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# Backward Difference Recursions and Infinite Series Representations in Computational Risk Theory

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**Abstract**. *Some new developments in computational discrete probability are carried out. The obtained recursions exploit the one-to-one correspondence between the probability function and the stop-loss transform of non-negative discrete arithmetic random variables. The following applications are noticed. Beekman's convolution formula for the two-stage nested evaluation of probabilities of ultimate ruin from compound Poisson risk processes is formulated as a one-stage recursion. Some general infinite series representations for the probabilities and stop-loss transform in terms of factorial moments are obtained. Useful examples include the mixed Poisson lognormal as well as Euler and generalized Euler distributions. A convenient recursive formula to compute the stop-loss transform of a compound distribution with arbitrary claim size but with Sundt type counting distribution is also derived. As a consequence, a very simple second-order backward recursion to evaluate the stop-loss transform of an arbitrary discrete random variable with non-negative support is displayed.*

**Key words** : Recursion, Backward Difference, Stop-Loss Transform, Factorial Moment, Compound Poisson, Beekman's Convolution Formula.

**AMS Subject Classifications** : 62E15, 65C50, 62P05

## 1. Introduction

The present study reports on some new developments about recursion formulas that found their origin and main applications in computational risk theory. Some early and essential papers in this area include Panjer [25], Beekman [1], De Pril [10] and Sundt [31]. A useful introductory textbook is Panjer and Willmot [26] and a specialized monograph on this topic is Sundt and Vernic [32].

The main recurring theme here is the one-to-one correspondence between the probability function and the stop-loss transform of discrete arithmetic random variables, which is exploited to derive various recursive relationships that include first- and second-order backward differences. A more detailed account of the content follows.

Section 2 introduces the fundamental relationships that are used throughout. They include the equivalence of second-order backward stop-loss differences with probabilities (lemma 2.1 and equation (3)) as well as the equivalence of first-order backward stop-loss differences with cumulative probabilities (4). As an important application, Beekman's convolution formula to compute the probabilities of ultimate ruin for compound Poisson risk processes is analyzed in Section 3. Although quite simple, Beekman's convolution formula has the disadvantage to be a two-stage nested recursive algorithm. Indeed, besides the probabilities of ultimate ruin the recursion depends upon the integrated tail distribution of a specific random variable, which must itself be computed recursively. In theorem 3.1, this inconvenience is removed and Beekman's convolution is formulated as a one-stage recursion. Section 4 considers some general infinite series representations for the probabilities and the stop-loss transform in terms of factorial moments. These formulas are especially useful in case the latter have simple expressions. Examples include the mixed Poisson lognormal as well as Euler and generalized Euler distributions. One should note that variants of these formulas were already used by Cramér et al. [8]. Finally, Section 5 contains recursive formulas for the stop-loss transform of compound distributions. The claim sizes of these distributions are arbitrary but the counting distribution is assumed to be of the type introduced by Sundt [32]. Theorem 5.1 is a second-order backward difference formula to calculate the stop-loss transform of these distributions. Example 5.1 demonstrates its application to arbitrary discrete arithmetic distributions with non-negative support by making use of the pseudo compound Poisson representation introduced by Hürlimann in [13,14] and recently revisited in [16]. Example 5.2 is an extension of Panjer's recursion already noticed by Sundt [31].

## 2. Fundamental Relationships

Let $X$ be a discrete arithmetic random variable defined on the natural numbers with probabilities $f_n = \Pr(X = n)$, $n \geq 0$. The *stop-loss transform* is defined and denoted by $\pi_n = E[(X-n)_+]$. In actuarial mathematics it represents excess-of-loss and net stop-loss premiums, and in financial mathematics it is related to option prices. The probability generating function (pgf) is denoted by $P(z) = \sum_{n=0}^{\infty} f_n z^n$ and the stop-loss generating function (slgf) by $\Pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$. The two generating functions are linked by the following power series identity.

**Lemma 2.1**. *Let* $\mu$ *be the mean of* $X$*. Then, the pgf and the slgf are linked by the identity*

$$(1-z^2)\Pi(z) = (1-z)\mu - z(1-P(z)). \tag{1}$$

*Proof*. In the space of discrete arithmetic functions, let $e$ and $id$ be defined by $e(n) = 1$, $id(n) = n$, $n \geq 0$. Then, one has the relation

$$\pi_n = \mu \cdot e(n) - id(n) + (id * n),\ n \geq 0,$$

where $*$ denotes the convolution operator. By taking generating functions, one obtains (1). ■

A lot of results in the algebraic theory of recursions depend upon the *first* and *second-order backward difference* operators defined by

$$\nabla x_n = x_n - x_{n-1}, \quad \nabla^2 x_n = x_n - 2x_{n-1} + x_{n-2}, \quad n \ge 0,$$

for any sequence of real numbers $(x_n)_{n\ge 0}$, with the convention that $x_{-2} = x_{-1} = 0$. Using the operator $\nabla^2$, the power series development of the identity (1) reads

$$\sum_{n=0}^{\infty} (\nabla^2 \pi_n) z^n = \mu - (1+\mu)z + \sum_{n=1}^{\infty} f_{n-1} z^n, \tag{2}$$

and yields the recursive relationship

$$\nabla^2 \pi_n = f_{n-1}, n \ge 2, \quad \pi_1 = \mu - 1 + f_0, \quad \pi_0 = \mu. \tag{3}$$

The relation (3) provides a one-to-one correspondence between the probabilities and the stop-loss transform. Given the probabilities, (3) is a *second-order linear recurrence* to evaluate the stop-loss transform. In this respect, one has also the *first-order linear recurrence*

$$\nabla \pi_n = -\overline{F}_{n-1}, \; n \ge 1, \quad \pi_0 = \mu, \tag{4}$$

where $F_n = \sum_{j=0}^{\infty} f_j$ is the cumulative probability, and $\overline{F}_n = 1 - F_n$ is the tail or survival probability. Let us mention here that (3) and (4) are discrete analogues of the Breeden and Litzenberger [6] relation in financial economics, which relate stock price density, distribution function and call price (see also Talponen and Viitasaari [33]). Note that (4) can be obtained from (3) using the *addition formula* $\sum_{j=0}^{n} \nabla x_j = x_n$ by setting $x_j = \nabla \pi_j$. Closely related to the stop-loss transform is the *integrated tail distribution* defined by

$$T_n = \sum_{j=0}^{n} \overline{F}_j \Big/ \sum_{j=0}^{\infty} \overline{F}_j, \; n \ge 0. \tag{5}$$

**Lemma 2.2**. *The integrated tail distribution is given by*

$$T_n = 1 - \frac{\pi_{n+1}}{\mu}, \; n \ge 0. \tag{6}$$

*Proof*. Use (4) and the addition formula $\sum_{j=0}^{n} \nabla x_j = x_n$ to get the formula

$$\sum_{k=0}^{n} \overline{F}_k = -\sum_{k=1}^{n+1} (-\overline{F}_{k-1}) = -\left( \sum_{k=0}^{n+1} \nabla \pi_k - \nabla \pi_0 \right) = \mu - \pi_{n+1}, \; n \ge 0.$$

In particular, since $\lim_{n\to\infty} \pi_n = 0$ one has $\sum_{k=0}^{n} \overline{F}_k = \mu$. Insertion of this into (5) leads to (6). ■

Although the above material is part of folklore in the field, and the elementary proofs are straightforward, no precise reference for this could be located by the author. Several applications of these basic facts and/or its variants, in particular a wide variety of recursive formulas derived from them, are presented in the subsequent analysis. Some of them are of general interest, but their main importance lies within risk theory.

## 3. One-Stage Recursion for Probabilities of Ultimate Ruin

First, the probability of ultimate ruin must be defined. Given is a compound Poisson random process $X_t = \sum_{i=1}^{N_t} Y_i$, which describes the aggregate claims incurred up to time $t$, where the claim sizes $Y_i$ are independent and identically distributed, and independent from the Poisson claim number process $N_t$ with intensity $\lambda$.

**Definition 3.1**. Let $(S_t)_{t\geq 0}$ be the standard surplus process given by $S_t = u + ct - X_t$, where $u$ is the initial capital, and $c$ the constant premium rate per unit time. The *probability of ultimate ruin* (of ever having a negative surplus) depends on $u$ and is defined by $\psi(u) = P(S_t < 0 \text{ for some } t \geq 0)$.

By ruin theory one has $\psi(u) = 1 - P(L \leq 0)$, with $L = \max\left\{X_t - ct,\ t \geq 0\right\}$ the maximum aggregate loss associated to $X_t$. The random quantity $L$ can be shown equal to a random sum $L = L_1 + L_2 + \ldots + L_M$, where the $L_i's$ are independent and identically distributed, and independent from the geometrically distributed random variable $M$ with parameter $1-\rho$, $\rho = \lambda\mu/c$, $\mu$ the mean of the claim size $Y = Y_i$. The sequence $\Phi_m = 1 - \psi(m), m \geq 0$, of *ultimate survival probabilities* can be computed by means of Beekman's convolution formula (e.g. Beekman [1], Seah [30], Panjer and Willmot [26], pp. 371-372, Kaas [19]) as follows:

$$\Phi_m = (1-\rho) \cdot \sum_{k=0}^{\infty} \rho^k (T^{*k})_m, m \geq 0, \quad \Phi_0 = 1-\rho, \tag{7}$$

where $T^{*k}$ denotes the $k$-fold convolution of a suitable discrete arithmetic approximation to the integrated tail distribution $(T_n)_{n\geq 0}$ that belongs to $L_1 = L_i$, which has probability density $f_{L_1}(y) = (1 - P(Y = y))/\mu$. In practice, rounding down (up) the random variable $L_1$ to multiples of some sufficiently small $\delta$ yields a lower (upper) bound to (7). Denote by $L_1^{\delta}$ such an integer valued approximation to $L_1$. By the form of (7) the sequence $(\Phi_m)_{m\geq 0}$ is nothing else than a geometric compound distribution with jump distribution $(T_n)_{n\geq 0}$. The power series $\Phi(z) = \sum_{m=0}^{\infty} \Phi_m z^m$ and $T(z) = \sum_{n=0}^{\infty} T_n z^n$ satisfy the identity $\Phi(z) = (1-\rho)(1-\rho T_0) + \rho T(z)\Phi(z)$, which implies the recursion

$$(1-\rho T_0)\Phi_m = \rho \cdot \sum_{j=1}^{m} T_j \Phi_{m-j},\ m \geq 1, \quad \Phi_0 = 1-\rho. \tag{8}$$

Next, express the integrated tail distribution in terms of the stop-loss transform $\pi_n = E[(L_1^{\delta} - n)_+], n \geq 0$, with slgf $\Pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$. In the following, let $Q(z) = \sum_{n=0}^{\infty} h_n z^n$ be the pgf of the discrete arithmetic approximation $L_1^{\delta}$. The cumulative distribution of $L_1^{\delta}$ is denoted by $H_n = \sum_{j=0}^{\infty} h_j$, and the survival function by $\overline{H}_n = 1 - H_n$. Similarly to lemma 2.2 one shows that

$$T_n = 1 - \frac{\pi_{n+1}}{\upsilon}, n \geq 0,\ \upsilon \text{ the mean of } L_1^{\delta}. \tag{9}$$

A further application of (4) shows that the *integrated tail probabilities* are given by

$$t_n = \nabla T_n = \frac{\overline{H}_n}{\upsilon}, n \geq 0, \quad T_{-1} = 0. \tag{10}$$

The corresponding generating functions satisfy the power series identity

$$(1-z)\Pi(z) = \upsilon - \upsilon z(1-z)T(z), |z| < 1. \tag{11}$$

By insertion into lemma 2.1 one sees that the generating function of the integrated tail distribution and the pgf are linked through the identity

$$\upsilon(1-z)^2 T(z) = 1 - Q(z). \tag{12}$$

From this, one gets similarly to (3) and (4) the recursion relations:

$$-\upsilon \bullet \nabla^2 T_n = h_n, n \geq 2, \quad \upsilon \bullet T_1 = \overline{H}_0 + \overline{H}_1, \quad \upsilon \bullet T_0 = \overline{H}_0, \tag{13}$$

$$\upsilon \bullet \nabla T_n = H_n, n \geq 1, \quad \upsilon \bullet T_0 = \overline{H}_0. \tag{14}$$

Although quite simple, Beekman's convolution formula has the disadvantage to be a two-stage nested recursive algorithm. Indeed, before being inserted into (8), the integrated tail distribution (7) must first be evaluated using the recursion (3), which reads here as $\nabla^2 \pi_{n+1} = h_n, \; n \geq 2, \; \pi_1 = \upsilon - 1 + h_0, \; \pi_0 = \upsilon$. However, based on the identity (12), the following simpler one-stage recursion is obtained.

**Theorem 3.1**. *(One-stage Beekman recursion formula) The ultimate survival probabilities of a compound Poisson risk process* $X_t = \sum_{i=1}^{N_t} Y_i$ *approximates the following one-stage recursion*

$$(1-\rho T_0)\Phi_m = \upsilon \bullet (\Phi_{m-1} + \nabla \Phi_{m-1}) - \rho \bullet \sum_{j=1}^{m} h_j \Phi_{m-j}, \; m \geq 3, \tag{15}$$

*where the first few* $\Phi_m{}'s$ *are determined by*

$$(1-\rho T_0)\Phi_m = \upsilon \bullet (T_1\Phi_1 + T_2\Phi_0), \quad (1-\rho T_0)\Phi_1 = \rho \bullet T_1 \Phi_0,$$

$$\Phi_0 = 1-\rho, \; T_j = 1 - \frac{\pi_{j+1}}{\upsilon}, \; \nabla^2_{j+1}\pi = h_j, \; \pi_0 = \upsilon, \; j = 0,1,2. \tag{16}$$

*Proof*. From Beekman's convolution formula one knows that

$$T(z) = \frac{\Phi(z) - \Phi_0(1-\rho T_0)}{\rho\Phi(z)}.$$

By insertion into (12) one gets the power series identity

$$\rho\Phi(z) - \upsilon(1-z)^2\Phi(z) + \Phi_0(1-\rho T_0)\,\upsilon(1-z)^2 = \rho\Phi(z)Q(z).$$

Comparison of coefficients yields after rearrangement the one-stage recursion (15). ■

## 4. Factorial Moment Based Series Representations

Notations to be employed here are those of the preliminary Section 2. The existence of all factorial moments of the random variable $X$ is assumed. They are denoted by $\mu_{[k]} = P^{(k)}(1)$, where by convention $\mu_{[0]} = 1$. The starting point is lemma 2.1, that is the identity

$$(1-z^2)\Pi(z) = (1-z)\mu - z(1-P(z)). \tag{17}$$

By taking derivatives one obtains through induction the power series identities

$$(1-z^2)\Pi'(z) - 2(1-z)\Pi(z) = -(1+\mu) + P(z) + zP'(z), \tag{18}$$

$$(1-z^2)\Pi^{(k)}(z) - 2k(1-z)\Pi^{(k-1)}(z) + (k-1)k\Pi^{(k-2)}(z) = kP^{(k-1)}(z) + zP^{(k)}(z),\ k \geq 2. \tag{19}$$

**Lemma 4.1**. *The stop-loss transform satisfies the following alternating series representation*

$$\pi_n = \sum_{k=n}^{\infty}(-1)^{k-n}\binom{k}{n}(m_{[k+1]} + m_{[k+2]}),\ n \geq 0, \tag{20}$$

*where* $m_{[k]} = \mu_{[k]}/k!, k \geq 0$.

*Proof*. Setting $z = 1$ in (19) one obtains

$$\Pi^{(k)}(1) = \frac{1}{k+1}\left\{P^{(k+1)}(1) + \frac{P^{(k+2)}(1)}{k+2}\right\}, k \geq 0.$$

The result follows from the rearranged Taylor expansion

$$\Pi(z) = \sum_{k=0}^{\infty}\frac{\Pi^{(k)}(1)}{k!}(z-1)^k = \sum_{k=0}^{\infty}\frac{\Pi^{(k)}(1)}{k!}\sum_{n=0}^{k}(-1)^{k-n}\binom{k}{n}z^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}(-1)^{k-n}\binom{k}{n}\frac{\Pi^{(k)}(1)}{k!}\right)z^n .$$

■

**Remark 4.1**. The first formula in the proof contains in particular the relations

$$\Pi(1) = \sum_{n=0}^{\infty}\pi_n = \tfrac{1}{2}(\mu_1 + \mu_2),\quad \Pi'(1) = \sum_{n=0}^{\infty}n\pi_n = \tfrac{1}{6}(\mu_3 - \mu_1), \tag{21}$$

where the $\mu_k{}'s$ are the $k$-th moments about the origin. The first one is useful in the evaluation of stop-loss distances between ordered random variables (see e.g. Kaas et al. [20], Section VI.4).

As an even simpler result, we may show that the probabilities and the stop-loss transform are alternating series in the factorial moments.

**Theorem 4.1**. *(Factorial moment series representations) Suppose that all factorial moments* $\mu_{[k]}$ *exist, then one has the series representations*

$$f_n = (-1)^n\sum_{k=n}^{\infty}(-1)^k\binom{k}{n}m_{[k]} = \frac{1}{n!}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\mu_{[n+j]},\ n \geq 0, \tag{22}$$

$$\pi_n = (-1)^n\sum_{k=n}^{\infty}(-1)^k\binom{k-1}{n-1}m_{[k+1]}, n \geq 0, \tag{23}$$

*with the convention*

$$\binom{k}{-1} = 1 \ \text{if}\ k = -1 \ \text{and}\ \binom{k}{-1} = 0 \ \text{if}\ k = 0,1,2,\ldots. \tag{24}$$

*Proof*. Rearrange the Taylor expansion of $P(z)$ as in the proof of lemma 4.1 to get immediately (22). Then the expression (23) follows from (22) by applying (3) and using binomial coefficient identities. Alternatively, reorder the summands in (20) to get

$$\pi_n = m_{[n+1]} + (-1)^n \sum_{k=n+1}^{\infty} (-1)^{k-1}\{\binom{k-1}{n} - \binom{k}{n}\} m_{[k+1]}$$
$$= m_{[n+1]} - (-1)^n \sum_{k=n+1}^{\infty} (-1)^{k-1}\binom{k-1}{n-1} m_{[k+1]}.$$

Shifting the summation index yields (23). ■

**Example 4.1**. An attractive counting distribution, for which these representations are useful, is the mixed Poisson lognormal distribution discussed by Shaban in Crow and Shimizu [9], where one misses expressions for the probabilities and the pgf. Theorem 4.1 applies with the factorial moments (e.g. Hürlimann [15], Section 5)

$$\mu_{[k]} = \alpha^k \beta^{k^2} = \Pi_{j=1}^{k} \alpha \beta^{2j-1}, \quad \alpha = e^{\mu}, \quad \beta = \sqrt{e^{\sigma^2}}, \tag{25}$$

where $\mu, \sigma$ are the natural parameters of the lognormal distribution. Other examples include the Euler and generalized Euler distributions (e.g. Benkherouf and Bather [2], Ramsay [27,28], Charalambides [7], Kemp [23], Benkherouf and Alzaid [3], Janardan [17], Hürlimann [15]).

**Remark 4.2**. The alternating series (22) and (23) are kind of inversion formulas. For a discrete arithmetic distribution with finite support $\{0,1,2,\ldots,m\}$ the series terminates at $j = m$. In this special case, one finds the representation (22) in Johnson and Kotz [18], formula (1.162), p. 62. Variants of (22), including convergence questions, have been used by Cramér et al. [8].

## 5. Recursions for Stop-Loss Transforms of Compound Distributions

In the present section, let $X = \sum_{i=1}^{N} Y_i$, where the $Y_i{}'s$ are independent and identically distributed, and independent of the counting random variable $N$. The (claim size) random variable $Y = Y_i$ has pgf $Q(z) = \sum_{n=0}^{\infty} h_n z^n$ and the counting random variable $N$ has pgf $P_N(z) = \sum_{n=0}^{\infty} p_n z^n$. For simplicity, suppose that $h_0 = 0$, a convenient assumption, which can be removed if necessary. It is well-known that the random variable $X$ has pgf $P(z) = P_N(Q(z))$. The means $\mu = P'(1)$ and $\upsilon = Q'(1)$ are assumed to be finite. Moreover, it is assumed that $N$ belongs to Panjer's extended family of counting distributions first considered in Sundt [31] (see also Sundt and Vernic [32]). More precisely, $N$ is said to be of type $R_k[a;b]$, with $a = (a_1,\ldots,a_k), b = (b_1,\ldots,b_k)$ being some parameter vectors, if the following recursion holds:

$$p_n = \sum_{i=1}^{k} \left(a_i + \frac{b_i}{n}\right) p_{n-i}, \ n \geq 1, \ \ p_0 > 0, \ p_n = 0, \ n < 0. \tag{26}$$

The type $R_1[a;b]$ describes the family of distributions by Katz [22] and Panjer [25], and the type $R_2[(a,0);(b,c)]$ is studied by Schröter [29]. Letting $k \to \infty$, the type $R_\infty[a;b]$ is defined by the recursion

$$p_n = \sum_{i=1}^{\infty}\left(a_i + \frac{b_i}{n}\right)p_{n-i},\ n \ge 1,\quad p_0 > 0, p_n = 0,\ n < 0. \tag{27}$$

In particular, the class $R_\infty[0;b]$ coincides with the pseudo compound Poisson distribution introduced in Hürlimann [13,14] and recently revisited in [16].

In the following, recursions for compound distributions with counting function of the type $R_k[a;b]$ are considered. Use again the notation of section 2, let $P(z) = \sum_{n=0}^{\infty} f_n z^n$, $F_n = \sum_{j=0}^{\infty} f_j$ the cumulative probability, $\overline{F}_n = 1 - F_n$ the survival probability, and $\pi_n = E[(X-n)_+]$, $n \ge 0$, the stop-loss transform. Consider the integrated tail distribution of $X$, which by lemma 2.2 is equal to $T_n = 1 - \frac{\pi_{n+1}}{\mu}, n \ge 0$, and let $T(z) = \sum_{n=0}^{\infty} T_n z^n$ be the corresponding pgf. Similarly to (11) one has the identity $(1-z)\Pi(z) = \mu - \mu z(1-z)T(z)$, $|z| < 1$, which when inserted in lemma 2.1 yields the relationship

$$P(z) = 1 - (1-z)^2 V(z),\quad V(z) = \mu \cdot T(z). \tag{28}$$

Since $\lim_{n\to\infty} T_n = 1$, the power series $T(z)$ and $V(z)$ are divergent. Therefore, the validity of formulas derived from $V(z)$ must be analyzed carefully. However, as a rule, no mathematical objection arises in case only formal power series manipulations and subsequent comparisons of coefficients are made. This convenient procedure follows from an "old" theorem of Borel [5], p. 44, that one finds in Boas [4] as explained by Gould [12] (see also Niven [24] for a readable account). With this, it is possible to take derivatives in (28) and build the ratio of power series

$$\frac{P'(z)}{P(z)} = \frac{\left[(1-z)^2 V(z)\right]'}{(1-z)^2 V(z) - 1}. \tag{29}$$

Recall that $P(z) = P_N(Q(z))$ and $N$ is of type $R_k[a;b]$. Using (26) one obtains

$$\frac{P'(z)}{P(z)} = \frac{\sum_{i=1}^{k}\left(a_i + \frac{b_i}{n}\right)\left[Q^i(z)\right]'}{1 - \sum_{i=1}^{k} a_i Q^i(z)}. \tag{30}$$

Comparing (29) and (30) one obtains after rearrangement the identity

$$\begin{aligned}\left[(1-z)^2 V(z)\right]' &= -\sum_{i=1}^{k}\left(a_i + \frac{b_i}{n}\right)\left[Q^i(z)\right]' \\ &+ \sum_{i=1}^{k}\left(a_i + \frac{b_i}{n}\right)\left[Q^i(z)\right]'\left[(1-z)^2 V(z)\right] + \sum_{i=1}^{k} a_i Q^i(z)\left[(1-z)^2 V(z)\right]'.\end{aligned} \tag{31}$$

Comparing coefficients one obtains the recursion

$$\begin{aligned} n\nabla^2 V_n &= -\sum_{i=1}^{k}\left(a_i + \frac{b_i}{n}\right) n h_n^{*i} \\ &+ \sum_{i=1}^{k}\sum_{j=0}^{n}\left\{n a_i + \frac{b_i}{i} j\right\} h_j^{*i}(\nabla^2 V_{n-j}),\ n \ge 1.\end{aligned} \tag{32}$$

But, the multiplication of power series is associative. Therefore, if $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$, then the condition $A(z) \cdot [(1-z)^2 B(z)] = [(1-z)^2 A(z)] \cdot B(z)$ implies the relation

$$\sum_{j=0}^{n} a_j(\nabla^2 b_{n-j}) = \sum_{j=0}^{n} (\nabla^2 a_j) b_{n-j}. \tag{33}$$

Using this property, the recursion (32) can be rewritten in the synthetic form

$$n\nabla^2 V_n = -R_n^n + \sum_{j=0}^{n} (\nabla^2 R_j^n) V_{n-j},$$

$$V_1 = \mu - \pi_2 = \overline{F}_0 + \overline{F}_1, \quad V_0 = \mu - \pi_1 = 1 - f_0 = \overline{F}_0, \tag{34}$$

where use has been made of the following auxiliary numbers

$$R_j^n = \sum_{i=1}^{k} R_j^n(i), \quad R_j^n(i) = \left(na_i + \frac{b_i}{i} j\right) h_j^{*i}, \quad j = 0, 1, \ldots, n. \tag{35}$$

With the simplifying assumption $h_0 = 0$, one has $R_0^n(i) = 0, i = 1, \ldots, k$, hence $R_0^n = 0$. A recursion in terms of the stop-loss transform is obtained using the relations $V_n = \mu - \pi_{n+1}, \nabla^2 V_n = -\nabla^2 \pi_{n+1}$, and the addition formula $\sum_{j=0}^{n} \nabla x_j = x_n$.

**Theorem 5.1**. *(Second-order backward difference recursion for stop-loss transform of compound distributions) Let* $X = \sum_{i=1}^{N} Y_i$ *be a compound random variable with* $N$ *of type* $R_k[a;b]$. *The stop-loss transform* $\pi_n = E[(X-n)_+], n \geq 0$, *satisfies the recursion formula*

$$n(\nabla^2 \pi_{n+1}) = R_n^n - \mu(\nabla R_n^n) + \sum_{j=0}^{n} (\nabla^2 R_j^n) \pi_{n+1-j}, \; n \geq 1, \tag{36}$$

$$\pi_1 = \mu - 1 + f_0, \quad \pi_0 = \mu,$$

*where the sequence* $(R_j^n)$ *is defined by (35).*

**Example 5.1**. (Stop-loss transform of a discrete arithmetic distribution) The pgf $P(z) = \sum_{n=0}^{\infty} f_n z^n$, $f_0 > 0$, has the so-called pseudo compound Poisson representation $P(z) = P_N(Q(z))$ with $N$ Poisson($\lambda$) distributed, $Q(z) = \sum_{n=0}^{\infty} h_n z^n$ with $h_n = r_n/\lambda n, n \geq 1$, $\lambda = -\ln(f_0)$, and the sequence $(r_n)_{n \geq 1}$ is determined by the recursion (e.g. Hürlimann [16], Proposition 1)

$$r_n f_0 = nf_n - \sum_{j=1}^{n-1} r_j f_{n-j}, \; n \geq 1. \tag{37}$$

In particular, the distribution is compound Poisson, or equivalently infinitely divisible, if, and only if, the sequence $(r_n)_{n\geq 1}$ is non-negative (e.g. Katti [21], Feller [11]). In the Poisson case one has $k=1, a_1=0, b_1=\lambda$, in (27), hence $R_j^n = r_j = \lambda j h_j$ does not depend on $n$. In this important compound Poisson situation one has the very simple stop-loss transform recursion

$$n(\nabla^2\pi_{n+1}) = r_n - \mu(\nabla r_n) + \sum_{j=0}^{n}(\nabla^2 r_j)\pi_{n+1-j},\ n\geq 1, \qquad (38)$$
$$\pi_1 = \mu - 1 + f_0,\quad \pi_0 = \mu.$$

Similar but more complex recursions can be formulated by taking notice of the fact that discrete arithmetic distributions are not only of pseudo compound Poisson type $R_\infty[0;r]$, but also of some type $R_\infty[a;b]$ for any choice of $a$ (see Sundt [31], Section 4B).

**Example 5.2**. (Recursion for the probabilities of a compound distribution) From (30) one derives also the following recursive scheme for the probabilities of a compound distribution with $N$ of type $R_k[a;b]$:

$$nf_n = \sum_{j=1}^{n} R_j^n f_{n-j}, n\geq 1,\ f_0 = p_0. \qquad (39)$$

This generalization of the recursion by Panjer [25] is found in Sundt [31], Section 6A, Corollary 5. Clearly, for a pseudo compound Poisson distribution of the type $R_\infty[0;r]$ in Example 5.1, one recovers the recursion $nf_n = \sum_{j=1}^{n} r_j f_{n-j},\ n\geq 1,\ f_0 = p_0 = \exp(-\lambda)$.

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