) &\geq \gamma_{\tilde{y}}\big((y_0^u)',y_0^u\big)(y_0^v-y_0^u)' \\ &+\gamma_y\big((y_0^u)',y_0^u\big)(y_0^v-y_0^u). \end{aligned} \tag{23}$$

By including (22) and (23) in (21), we obtain

$$\begin{aligned} J(v)-J(u) \geq\ & \mathbb{E}\Big[\mathbb{E}'\big[h_{\tilde{x}}\big(x_T^u,(x_T^u)'\big)+h_x\big((x_T^u)',x_T^u\big)\big](x_T^v-x_T^u) \\ &+\mathbb{E}'\big[\gamma_{\tilde{y}}\big(y_0^u,(y_0^u)'\big)+\gamma_y\big((y_0^u)',y_0^u\big)\big](y_0^v-y_0^u) \\ &+\int_0^T\mathbb{E}'\int_E(l(t,v_t)-l(t,u_t))\Big]\pi(de)dt. \end{aligned} \tag{24}$$

Application of Itô's formula to $q_t(x_t^v-x_t^u)$ and $p_t(y_t^v-y_t^u)$ and then taking the expectation leads to

$$\begin{aligned} &\mathbb{E}\Big[\mathbb{E}'\big[h_{\tilde{x}}\big(x_T^u,(x_T^u)'\big)+h_x\big((x_T^u)',x_T^u\big)\big]\Big](x_T^v-x_T^u) \\ &= \mathbb{E}[q_T](x_T^v-x_T^u)+\mathbb{E}'\big[L_Tp_T+M_T(p_T)'\big](x_T^v-x_T^u) \\ &= \mathbb{E}\int_0^T\mathbb{E}'\big[f_x\big(t,(x_t^u)',x_t^u,u_t\big)q_t+\sigma_x\big(t,(x_t^u)',x_t^u,u_t\big)k_t \\ &+\int_E c_x\big(t,(x_{-t})',x_{-t},u_t,e\big)R_t(e)\pi(de) \\ &-\int_E g_x\big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t\big)p_t\pi(de) \\ &+\int_E l_x\big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t\big)\pi(de) \\ &+f_{\tilde{x}}\big(t,x_t,(x_t)',u_t\big)(q_t)'+\sigma_{\tilde{x}}\big(t,x_t,(x_t)',u_t\big)(k_t)' \\ &+\int_E l_{\tilde{x}}\big(t,x_t,y_t,z_t,r_t(e),(x_t)',(y_t)',(z_t)',(r_t(e))',u_t\big)\pi(de) \end{aligned}$$

$$
\begin{aligned}
&+\int_E c_{\tilde{x}}\Big(t,(x_{-t})',x_{-t},u_t,e\Big)(R_t(e))'\pi(de)\Big](x_t^v-x_t^u)\\
&+q_t\mathbb{E}'\Big[f(t,(x_t^v)',x_t^v,v_t)-f(t,(x_t^u)',x_t^u,u_t)\Big]\\
&+k_t\mathbb{E}'\Big[\sigma(t,(x_t^v)',x_t^v,v_t)-\sigma(t,(x_t^u)',x_t^u,u_t)\Big]\\
&+\int_E R_t(e)\mathbb{E}'\Big[c(t,(x_t^v)',x_t^v,v_t,e)-c(t,(x_t^u)',x_t^u,u_t,e)\Big]\pi(de)dt\\
&+\mathbb{E}'\Big[L_Tp_T+M_T(p_T)'\Big](x_T^v-x_T^u)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}'\Big[\gamma_{\tilde{y}}\Big(y_0^u,(y_0^u)'\Big)+\gamma_y\Big((y_0^u)',y_0^u\Big)\Big](y_0^v-y_0^u)\Big]\\
&=-\mathbb{E}[p_0(y_0^v-y_0^u)]\\
&=-\mathbb{E}\Big[\mathbb{E}'\Big[L_Tp_T+M_T(p_T)'\Big](x_T^v-x_T^u)\Big]\\
&+\mathbb{E}\int_0^T\Big[(y_t^v-y_t^u)\mathbb{E}'\int_E\Big[g_y(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)p_t\\
&-l_y(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)\Big]\\
&+\mathbb{E}'\int_E\Big[g_{\tilde{y}}\Big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',u_t\Big)(p_t)'\\
&-l_{\tilde{y}}\Big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',u_t\Big)\Big]\pi(de)dt\\
&-p_t\mathbb{E}'\int_E(g(t,v_t)-g(t,u_t))\pi(de)dt\\
&+(z_t^v-z_t^u)\Big[\mathbb{E}'\int_E\Big[g_z(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)p_t\\
&-l_z(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t)\Big]\\
&+\mathbb{E}'\int_E\Big[g_{\tilde{z}}\Big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x^u(t))',(y^u(t))',(z^u(t))',(r_t^u(e))',u_t\Big)(p_t)'\\
&-l_{\tilde{z}}\Big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x^u(t))',(y^u(t))',(z^u(t))',(r_t^u(e))',u_t\Big)\Big]\Big]\pi(de)dt\\
&+(r_t^v(e)-r_t^u(e))\Big[\mathbb{E}'\int_E\Big[g_r\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t\Big)p_t\\
&-l_r\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',r_t^u(e)',x_t^u,y_t^u,z_t^u,r_t^u(e),u_t\Big)\Big]\\
&+\mathbb{E}'\int_E\Big[g_{\tilde{r}}\Big(t,x_t^u,y_t^u,z_t^u,r_t^u(e),(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',u_t\Big)(p_t)'\\
&-l_{\tilde{r}}\Big(t,x_{-t}^u,y_{-t}^u,z_{-t}^u,r_{-t}^u(e),(x_{-t}^u)',(y_{-t}^u)',(z_{-t}^u)',(r_{-t}^u(e))',u_t\Big)\Big]\Big]\pi(de)dt
\end{aligned}
$$

Cosideration of the two above expectations in (24) yields

$$\begin{aligned}
J(v)-J(u) \geq\ & \mathbb{E}\int_0^T \mathbb{E}'\Big[H\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),v)\\
&-H\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)]dt\\
&-\mathbb{E}\int_0^T \mathbb{E}'\Big[H_{\tilde{x}}\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(x_t^v-x_t^u)'\Big]dt\\
&-\mathbb{E}\int_0^T \mathbb{E}'\Big[H_{\tilde{y}}\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(y_t^v-y_t^u)'\Big]dt\\
&-\mathbb{E}\int_0^T \mathbb{E}'\Big[H_{\tilde{z}}\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(z_t^v-z_t^u)'\Big]dt\\
&-\mathbb{E}\int_0^T \mathbb{E}'\Big[H_{\tilde{r}}\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(r_t^v(e)-r_t^u(e))'\Big]dt\\
&-\mathbb{E}\int_0^T \mathbb{E}'\Big[H_x\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(x_t^v-x_t^u)]dt\\
&-\mathbb{E}\int_0^T \mathbb{E}'\Big[H_y\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(y_t^v-y_t^u)]dt\\
&-\mathbb{E}\int_0^T \mathbb{E}'\Big[H_z\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(z_t^v-z_t^u)]dt\\
&-\mathbb{E}\int_0^T \mathbb{E}'\Big[H_r\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\\
&,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(r_t^v(e)-r_t^u(e))]dt.
\end{aligned}$$

Moreover, by convexity of H with respect to $(\tilde{x},\tilde{y},\tilde{z},\tilde{r}(.),x,y,z,r(.),v)$, we have

$$\begin{aligned}
&H\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),v)\\
&-H\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)\\
&\geq H_{\tilde{x}}\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(x_t^v-x_t^u)'\\
&+H_{\tilde{y}}\Big(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t)(y_t^v-y_t^u)'
\end{aligned}$$

$$+H_{\tilde{z}}\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)(z_t^v-z_t^u)'$$
$$+H_{\tilde{r}}\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)(r_t^v(e)-r_t^u(e))'$$
$$+H_x\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)(x_t^v-x_t^u)$$
$$+H_y\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)(y_t^v-y_t^u)$$
$$+H_z\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)(z_t^v-z_t^u)$$
$$+H_r\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)(r_t^v(e)-r_t^u(e))$$
$$+H_v\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)(v_t-u_t). \quad (25)$$

This leads to

$$J(v)-J(u) \geq \mathbb{E}\int_0^T \mathbb{E}'\Big[H_v\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))'\right.$$
$$\left.,x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)(v_t-u_t)\Big]dt$$

Now since $\mathbb{E}'\left[H\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)\right]$ is convex, and by convex optimization, we obtain

$$\mathbb{E}'\left[H_v\left(t,(x_t^u)',(y_t^u)',(z_t^u)',(r_t^u(e))',x_t,y_t,z_t,r_t(e),p_t,q_t,k_t,R_t(e),u_t\right)(v_t-u_t)\right]$$
$$\geq 0,\ dtdp-a.e.\ \text{on } [0,T]\times\Omega.$$

Hence we may deduce that
$J(v)-J(u)\geq 0$,
for all $v(.)\in U_{ad}$, and this demonstrates that $u(.)$ is optimal. ■

7. Application: A Linear-Quadratic Control Problem

For the sake of simplicity, we restrict ourselves in this application to the one dimensional case, i.e. $n=d=m=k=1$. Then the coefficients of equations are linear with respect to all of the variables according to

$$f(t,\tilde{x},x,v) = \tilde{A}\tilde{x}+Ax+A_1v$$

$$\sigma(t,\tilde{x},x,v) = \tilde{B}\tilde{x}+Bx+B_1v$$

$$c(t,\tilde{x},x,v,e) = \tilde{C}(e)\tilde{x}+C(e)x+C_1(e)v$$

$$g(t,\tilde{x},\tilde{y},\tilde{z},x,y,z,v) = \tilde{a}\tilde{x}+\tilde{b}\tilde{y}+\tilde{c}\tilde{z}+ax+by+cz+a_1v$$

$$\Phi(x) = Nx$$

$$l(t,\widetilde{x},\widetilde{y},\widetilde{z},x,y,z,,v) = \frac{1}{2}(Rx^2+Dy^2+Hz^2+Mv^2)$$

$$h(x) = \frac{1}{2}Qx^2$$

$$\gamma(y) = \frac{1}{2}Ly^2,$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, A, B, a, b, c, A_1, B_1, a_1 are constant and $\tilde{C}(e), C(e)$ and $C_1(e)$ are functions in $L^2(E,\mathcal{B}(E),\pi;\mathbb{R}))$. Then the Forward Backward SDE writes as follows :

$$\begin{aligned}
dx_t^v &= (\tilde{A}\mathbb{E}[x_t]+Ax_t+A_1v_t)dt+(\tilde{B}\mathbb{E}[x_t]+Bx_t+B_1v_t)dB_t \\
&\quad+\int_E\Big(\tilde{C}(e)\mathbb{E}[x_{t^-}]+C(e)x_{t^-}+C_1(e)v_t\Big)\widetilde{N}(de,dt) \\
-dy_t^v &= \Big(\tilde{a}\mathbb{E}[x_t]+\tilde{b}\mathbb{E}[y_t]+\tilde{c}\mathbb{E}[z_t]+ax_t+by_t+cz_t+a_1v_t\Big)dt \\
&\quad-z_t^v dB_t-\int_E r_t(e)\widetilde{N}(de,dt), \\
x_0^v &= x_0, \\
y_T^v &= Nx_T,
\end{aligned}\tag{26}$$

$v_t\in U_{ad}$ and the cost functional is a quadratic, as

$$\begin{aligned}
J(v) &= \int_0^T \frac{1}{2}(R\mathbb{E}[x_t^2]+D\mathbb{E}[y_t^2]+H\mathbb{E}[z_t^2]+M\mathbb{E}[v_t^2])dt \\
&\quad+\frac{1}{2}Q\mathbb{E}[x_T^2]+\frac{1}{2}L\mathbb{E}[y_0^2],
\end{aligned}\tag{27}$$

where $R\geq 0$, $D\geq 0$, $M>0$, $Q\geq 0$, H, $L\geq 0$ are constants. Let u be an optimal admissible control and $(x_t^u,y_t^u,z_t^u,r_t^u)$ be the corresponding optimal trajectory. The adjoint equations are defined by

$$\left.\begin{aligned}
dp_t &= \Big(\tilde{b}\mathbb{E}[p_t]+bp_t-Dy_t\Big)dt+(\tilde{c}\mathbb{E}[p_t]+cp_t-Hp_t)dB_t \\
p_0 &= -Ly_0,
\end{aligned}\right\}\tag{28}$$

$$\left.\begin{aligned}
-dq_t &= \Big[Aq_t+Bk_t+\int_E C(e)R(t,e)\pi(de)-ap_t+Rx_t \\
&\quad+\tilde{A}\mathbb{E}[q_t]+\tilde{B}\mathbb{E}[k_t]+\int_E C(e)\mathbb{E}[R(t,e)]\pi(de)-\tilde{a}\mathbb{E}[p_t]\Big]dt \\
&\quad-k(t)dB_t-\int_E R(t,e)\widetilde{N}(de,dt). \\
q_T &= Qx_T-Np_T,
\end{aligned}\right\}\tag{29}$$

and the Hamiltonian function is defined by

$$\begin{aligned} &H(t,\tilde{x},\tilde{y},\tilde{z},,x,y,z,p,q,k,R(.),v) \\ &\quad = qf(t,\tilde{x},x,v) + k\sigma(t,\tilde{x},x,v) - pg(t,\tilde{x},\tilde{y},\tilde{z},x,y,z,v) \\ &\quad + l(t,x,y,z,v) + \int_E R(e)c(t,\tilde{x},x,v,e)\pi(de). \end{aligned} \tag{30}$$

From the necessary condition of optimality we have

$$u_t = \tfrac{1}{M}\left(-A_1 q_t - B_1 k_t + a_1 p_t - \int_E C_1(e)R(e)\pi(de)\right). \tag{31}$$

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