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Mixed Sub-Fractional-White Heat Equation

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Abstract. *We introduce a new stochastic heat equation with a colored-white fractional noise, which behaves as a Wiener process in the spatial variable and as mixed sub-fractional Brownian motion in time. A necessary and sufficient condition for the existence of its solution is reported. We also analyze regularity properties of this equation, with respect to the temporal and spatial variables, respectively. Some fractal dimensions of the graphs and ranges of the associated sample paths are determined.*

Key words : Mixed Sub-Fractional Brownian Motion, Stochastic Heat Equation, Gaussian Noise, Fractal Dimension.

AMS Subject Classifications : Primary 60H15, 60G15, 60G17; Secondary 28A78, 28A80

1. Introduction

Recently, stochastic heat equations driven by different kinds of noise have widely been studied, and a lot of interesting results have been obtained (see, for instance, [2], [10], [12], [15]). Especially in [10], the author studied the solutions to the stochastic heat equations with fractional-white noise; that is with additive Gaussian noise that behaves as a Brownian motion with respect to the spatial variable and as fractional Brownian motion with respect to the temporal variable. These kinds of equations can be used to model a variety of physical phenomena which are subject to random perturbations. In these models, the noise is added to the partial differential equation to recover the chaotic nature of the process in question. However, there are no strict rules which decide the choice of the noise term, and the choice of a reasonable stochastic process really depends on the equation of motion in question, and on its physical meaning as far as possible.

In this paper, we will introduce a stochastic heat equation with a new noise, which behaves as a Wiener process in the spatial variable and as mixed-sub-fractional Brownian motion in time. That is why we will call this equation *Mixed-Sub-Fractional-White Heat Equation*. The concept of mixed sub-fractional Brownian motion was introduced and investigated in [18] and

[3]. It is a linear combination of a finite number of sub-fractional Brownian motions. So it is, in fact, an extension of both the sub-fractional Brownian motion and the Brownian motion; and this is an advantage of this process. It preserves many properties of the well known mixed fractional Brownian motion (see e.g. [4], [20]) but not the stationarity of the increments. It has been proven in [18] that the mixed sub-fractional Brownian motion could serve to get a good model of certain phenomena, taking not only the sign (as in the case of the fractional Brownian motion and the sub-fractional Brownian motion), but also the strength of dependence between the increments of the phenomena into account; and this is another reason making the *Mixed-Sub-Fractional-White Heat Equation* interesting to be investigated. For more information about the mixed sub-fractional Brownian motion see [18], [19] and [3]. A main purpose of this paper is to study some fine properties of the solution to the Mixed-Sub-Fractional-White Heat Equation. More precisely we will give a necessary and sufficient condition for the existence of its solution. Then we will analyze its regularity properties, with respect to the temporal and to the spatial variables. Given the importance of the fractal dimensions of subsets of $\mathbb{R}^d$ for their geometrical complexities, we will also investigate the Hausdorff and Packing dimensions of the graphs and the ranges of the pertaining solution sample paths.

The rest of this paper is organized as follows. In Section 2, we will state some necessary definitions and some important properties of the mixed-sub fractional Brownian motion, and we will present some new characteristics of this process, useful for this study.

In section 3, we introduce our new *mixed-sub-fractional Gaussian* noise and define the Wiener integral with respect to it. Then, we will introduce the Mixed-Sub-Fractional-White Heat Equation, and investigate the existence and mixed-self similarity property of its solution. The last section will be devoted to the study of the regularity of the solution to this equation, in time then in space, and to the investigation of the Hausdorff and Packing dimensions of the graphs and the ranges of the solution sample paths.

2. Mixed Sub-Fractional Brownian Motion

The sub-fractional Brownian motion (sfBm) is an extension of a Brownian motion, which was investigated in many papers (e.g. [11], [13]). It is a stochastic process $\xi^H = \{\xi_t^H; t \geq 0\}$, defined on a probability space $(\Omega, F, \mathbb{P})$ by

$$\forall t \in \mathbb{R}_+, \quad \xi_t^H = \frac{B_t^H + B_{-t}^H}{\sqrt{2}}, \tag{1}$$

where $H \in (0,1)$, and $\{B^H(t), t \in \mathbb{R}\}$ is a fractional Brownian motion (fBm) on the whole real line; i.e. B^H is a continuous and centered Gaussian process with the covariance function

$$Cov(B^H(t), B^H(s)) = 1/2\ (|\ t\ |^{2H} + |\ s\ |^{2H} - |\ t-s\ |^{2H}). \tag{2}$$

The index H is the Hurst parameter of B^H. The sfBm arises from occupation time fluctuations of branching particle systems with a Poisson initial condition [11].

For $N \in \mathbb{N} \setminus \{0\}$, $H = (H_1, H_2, .., H_N) \in (0,1)^N$ & $a = (a_1, a_2, .., a_N) \in \mathbb{R}^N \setminus \{(0, .., 0)\}$, the mixed sub-fractional Brownian motion (msfBm), of parameters N, a and H, is the process $S = \{S_t^H(N,a);\ t \geq 0\} = \{S_t^H;\ t \geq 0\}$, defined on the probability space $(\Omega, F, \mathbb{P})$ by

$$\forall t \in \mathbb{R}_+, \quad S_t^H(N,a) = \sum_{i=1}^{N} a_i \xi^{H_i}(t), \tag{3}$$

where $(\xi^{H_i})_{i\in\{1,\dots,N\}}$ is a family of independent sub-fractional Brownian motions of Hurst parameters H_i defined on $(\Omega, F, \mathbb{P})$.

If $N = 1$, and $a_1 = 1$, $S^H = \xi^H$ is a sub-fractional Brownian motion, and if $N = 1$, $H_1 = 1/2$ and $a_1 = 1$, S^H is a standard Brownian motion. So, the msfBm is more general, and mainly for that reason, (see [18]), this process is interesting to be investigated.

The msfBm has been introduced by El-Nouty and Zili in [3], in the particular case where $N = 2$ and $H_1 = \frac{1}{2}$. Then, it has been generalized and further investigated by Zili in [18]. In the following lemma, we state some properties of this process, which will be useful in this paper. For proofs of these properties and for more information on this process, the reader can see [18].

Lemma 2.1. *The msfBm satisfies the following properties:*
1. S_t^H *is a centered Gaussian process.*
2. $\forall s \in \mathbb{R}_+, \forall t \in \mathbb{R}_+,$

$$R_{a,H}(t,s) = Cov(S_t^H(a), S_s^H(a))$$
$$= \sum_{i=1}^{N} a_i^2 \{t^{2H_i} + s^{2H_i} - 1/2[(s+t)^{2H_i} + |\ t-s\ |^{2H_i}\ ]\}. \tag{4}$$

3. (*Mixed-self-similarity* property*) *For any* $h > 0$*, the processes* $\{S_{ht}^H(a)\}$ *and* $\{S_t^H(a_1 h^{H_1}, a_2 h^{H_2}, \dots, a_N h^{H_N})\}$ *have the same law.*

Let us further investigate this process, to enable a study the heat equation driven by it.

Remark 2.1. If $H_i = \frac{1}{2}$ for every non-zero parameter a_i and $i \in \{1,\dots,N\}$, the processes $S^H(a)$ and $\sqrt{\sum_{i=1}^N a_i^2}\, B_t$ have the same law, where B denotes the standard Brownian motion. In this case, the associated heat equation had earlier been investigated in many works (e.g. [10]). That is why, in this paper, we will be interested only on the case when

$$\textit{for all } i \in \{1,\dots,N\} \textit{ such that } a_i \neq 0, \textit{we have } H_i \geq 1/2\,, \tag{5}$$

and

$$\textit{we have at least } i \in \{1,\dots,N\} \textit{ such that } a_i \neq 0 \textit{ and } H_i > 1/2. \tag{6}$$

Moreover, some of our results will be true only under the condition:

$$\textit{for all } i \in \{1,\dots,N\} \textit{ such that } a_i \neq 0, \textit{we have } H_i > 1/2. \tag{7}$$

And in all the sequel of this paper we will denote

$$H_{i_0} = \min\{H_i\,; i \in \{1,\dots,N\};\ H_i > 1/2 \textit{ and } a_i \neq 0\}. \tag{8}$$

Furthermore, the following lemma gives a moving average representation of the msfBm.

Lemma 2.2. *If condition (7) holds, then for every t,*

*The notion of Mixed-self-similarity property was first introduced in 2006 by Zili [20].

$$S_t^H(a) = \sum_{i=1}^{N} a_i C(H_i)(H_i - \tfrac{1}{2}) \int_{\mathbb{R}} \int_0^t \Big[(u-s)_+^{H_i - \frac{3}{2}} + (u+s)_-^{H_i - \frac{3}{2}} \Big] du \, dW_i(s), \tag{9}$$

where W_i ; $i \in \{1, \dots, N\}$ are independent Brownian measures on $\mathbb{R}$, and

$$C(H_i) = \left(2 \int_0^\infty \Big((1+s)^{H_i - \frac{1}{2}} - s^{H_i - \frac{1}{2}} \Big)^2 ds + \tfrac{1}{2H_i} \right)^{-\frac{1}{2}}. \tag{10}$$

Proof. By the independence of the processes ξ_{H_i}; $i \in \{1, \dots, N\}$ and by the moving average representation of the sfBm given in [11] we get

$$S_t^H(a) = \sum_{i=1}^{N} C(H_i) \int_{\mathbb{R}} \Big[(t-s)_+^{H_i - \frac{1}{2}} + (t+s)_-^{H_i - \frac{1}{2}} - 2(-s)_+^{H_i - \frac{1}{2}} \Big] dW_i(s), \tag{11}$$

where W_i ; $i \in \{1, \dots, N\}$ are independent Brownian measures on $\mathbb{R}$, and $C(H_i)$ is defined by (10). Then, we easily check that

$$(t-s)_+^{H_i - \frac{1}{2}} + (t+s)_-^{H_i - \frac{1}{2}} - 2(-s)_+^{H_i - \frac{1}{2}} = (H_i - \tfrac{1}{2}) \int_0^t \Big[(u-s)_+^{H_i - \frac{3}{2}} + (u+s)_-^{H_i - \frac{3}{2}} \Big] du, \tag{12}$$

for every $H_i \in (\frac{1}{2}, 1)$ and $(t,s) \in \mathbb{R}^2$. Finally, from (11) and (12) we deduce trivially equation (9). ■

The following proposition will play a very important role in all the rest of this paper.

Proposition 2.1. *If Conditions(5) and (6) are satisfied then, there exist two positive constants C_1 and C_2 such that,*

$$C_1 \mid u - v \mid^{2H_{i_0} - 2} \leq \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} \leq C_2 \mid u - v \mid^{2H_{i_0} - 2}, \tag{13}$$

for every $u, v \in [0, T]$.

Proof. By Equation (4) we easily get

$$\frac{\partial^2 R_{a,H}(s,t)}{\partial s \partial t} = \sum_{i=1}^{N} a_i^2 H_i (2H_i - 1) [\mid t - s \mid^{2H_i - 2} - (t+s)^{2H_i - 2}]. \tag{14}$$

Hence, a simple calculation allows us to arrive at equation (13) with

$$C_1 = a_{i_0}^2 H_{i_0} (2H_{i_0} - 1) \quad and \quad C_2 = 2 \sum_{i=1}^{N} a_i^2 H_i (2H_i - 1) T^{2(H_i - H_{i_0})}.$$

3. The Heat Equation Driven by Mixed-Sub-Fractional Noise

The aim of this paper is to study the stochastic partial differential equation

$$\left. \begin{aligned} \frac{\partial u_{a,H}}{\partial t} &= \tfrac{1}{2} \Delta u_{a,H} + W_{a,H}, \; t \in [0,T], \; x \in \mathbb{R}^d, \\ u_{a,H}(.,0) &= 0, \end{aligned} \right\} \tag{15}$$

where $W_{a,H} = \{ W_{a,H}(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d) \}$ is a centered random noise with covariance given by:

$$\mathbb{E}(W_{a,H}(t,A)W_{a,H}(s,B)) \;= R_{a,H}(t,s)\lambda(A\cap B), \tag{16}$$

where λ is the Lebesgue measure, and $R_{a,H}$ is the covariance defined by (4). We will call this noise *mixed-sub-fractional-white* because it behaves as a msfBm in time and as a Wiener process (white) in space. The stochastic partial differential equation (15) will be called *mixed-sub-fractional heat equation.*

The canonical Hilbert space associated to the noise $W_{a,H}$ is defined as follows. First, consider $\mathcal{E}$ to be the set of linear combinations of elementary functions $\mathbf{1}_{[0,t]}\times A$, $t\in[0,T]$, $A\in\mathcal{B}_b(\mathbb{R}^d)$, and $\mathcal{H}_{a,H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product

$$<\mathbf{1}_{[0,t]}\times A,\mathbf{1}_{[0,s]}\times B>_{\mathcal{H}} \;:= \mathbb{E}(W_{a,H}(t,A)W_{a,H}(s,B)).$$

We have for $g,h\in\mathcal{H}_{a,H}$, smooth enough,

$$<g,h>_{\mathcal{H}_{a,H}} \;= \int_0^T\int_0^T dudv\int_{\mathbb{R}^d}dy\frac{\partial^2 R_{a,H}(u,v)}{\partial u\partial v}g(y,u)h(y,v). \tag{17}$$

By a routine extension of the construction described, in an example in [10] and [15], it is possible to define Wiener integrals with respect to the process $W_{a,H}$. This Wiener integral will act as an isometry between the Hilbert space $\mathcal{H}_{a,H}$ and $L^2(\Omega)$ defined by

$$\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi(u,y)W_{a,H}(du,dy)\int_0^T\int_{\mathbb{R}^d}\psi(u,y)W_{a,H}(du,dy)$$

$$=\int_0^T\int_0^T\mu_{a,H}(du,dv)\int_{\mathbb{R}^d}dy\varphi(u,y)\psi(v,y), \tag{18}$$

for any function φ,ψ such that

$$\int_0^T\int_0^T d\;|\;\mu_{a,H}\;|\;(u,v)\int_{\mathbb{R}^d}dy\;|\;\varphi(u,y)\;|\quad|\;\varphi(v,y)\;|<\infty\,,$$

and

$$\int_0^T\int_0^T d\;|\;\mu_{a,H}\;|\;(u,v)\int_{\mathbb{R}^d}dy\;|\;\psi(u,y)\;|\quad|\;\psi(v,y)\;|<\infty,$$

where $\mu_{a,H}$ is the measure

$$d\mu_{a,H}(u,v) = \frac{\partial^2 R_{a,H}}{\partial u\partial v}(u,v)dudv, \tag{19}$$

and $|\mu_{a,H}|$ denotes the total variation measure associated to $\mu_{a,H}$.

The following transfer formula will be needed in the sequel.

Proposition 3.1. *If Condition (7) holds, for every* $g\in H_{a,H}$ *then*

$$\int_0^T\int_{\mathbb{R}^d}g(s,y)dW_a^H(s,y) = \sum_{i=1}^N a_iC(H_i)(H_i-1/2)$$

$$\times\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\mathbf{1}_{(0,T)}(u)g(u,y)\Big[(u-s)_+^{H_i-\frac{3}{2}}+(u+s)_-^{H_i-\frac{3}{2}}\Big]du\,dW_i(s,y), \tag{20}$$

where Wi, for $i\in\{1,\ldots,N\}$, *are independent space-time white noises with covariance*

$$\mathbb{E}(W_i(s,A)W_i(t,B)) \;= (t\wedge s)\lambda(A\cap B),$$

and

$$C(H_i) = \left[2\int_0^\infty\Big((1+s)^{H_i-\frac{1}{2}}-s^{H_i-\frac{1}{2}}\Big)^2ds+\frac{1}{2H_i}\right]^{-\frac{1}{2}}. \tag{21}$$

Proof. This proposition is a straightforward consequence of the moving average expression for the msfBm (11). ■

Next let us define the mild solution of the sub-mixed heat equation (15).

Definition 3.1. If we denote by $\{u_{a,H}(t,x); t \in [0,T], x \in \mathbb{R}^d\}$, the process defined by

$$u_{a,H}(t,x) := \int_0^t \int_{\mathbb{R}^d} G(t-v, x-y) W_{a,H}(dv,dy), \tag{22}$$

where the above integral is a Wiener integral with respect to the noise $W_{a,H}$ and G is the Green kernel of the heat equation given by

$$G(t,x) = \begin{cases} (2t)^{-d/2} \exp\left(\frac{|x|^2}{2t}\right), & if \quad t > 0, x \in \mathbb{R}^d \\ 0, & if \quad t \leq 0, x \in \mathbb{R}^d, \end{cases} \tag{23}$$

then the process $u_{a,H}$ is called the *mild solution* of the stochastic heat equation (15).

In the following proposition, we will give a necessary and sufficient condition for existence of the mixed-sub-fractional heat equation mild solution.

Proposition 3.2. *If Conditions(5) and (6) are satisfied, then the solution to the mixed-sub-fractional heat equation (15) exists if and only if* $d < 4H_{i_0}$.

Proof. By (18) and (22),

$$\mathbb{E}(u_{a,H}^2(t,x)) = \int_0^t \int_0^t \mu_{a,H}(du,dv) \int_{\mathbb{R}^d} dy\, G(x-y,t,u)G(x-y,t,v). \tag{24}$$

So it follows from (19) and Proposition 2.1 that

$$C_1 I(t,x) \leq \mathbb{E}(u_{a,H}^2(t,x)) \leq C_2 I(t,x), \tag{25}$$

where

$$I(t,x) = \int_0^t \int_0^t dudv \,|\, u-v \,|^{2H_{i_0}-2} \int_{\mathbb{R}^d} dy G(x-y,t,u)G(x-y,t,v). \tag{26}$$

Using (23) and making a suitable change of variables, we obtain

$$\int_{\mathbb{R}^d} dy\, G(x-y,t,u)G(x-y,t,v) = \left(\frac{\pi}{2(2t-u-v)}\right)^{d/2}. \tag{27}$$

Hence, from (25), (26) and (27), we deduce that the solution to the mixed-sub-fractional heat equation (15) exists if and only if

$$\int_0^t \int_0^t dudv \,|\, u-v \,|^{2H_{i_0}-2} (2t-u-v)^{-d/2} < \infty. \tag{28}$$

And we easily check that (28) is true if and only if $d < 4H_{i_0}$. ■

Throughout the rest of this paper, we shall assume that

$$d < 4H_{i_0}, \tag{29}$$

and in the following proposition we will give an explicit expression for the covariance of the mild solution.

Proposition 3.3. *If Condition (7) holds, then for fixed* $x \in \mathbb{R}^d$, *and for* $s \leq t$,

$$\mathbb{E}(u_{a,H}(t,x)u_{a,H}(s,x)) = \left(\frac{\pi}{2}\right)^{d/2} \sum_{i=1}^N a_i^2 H_i (1-2H_i)$$

$$\times \int_0^t \int_0^s dudv[(u+v)^{2H_i-2} - |\ u-v\ |^{2H_i-2}\ ](t+s-u-v)^{-\frac{d}{2}}. \qquad (30)$$

Proof. This result follows from (14), (18), (27) and some simple calculus. ■

The next proposition deals with the mixed-self-similarity of the solution sample paths.

Proposition 3.4. *If Condition (7) holds, then the process* $u_{a,H} : t \mapsto u_{a,H}(t,x)$ *is mixed-self-similar of order* $H - \frac{d}{4}$. *That is, for every* $h > 0$, *the processes* $(u_{a,H}(ht,x))_{t\in\mathbb{R}_+}$ *and* $(u_{ah^{H-\frac{d}{4}},H}(t,x))_{t\in\mathbb{R}_+}$ *have the same law, where* $ah^{H-\frac{d}{4}} = (a_1 h^{H_1-\frac{d}{4}}, \ldots, a_N h^{H_N-\frac{d}{4}})$.

Proof. For fixed $h > 0$, the processes $(u_{a,H}(ht,x))_{t\in\mathbb{R}_+}$ and $(u_{ah^{H-\frac{d}{4}},H}(t,x))_{t\in\mathbb{R}_+}$ are Gaussian and centered. Moreover, by proposition 3.4,

$$\mathbb{E}(u_{a,H}(ht,x)u_{a,H}(hs,x)) = \left(\frac{\pi}{2}\right)^{d/2} \sum_{i=1}^N a_i^2 H_i(1-2H_i)$$

$$\times \int_0^{ht} \int_0^{hs} dudv[(u+v)^{2H_i-2} - |\ u-v\ |^{2H_i-2}\ ](ht+hs-(u+v))^{-\frac{d}{2}}. \qquad (31)$$

So, by the change of variables $u' = \frac{u}{h}$, $v' = \frac{v}{h}$ in the integral $dudv$ we directly get

$$\mathbb{E}(u_{a,H}(ht,x)u_{a,H}(hs,x)) = \mathbb{E}\Big(u_{ah^{H-\frac{d}{4}},H}(t,x)u_{ah^{H-\frac{d}{4}},H}(s,x)\Big).$$

This immediately implies proposition 3.4. ■

4. Regularity and Fractal Properties

4.1. Study of the regularity of the solution in time

In this paragraph our attention will focus on the behavior of the increments of the solution $u_{a,H}(t,x)$ to (15) with respect to the variable t . We will give sharp upper and lower bounds for the L^2-norm of these increments. First, let us state the following technical lemma.

Lemma 4.1. *For every* $\gamma \in (\frac{d}{4}, 1)$, *there exist two positive constants* $c_1(d,\gamma)$ *and* $c_2(d,\gamma)$, *depending only on d and* γ, *such that, for every* $s,t \in [0,T]$,

1. $\displaystyle\int_s^t \int_s^t dudv\ |\ u-v\ |^{2\gamma-2}(2t-u-v)^{-d/2} = c_1(d,\gamma)\ |\ t-s\ |^{2\gamma-d/2}.$

2. $\displaystyle\int_0^s \int_0^s dvdu\ |\ u-v\ |^{2\gamma-2}\ [(2t-u-v)^{-d/2} - 2(t+s-u-v)^{-d/2}$

$+(2s-u-v)^{-d/2}] \le c_2(d,\gamma)\ |\ t-s\ |^{2\gamma-d/2}$.

Proof. The proofs of both assertions invoke in their first stage the change of variables $u' = t-u, v' = s-v$ and then $u' = \frac{u}{t-s}$, $v' = \frac{v}{t-s}$. For the second assertion, we use also the fact that the integral

$$\int_0^\infty \int_0^\infty dudv \ \mid \ u-v \ \mid^{2\gamma-2} \ [(2+u+v)^{-d/2} - (1+u+v)^{-d/2} + (u+v)^{-d/2}]$$

is finite. ■

The main result in this paragraph is the following proposition.

Proposition 4.1. *If conditions (5) and (6) are satisfied, then there exist two positive constants C_3 and C_4 such that, for any $s,t \in [0,T]$ and for any $x \in R^d$,*

$$C_3|t-s|^{2H_{i_0}-\frac{d}{2}} \leq \mathbb{E}|u_{a,H}(t,x) - u_{a,H}(s,x)|^2 \leq C_4|t-s|^{2H_{i_0}-\frac{d}{2}}.$$

Proof. We have

$$\mathbb{E}|u_{a,H}(t,x) - u_{a,H}(s,x)|^2 = R_{u,a,H}(t,t) - 2R_{u,a,H}(t,s) + R_{u,a,H}(s,s),$$

where $R_{u,a,H}$ denotes the covariance of the process $u_{a,H}$ with respect to the time variable for fixed $x \in \mathbb{R}^d$, i.e.

$$\begin{aligned} R_{u,a,H}(t,s) &= \mathbb{E}(u_{a,H}(t,x)u_{a,H}(s,x)) \\ &= \left(\frac{\pi}{2}\right)^{d/2} \int_0^t \int_0^s dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} (t+s-u-v)^{-d/2} \end{aligned}$$

for every $s,t \in [0,T]$. So,

$$\begin{aligned} \mathbb{E}|u_{a,H}(t,x) - u_{a,H}(s,x)|^2 &= \left(\frac{\pi}{2}\right)^{d/2} \int_0^t \int_0^t dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} (2t-u-v)^{-d/2} \\ &\quad - 2\left(\frac{\pi}{2}\right)^{d/2} \int_0^t \int_0^s dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} (t+s-u-v)^{-d/2} \\ &\quad + \left(\frac{\pi}{2}\right)^{d/2} \int_0^s \int_0^s dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} (2s-u-v)^{-d/2} \end{aligned}$$

which can also be written as

$$\mathbb{E}|u_{a,H}(t,x) - u_{a,H}(s,x)|^2 = A_{a,H}(t,s) + B_{a,H}(t,s) + C_{a,H}(t,s),$$

where

$$A_{a,H}(t,s) = \left(\frac{\pi}{2}\right)^{d/2} \int_s^t \int_s^t dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} (2t-u-v)^{-d/2},$$

$$\begin{aligned} B_{a,H}(t,s) = \left(\frac{\pi}{2}\right)^{d/2} \int_0^s \int_0^s dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} [&(2t-u-v)^{-d/2} \\ &-2(t+s-u-v)^{-d/2} + (2s-u-v)^{-d/2}], \end{aligned}$$

and

$$C_{a,H}(t,s) = 2\left(\frac{\pi}{2}\right)^{d/2} \int_s^t \int_0^s dudv \frac{\partial^2 R_{H,a}(u,v)}{\partial u \partial v} [(2t-u-v)^{-d/2} - 2(t+s-u-v)^{-d/2}].$$

Since, $C_{a,H}(t,s) \leq 0$ and since

$$\frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} \leq C_2 \ \mid \ u-v \ \mid^{2H_{i_0}-2} \ ,$$

$$\mathbb{E}|u_{a,H}(t,x)-u_{a,H}(s,x)|^2$$
$$\leq c\left[\int_s^t\int_s^t dudv \mid u-v \mid^{2H_{i_0}-2} (2t-u-v)^{-d/2}+\int_0^s\int_0^s dudv \mid u-v \mid^{2H_{i_0}-2}\right.$$
$$\left.\times\left((2t-u-v)^{-d/2}-2(t+s-u-v)^{-d/2}+(2s-u-v)^{-d/2}\right)\right], \tag{32}$$

where c denotes a positive constant. Consequently, by (29), (32) and lemma 4.1 we get the upper bound.

We now prove the lower bound. The mild solution is expressible as

$$u_{a,H}(t,x) := \int_0^t\int_{\mathbb{R}^d} G(t-v,x-y)W_{a,H}(dv,dy), \tag{33}$$

So, for every $x\in\mathbb{R}^d$ and $(s,t)\in[0,T]^2$,

$$u(t,x)-u(s,x)$$
$$=\int_0^T\int_{\mathbb{R}^d}[G(t-\sigma,x-y)\mathbf{1}_{(0,t)}(\sigma)-G(s-\sigma,x-y)\mathbf{1}_{(0,s)}(\sigma)]\,dW_{a,H}(\sigma,y). \tag{34}$$

By the transfer formula (20) we get:

$$u_{a,H}(t,x)-u_{a,H}(s,x) = \sum_{i=1}^N D_i(a_i,H_i)\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_i(\sigma,y)F_i(\sigma,y), \tag{35}$$

where, for every $i\in\{1,\ldots,N\}$,
$D_i(a_i,H_i) = C(H_i)(H_i-1/2)$
and

$$F_i(\sigma,y) = \left[\int_{\mathbb{R}} duG(t-u,x-y)\mathbf{1}_{(0,t)}(u)T_i(u,\sigma)-\int_{\mathbb{R}} duG(s-u,x-y)\mathbf{1}_{(0,s)}(u)T_i(u,\sigma)\right]$$

with

$T_i(u,\sigma) = (u-\sigma)_+^{H_i-\frac{3}{2}}+(u+\sigma)_-^{H_i-\frac{3}{2}}$.

Equation (35) and the independence of the W_i's allow us to write

$$\mathbb{E}(u_{a,H}(t,x)-u_{a,H}(s,x))^2 = \sum_{i=1}^N D_i^2(a_i,H_i)\mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_i(\sigma,y)F_i(\sigma,y)\right)^2$$
$$= D_{i_0}^2(a_i,H_{i_0})\,\mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_{i_0}(\sigma,y)F_{i_0}(\sigma,y)\right)^2$$
$$+\sum_{i=1;i\neq i_0}^N D_i^2(a_i,H_i)\,\mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_i(\sigma,y)F_i(\sigma,y)\right)^2$$
$$\geq D_{i_0}^2(a_i,H_{i_0})\,\mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_{i_0}(\sigma,y)F_{i_0}(\sigma,y)\right)^2, \tag{36}$$

where the last inequality is due to the fact that

$$\sum_{i=1;i\neq i_0}^N D_i^2(a_i,H_i)\,\mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_i(\sigma,y)F_i(\sigma,y)\right)^2 \geq 0.$$

Moreover, by isometry of the Wiener process W we get

$$\mathbb{E}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_{i_0}(\sigma,y)F_{i_0}(\sigma,y)\Big)^2 = \int_{\mathbb{R}}\int_{\mathbb{R}^d} d\sigma dy F^2_{i_0}(\sigma,y)$$
$$\geq \int_s^t\int_{\mathbb{R}^d} d\sigma dy F^2_{i_0}(\sigma,y). \quad (37)$$

Since,
$T_{i_0}(\sigma,u) = (u-\sigma)^{H_{i_0}-3/2}\mathbf{1}_{u\geq\sigma} + (-u-\sigma)^{H_{i_0}-3/2}\mathbf{1}_{u\leq-\sigma}$,
we have

$$F_{i_0}(\sigma,y) = \int_0^t duG(t-u,x-y)T_{i_0}(u,\sigma) - \int_0^s duG(s-u,x-y)T_{i_0}(u,\sigma)$$
$$= \int_\sigma^t duG(t-u,x-y)(u-\sigma)^{H_{i_0}-3/2},$$

for every $\sigma \in [s,t]$. Hence,

$$\mathbb{E}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_{i_0}(\sigma,y)F_{i_0}(\sigma,y)\Big)^2$$
$$= \int_s^t\int_{\mathbb{R}^d} d\sigma dy\Big(\int_\sigma^t duG(t-u,x-y)(u-\sigma)^{H_{i_0}-3/2}\Big)^2. \quad (38)$$

So, for every, $s,t \in [0,T]$; $s \leq t$,

$$\mathbb{E}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_{i_0}(\sigma,y)F_{i_0}(\sigma,y)\Big)^2$$
$$= \int_s^t d\sigma\int_{\mathbb{R}^d} dy\int_\sigma^t\int_\sigma^t dvduG(t-u,x-y)(u-\sigma)^{H_{i_0}-3/2}$$
$$\times G(t-v,x-y)(v-\sigma)^{H_{i_0}-3/2}, \quad (39)$$
$$= \int_s^t du\int_s^t dv\int_{\mathbb{R}^d} dyG(t-u,x-y)G(t-v,x-y)$$
$$\times\int_s^{u\wedge v}(u-\sigma)_+^{H_{i_0}-3/2}(v-\sigma)_+^{H_{i_0}-3/2}d\sigma,$$

where in the last equality we have used the fact that

$$(s \leq \sigma \leq t,\ \sigma \leq u \leq t,\ \sigma \leq v \leq t) \Leftrightarrow (s \leq u \leq t,\ s \leq v \leq t,\ s \leq \sigma \leq u\wedge v).$$

Equations (39) and (27) imply that

$$\mathbb{E}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_{i_0}(\sigma,y)F_{i_0}(\sigma,y)\Big)^2 = \Big(\frac{\pi}{2}\Big)^{d/2}\int_s^t du\int_s^t dv(2t-u-v)^{-d/2}$$
$$\times\int_s^{u\wedge v}(u-\sigma)_+^{H_{i_0}-3/2}(v-\sigma)_+^{H_{i_0}-3/2}d\sigma. \quad (40)$$

By the change of variable $z = \frac{u\wedge v-\sigma}{u\vee v-\sigma}$ and by some simple calculus we get

$$\int_s^{u\wedge v}(u-\sigma)^{H_{i_0}-3/2}(v-\sigma)^{1-2H_{i_0}}d\sigma$$

$$= \mid u-v \mid^{2H_{i_0}-2} \int_s^{\frac{u\wedge v-\sigma}{u\vee v-\sigma}}(1-z)^{1-2H_{i_0}}z^{H_{i_0}-3/2}dz. \tag{41}$$

Then, by (40) and (41),

$$\mathbb{E}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_{i_0}(\sigma,y)F_{i_0}(\sigma,y)\Big)^2 = \left(\frac{\pi}{2}\right)^{d/2}\int_s^t du\int_s^t dv(2t-u-v)^{-d/2}$$

$$\times \mid u-v \mid^{2H_{i_0}-2} \int_0^{\frac{u\wedge v-\sigma}{u\vee v-\sigma}}(1-z)^{1-2H_{i_0}}z^{H_{i_0}-3/2}dz. \tag{42}$$

Now, by the change of variables $u-s=u'$ and $v-s=v'$ we obtain:

$$\mathbb{E}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_{i_0}(\sigma,y)F_{i_0}(\sigma,y)\Big)^2 = \left(\frac{\pi}{2}\right)^{d/2}\int_0^{t-s} du\int_0^{t-s} dv(2(t-s)-u-v)^{-d/2}$$

$$\times \mid u-v \mid^{2H_{i_0}-2} \int_0^{\frac{u\wedge v}{u\vee v}}(1-z)^{1-2H_{i_0}}z^{H_{i_0}-3/2}dz. \tag{43}$$

Finally, by the change of variables $\tilde{u} = \frac{u}{t-s}$ and $\tilde{v} = \frac{v}{t-s}$, it follows that

$$\mathbb{E}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW_{i_0}(\sigma,y)F_{i_0}(\sigma,y)\Big)^2 = D(d,H_{i_0})(t-s)^{2H_{i_0}-d/2}, \tag{44}$$

where $D(d,H_{i_0})$ is the constant defined by

$$D(d,H_{i_0}) = c\int_0^1 du\int_0^1 dv(2-u-v)^{-d/2} \mid u-v \mid^{2H_{i_0}-2} \int_0^{\frac{u\wedge v}{u\vee v}}(1-z)^{1-2H_{i_0}}z^{H_{i_0}-3/2}dz.$$

Since $H_{i_0} > \frac{1}{2}$, the constant $D(d,H_{i_0})$ is clearly finite. . ■

Remark 4.1. Proposition 4.1 tells us that the process $(u_{a,H}(.,x))$ is an infinite dimensional quasi-helix (in the sense of Kahane [7]) of index $I = H_{i_0} - \frac{d}{4}$. Various properties of quasi-helices are known and again we refer to [7] for more detailed information.

In particular, as an immediate consequence of proposition 4.1, the following results hold.

Corollary 4.1. *If conditions (5) and (6) are satisfied, then for any* $x \in \mathbb{R}^d$*, the process* $t \to u_{a,H}(t,x)$ *is Hölder continuous of order* $\delta \in (0, H_{i_0} - \frac{d}{4})$.

As a second consequence of proposition 4.1, by proceeding as in the proof of Proposition 3.2 in [6], we arrive at the next result.

Corollary 4.2. *If conditions (5) and (6) are satisfied, then for any* $x \in \mathbb{R}^d$,

$$\lim_{\epsilon\to 0}\sup_{t\in[t_0-\epsilon,t_0+\epsilon]}\left|\frac{u_{a,H}(t,x)-u_{a,H}(t_0,x)}{t-t_0}\right| = +\infty .$$

with probability one for every t_0*. And consequently, the trajectories of the process* $u_{a,H}(.,x)$ *are not differentiable.*

4.2. **Sharp regularity of the solution in space**

In the spirit of [14], in this section we fix $t > 0$ and analyze the space regularity of the solution $\{u_{a,H}(t,x), x \in \mathbb{R}^d\}$. We will first prove the following lemma.

Lemma 4.2. *If conditions (5) and (6) are satisfied, then the Gaussian random field $\{u_{a,H}(t,x), x \in \mathbb{R}^d\}$ is stationary with the spectral measure*

$$\Delta(d\xi) = (2\pi)^{-d} \sum_{i=1}^{N} a_i^2 H_i(2H_i - 1) \int_0^t \int_0^t dudv[(u+v)^{2H_i-2} + | \ u-v \ |^{2H_i-2}\ ]$$
$$\times \int_{\mathbb{R}^d} \exp(-\frac{(2t-u-v)}{2} \ | \ \xi \ |^2 \ )d\xi.$$

Proof. By the Fourier transform of the Green kernel and Parseval's identity we get

$$\mathbb{E}(u_{a,H}(t,x)u_{a,H}(t,y)) = \int_0^t \int_0^t dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} \int_{\mathbb{R}^d} dz G(x-z,t,u)G(y-z,t,v)$$

$$= (2\pi)^{-d} \int_0^t \int_0^t dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} \int_{\mathbb{R}^d} d\xi \ \exp(-\frac{(t-u)}{2} \ | \ \xi \ |^2 \ ) \exp(i < x,\xi >)$$
$$\times \exp(-\frac{(t-v)}{2} \ | \ \xi \ |^2 \ ) \exp(-i < y,\xi >)$$
$$= \int_{\mathbb{R}^d} \exp(i < x-y,\xi >)$$
$$\times \left[(2\pi)^{-d} \int_0^t \int_0^t dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} \exp(-\frac{(2t-u-v)}{2} \ | \ \xi \ |^2 \ ) \right] d\xi.$$

$$\mathbb{E}(u_{a,H}(t,x)u_{a,H}(t,y)) = \int_0^t \int_0^t dudv \frac{\partial^2 R_{a,H}(u,v)}{\partial u \partial v} \int_{\mathbb{R}^d} dz G(x-z,t,u)G(y-z,t,v)$$

By the expression of $\frac{\partial^2 R_{a,H}}{\partial u \partial v}$ we get the result. ■

Corollary 4.3. *If Conditions(5) and (6) are satisfied, then there exist two positive constants $c_1(t,H_{i_0})$ and $c_2(t,H_{i_0})$, depending only on t and H_{i_0}, such that:*
$c_1(t,H_{i_0}) \ | \ \xi \ |^{-4H_{i_0}} \ d\xi \leq \Delta(d\xi)) \ \leq c_2(t,H_{i_0}) \ | \ \xi \ |^{-4H_{i_0}} \ d\xi$ *for all* $\xi \in R^d$ *with* $| \ \xi \ | \geq 1$.

Proof. By equation (13), we have

$$C_1(2\pi)^{-d} \int_0^t \int_0^t \ | \ u-v \ |^{2H_{i_0}-2} \ dudv \exp(-\frac{(2t-u-v)}{2} \ | \ \xi \ |^2 \ )d\xi$$
$$\leq \ \Delta(d\xi)$$

$$\leq \ C_2(2\pi)^{-d}\int_0^t\int_0^t \ | \ u-v \ |^{2H_{i_0}-2} \ dudv\exp(-\tfrac{(2t-u-v)}{2} \ | \ \xi \ |^2 \ )d\xi,$$

and by [2] (Proposition 4.3), there exist two strictly positive constants $c_{1,H_{i_0}}, c_{2,H_{i_0}}$ such that

$$c_{1,H_{i_0}}(t^{2H_{i_0}} \wedge 1)\left(\frac{1}{1+|\xi|^2}\right)^{2H_{i_0}}$$

$$\leq \ \int_0^t\int_0^t dudv \ | \ u-v \ |^{2H_{i_0}-2} \ \exp(-\tfrac{(u+v)}{2} \ | \ \xi \ |^2 \ ) \tag{45}$$

$$\leq \ c_{2,H_{i_0}}(t^{2H_{i_0}} \wedge 1)\left(\frac{1}{1+|\xi|^2}\right)^{2H_{i_0}}.$$

This allows to see the stated result. ■

Corollary 4.3 means, among other things, that the spectral measure $\Delta(d\xi)$ is comparable with an absolutely continuous measure with density function that is comparable to $| \ \xi \ |^{-4H_{i_0}}$ for all $\xi \in \mathbb{R}^d$ with $| \ \xi \ | \geq 1$. This is quite interesting for the study of the regularity of $\{u_{a,H}(t,x), x \in \mathbb{R}^d\}$. Indeed, as a first consequence of corollary 4.3, we get the following result.

Theorem 4.1. *Let* $\beta = \min\{1, 2H_{i_0} - \frac{d}{2}\}$ *and* $\rho = \begin{cases} 1 & if \quad \beta = 1 \\ 0 & \quad otherwise. \end{cases}$ *. If conditions (5) and (6) are satisfied, then for any* $M > 0$*, there exist positive and finite constants* c_3*,* c_4 *such that for any* $x,y \in [-M,M]^d$*,*

$$c_3 \ | \ x-y \ |^{2\beta} \left(Log\frac{1}{|x-y|}\right)^{\rho} \leq \mathbb{E}(| \ u_{a,H}(t,x) - u_{a,H}(t,y) \ |^2 \ )$$

$$\leq c_4 \ | \ x-y \ |^{2\beta} \left(Log\frac{1}{|x-y|}\right)^{\rho}. \tag{46}$$

Proof. Take $x,y \in [-M,M]^d$ and let $z := y-x \in \mathbb{R}^d$. By Parseval's identity, we can write

$$\mathbb{E}(| \ u_{a,H}(t,y) - u_{a,H}(t,x) \ |^2 \ )$$

$$= \ \int_0^t\int_0^t \mu_{a,H}(du,dv)\int_{\mathbb{R}^d} dy'[G(t-u,x+z-y') - G(t-u,x-y')]$$

$$\times [G(t-v,x+z-y') - G(t-v,x-y')]$$

$$= \ (2\pi)^{-d}\int_0^t\int_0^t \mu_{a,H}(du,dv)\int_{\mathbb{R}^d} d\xi \ \mathcal{F}(G(t-u,x+z-.) \ - G(t-u,x-.))(\xi)$$

$$\times \mathcal{F}\left(G(t-v,x+z-.) - G(t-v,x-.)\right)(\xi)$$

$$= (2\pi)^{-d}\int_0^t\int_0^t \mu_{a,H}(du,dv)\int_{\mathbb{R}^d} d\xi \exp\left(-(2t-u-v)\frac{|\xi|^2}{2}\right)(2-2\cos <\xi,z>),$$

where in the last equality we have used
$\mathcal{F}G(t,x-.)(\xi) = \exp\left(i<x,\xi> - \frac{t|\xi|^2}{2}\right)\mathbf{1}_{t>0}(\xi), \xi \in \mathbb{R}^d$.

Therefore,

$$\mathbb{E}(|u_{a,H}(t,x)-u_{a,H}(t,y)|^2) = 2(2\pi)^{-d}\int_{\mathbb{R}^d} d\xi\,(1-\cos <\xi,z>)\theta(t,\xi), \tag{47}$$

where

$$\theta(t,\xi) = \int_0^t\int_0^t \frac{\partial^2 R_{a,H}}{\partial u\partial v}(u,v)dudv\exp\left(-(2t-u-v)\frac{|\xi|^2}{2}\right).$$

By Equation (13),

$$C_1\int_0^t\int_0^t |u-v|^{2H_{i_0}-2}\,dudv\exp\left(-(2t-u-v)\frac{|\xi|^2}{2}\right) \le \theta(t,\xi) \tag{48}$$

$$\le C_2\int_0^t\int_0^t |u-v|^{2H_{i_0}-2}\,dudv\exp\left(-(2t-u-v)\frac{|\xi|^2}{2}\right),$$

where C_1 and C_2 are two positive constants. Then, following the same lines as those of the proof of Theorem 4 in [14], we show that there exist two strictly positive constants C_5 and C_6 such that

$$C_5 |x-y|^{2\beta}\left(Log\frac{1}{|x-y|}\right)^{\rho}$$

$$\le \int_{\mathbb{R}^d}(1-\cos <\xi,z>)d\xi \int_0^t\int_0^t |u-v|^{2H_{i_0}-2}\,dudv\exp\left(-(2t-u-v)\frac{|\xi|^2}{2}\right)$$

$$\le C_6 |x-y|^{2\beta}\left(Log\frac{1}{|x-y|}\right)^{\rho}. \tag{49}$$

Hence, the result is a straightforward consequence of equations (47), (48) and (49). ■

As direct consequence of theorem 4.1 we obtain the next result.

Corollary 4.4. *If conditions (5) and (6) are satisfied and* $2H_{i_0} - \frac{d}{2} > 1$, *then* $\{u_{a,H}(t,x), x \in \mathbb{R}^d\}$ *has a modification (still denoted by the same notation) such that almost surely the sample function* $x \mapsto u_{a,H}(t,x)$ *is continuously differentiable on* $\mathbb{R}^d$. *Moreover, for any* $M > 0$, *there exists a positive random variable K with all moments such that for every* $j = 1,\ldots,d$, *the partial derivative* $\frac{\partial}{\partial x_j}u_{a,H}(t,x)$ *has the following modulus of continuity on* $[-M,M]^d$:

$$\sup_{x,y\in[-M,M]^d,\,|x-y|\le\epsilon}\left|\frac{\partial}{\partial x_j}u_{a,H}(t,x)-\frac{\partial}{\partial y_j}u_{a,H}(t,y)\right|\le K\epsilon^{2H_{i_0}-\frac{d}{2}-1}\sqrt{Log\frac{1}{\epsilon}}\,. \tag{50}$$

Proof. With equation (13), we can apply exactly the same steps of the proof of Theorem 5 in [4]. ■

By lemma 4.1 and equation (13), and with the results of [17], we obtain the following result that corresponds to the case where $2H_{i_0}-\frac{d}{2}<1$.

Lemma 4.3. *Suppose that conditions (5) and (6) are satisfied and $2H_{i_0}-\frac{d}{2}<1$. Then, for every fixed $t>0$, the Gaussian field $\{u_{a,H}(t,x),x\in\mathbb{R}^d\}$ is strongly locally nondeterministic. Namely, for every $M>0$, there exits a constant $C_7>0$ (depending on t and M) such that for every $n\geq 1$ and for every $x,y_1,\dots,y_n\in[-M,M]^d$,*

$$Var(u_{a,H}(t,x)|u_{a,H}(t,y_1),\dots,u_{a,H}(t,y_n))\ \geq C_7\min_{0\le j\le n}\{|\ x-y_j\ |^{4H_{i_0}-d}\},$$

where $y_0=0$.

As a consequence of this lemma, and by [[9], Theorems 4.1 and 5.1], we obtain the following uniform and local moduli of the continuity characteristic.

Corollary 4.5. *Suppose that conditions (5) and (6) are satisfied and $2H_{i_0}-\frac{d}{2}<1$. Let $t>0$ and $M>0$ be fixed. Then, if we denote $\beta=2H_{i_0}-\frac{d}{2}$, we have*

- *Almost surely*

$$\lim_{\epsilon\to 0}\frac{\max_{x\in[-M,M]^d,\,|h|\le\epsilon}|u_{a,H}(t,x+h)-u_{a,H}(t,x)|}{\epsilon^{\beta}\sqrt{LogLog(1/\epsilon)}}=C_8.$$

- *For $x_0\in\mathbb{R}^d$,*

$$\overline{\lim_{\epsilon\to 0}}\frac{\max_{|h|\le\epsilon}|u_{a,H}(t,x_0+h)-u_{a,H}(t,x_0)|}{\epsilon^{\beta}\sqrt{LogLog(1/\epsilon)}}=C_9,$$

where C_8 and C_9 are positive constants.

Now by lemma 4.3 and according to [8] we get the following Chung's LIL characteristic.

Corollary 4.6. *Suppose that conditions (5) and (6) are satisfied and $2H_{i_0}-\frac{d}{2}<1$. Then, for every $t>0$ and $x_0\in\mathbb{R}^d$,*

$$\underline{\lim}_{\epsilon\to 0}\frac{\max_{|h|\le\epsilon}|u_{a,H}(t,x_0+h)-u_{a,H}(t,x_0)|}{\epsilon^{\beta}\sqrt{LogLog(1/\epsilon)}}=C_{10},$$

where C_{10} is a positive constant.

4.3. **Fractal characteristics of the sample paths**

For fixed $x\in\mathbb{R}^d$, we denote the range of the restriction of the process $u_{a,H}(.,x)$ on $[0,T]$ by

$$u_{a,H}([0,T],x) = \{u_{a,H}(t,x); t \in [0,T]\}, \quad (51)$$

and its graph by

$$Grf_T u_{a,H}(.,x) = \{(t, u_{a,H}(t,x)); t \in [0,T]\}, \quad (52)$$

where $T > \epsilon > 0$.

The aim of this paragraph is to study Hausdorff and Packing dimensions of the sets defined just above. These dimensions have extensively been used in describing thin sets and fractals. Hence we only recall briefly their definitions. The Hausdorff dimension of a set $E \subset \mathbb{R}^d$ is defined by

$$dim_H E = \inf\{\alpha > 0; M^\alpha(E) = 0\} = \sup\{\alpha > 0; M^\alpha(E) = +\infty\}, \quad (53)$$

where, for $\alpha > 0$, $M^\alpha(E)$ denotes the $\alpha-$ dimensional Hausdorff measure of E, defined by

$$M^\alpha(E) = \lim_{\delta \to 0} \inf\left\{ \mid E \mid^\alpha ; E \subset \bigcup_{k=1}^{\infty} E_k; \mid E_k \mid < \delta \right\}, \quad (54)$$

where $\mid E_k \mid$ is the diameter of the set E_k and the infinimum is taken over all coverings $(E_k)_{k\in\mathbb{N}}$ of E.

The packing dimension of a bounded set $F \subset \mathbb{R}^d$ is defined by:

$$dim_P F = \inf\left\{ \sup_n \overline{dim_B} F_n \, : \, F \subset \bigcup_{n=1}^{\infty} F_n \right\}. \quad (55)$$

where, $\overline{dim_B} F_n$ is the *upper box-counting dimension* of F_n defined by

$$\overline{dim_B} F_n = \lim_{\epsilon \to 0} \sup \frac{log N(F_n, \epsilon)}{-log \epsilon}, \quad (56)$$

and for any $\epsilon > 0$, $N(F_n, \epsilon)$ is the smallest number of balls of radius ϵ (in Euclidean metric) needed to cover F_n.

Among the properties of such dimensions, we recall that for any bounded set $F \subset \mathbb{R}^d$,

$$dim_H F \leq dim_P F \leq \overline{dim_B} F \leq d. \quad (57)$$

For more information on Hausdorff and Packing dimensions , the reader is referred to [5].

Let us start this study by the set $Grf_T u_{a,H}(.,x)$. Throughout all the rest of this paper, c denotes a generic positive constant that may be different from line to line.

Lemma 4.4. *Suppose that conditions (5) and (6) are satisfied. For any $T > 0$, with probability 1,*

$$dim_H Grf_T u_{a,H}(.,x) = dim_P Grf_T u_{a,H}(.,x) = 2 - H_{i_0} + \tfrac{d}{4}.$$

Proof. By corollary 4.1, for any $T > 0$ and $x \in \mathbb{R}^d$, $u_{a,H}(.,x)$ has a modification whose sample-paths have a Hölder continuity, with order $\gamma < H_{i_0} - \frac{d}{4}$ on the interval $[0,T]$. So by Lemmas 2.1 and 2.2 in [16], for any $T > 0$, with probability 1,

$$dim_H Grf_T u_{a,H}(.,x) \leq 2 - H_{i_0} + \tfrac{d}{4} \quad \text{and} \quad dim_P Grf_T u_{a,H}(.,x) \leq 2 - H_{i_0} + \tfrac{d}{4}.$$

Now, in order to get the lower bound, by (57) and by the Frostman's Theorem (see e.g. [5]), we only need to show that for any $T > 0$, the occupation measure ν of $t \mapsto (t, u_{a,H}(t,x))$, when t is restricted to the interval $[0;T]$, has with probability 1, a finite γ –dimensional energy, for any $\gamma \in \left(1,\, 2 - H_{i_0} + \frac{d}{4}\right)$. More precisely, for any Borel set $A \subset \mathbb{R}^2$, $\nu(A)$ is defined as the integral

$$\nu(A) = \int_0^T \mathbf{1}_{\{(t,u_{a,H}(t,x))\in A\}} dt, \tag{58}$$

where, for every set $V \subset \mathbb{R}^2$, $\mathbf{1}_V$ denotes the characteristic function of the set V. Then we need to prove that with probability 1, the integral

$$\int_{Grf_T u_{a,H}(.,x)} \int_{Grf_T u_{a,H}(.,x)} \mid x-y \mid^{-\gamma} \nu(dx)\nu(dy), \tag{59}$$

is finite. This is easily seen by a monotone class argument, to be equivalent to

$$\int_0^T \int_0^T (\mid s-t \mid + \mid u_{a,H}(s,x) - u_{a,H}(t,x) \mid)^{-\gamma} dsdt < +\infty. \tag{60}$$

In order to obtain (60), it suffices to show that

$$\int_0^T \int_0^T \mathbb{E}((\mid s-t \mid + \mid u_{a,H}(s,x) - u_{a,H}(t,x) \mid)^{-\gamma})\, dsdt < +\infty. \tag{61}$$

Since the process $u_{a,H}(.,x)$ is centered Gaussian, we easily check that for all $(s,t) \in \mathbb{R}^2, s \neq t$ and for every real $\gamma > 1$, we have

$$\mathbb{E}((\mid s-t \mid + \mid u_{a,H}(s,x) - u_{a,H}(t,x) \mid)^{-\gamma}) \leq C_{11} \mid t-s \mid^{1-\gamma} \sigma^{-1}_{a,H,x}(s,t), \tag{62}$$

where

$$\sigma^2_{a,H,x}(s,t) = \mathbb{E}(u_{a,H}(t,x) - u_{a,H}(s,x))^2 ,$$

and C_{11} is a positive constant.

Now, by (62) and by proposition 4.1, we arrive at

$$\begin{aligned} &\int_0^T \int_0^T \mathbb{E}((\mid s-t \mid + \mid u_{a,H}(s,x) - u_{a,H}(t,x) \mid)^{-\gamma}) dsdt \\ &\leq C_{11} \int_0^T \int_0^T \mid t-s \mid^{1-\gamma} \sigma^{-1}_{a,H,x}(s,t) dsdt \\ &\leq C_{12} \int_0^T \int_0^T \mid t-s \mid^{1+\frac{d}{4}-H_{i_0}-\gamma} dsdt \end{aligned}$$

where C_{12} is a positive constant. And since $\gamma \in (1,\ 2-H_{i_0}+\frac{d}{4})$, the last double integral is finite. Here the proof completes. ■

In the following last lemma, we will give the Hausdorff and Packing dimensions of the set $u_{a,H}([0,T],x)$.

Lemma 4.5. *Suppose that conditions (5) and (6) are satisfied. For any $T > 0$, with probability* 1,

$dim_H u_{a,H}([0,T],x) = 1$ and $dim_P u_{a,H}([0,T],x) = 1$.

Proof. By Lemmas 2.1 and 2.2 in [16], we clearly have

$dim_H u_{a,H}([0,T],x) \leq 1$ and $dim_P u_{a,H}([0,T],x) \leq 1$ $a.s.$

So, by (57), we only need to prove that

$1 \leq dim_H\, u_{a,H}([0,T],x)$ $a.s.$

Next we note that for $\epsilon \in (0,T)$,

$dim u_{a,H}([0,T],x) \geq dim u_{a,H}([\epsilon,T],x)$,

and that for any standard normal variable X and $0 < \gamma < 1$, we have

$$\mathbb{E}(\mid X \mid^{-\gamma}) < \infty. \tag{63}$$

Hence by Frostman's theorem (see e.g. [5]), it is sufficient to show that for all $0 < \gamma < 1$,

$$E_\gamma = \int_\epsilon^T \int_\epsilon^T \mathbb{E}(\mid u_{a,H}(s,x) - u_{a,H}(t,x) \mid^{-\gamma})dsdt < +\infty. \quad (64)$$

By proposition 4.1 and (64), there exists a positive and finite constant C_{13} such that

$$E_\gamma \le C_{13} \int_\epsilon^T \int_\epsilon^T \mid s-t \mid^{-\gamma(H_{i_0}-\frac{d}{4})} dsdt. \quad (65)$$

Since $0 < \gamma(H_{i_0} - \frac{d}{4}) < 1$, the second member of the inequality (65) is finite, which leads to the required result. ■

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