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A Newton-Type Method With Ninth-Order Convergence for Solving Nonlinear Equations

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Abstract. *In this paper we define a new two-point Newton-type method for finding a simple root of nonlinear equations. It is proved that the new method has the convergence order of nine requiring only four function evaluations per full iteration. The Kung and Traub conjecture states that multipoint iteration methods, without memory based on n function evaluations, could achieve maximum convergence order* 2^{n-1}. *The new method produces however a convergence order of nine, which is better than the expected maximum convergence order of eight. Therefore, we show that this conjecture can fail for a particular set of nonlinear equations. The drawback of this new method is its restricted utilization to a zero simple root. It is demonstrated though that this method is very competitive with eighth-order methods.*

Key words : Newton-Type Method, Nonlinear Equations, Kung-Traub Conjecture, Maximum Order of Convergence, Efficiency Index, Two-Point Method.

AMS Subject Classifications : 65H05

1. Introduction

It is well known that one of the most important problem in science and engineering [4,6,9] is to find solutions of a nonlinear equation. In this paper, we propose a new two-point ninth-order iterative method to find a simple root α of the nonlinear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$, for an open interval D, is a scalar function. Many higher order variants of the Newton-type method have been developed based on the famous Kung and Traub conjecture [3]. For the purpose of this paper, we improve the one-point Newton-type method and construct a new two-point ninth-order iterative method for finding simple roots of nonlinear equations. This new iterative method has a better efficiency index than the eighth-order methods described in [1,6,7,8,11]. Hence, the proposed ninth-order method should be significantly better in comparison with the previously established methods. The ninth-order method presented in this paper uses only four evaluations of the function per iteration. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could possibly achieve maximum convergence order 2^{n-1}. In fact, we have obtained a higher order of convergence than the maximum order of convergence suggested by the Kung and Traub conjecture [3]. We demonstrate that the Kung and Traub conjecture fails for the particular case when the simple root of a nonlinear equation is equal to zero. Actually it is a weakness of this new, two-point Newton-type, method that it is only usable when the simple root of the nonlinear equation is zero.

The structure of this paper is as follows: Some basic definitions relevant to the present work are stated in the section 2. In section 3 the new two-point ninth-order iterative method is constructed and proved. In section 4, four well established three-point eighth-order methods are revisited to exhibit the comparative effectiveness of the new two-point ninth-order iterative method. Finally, in section 5, numerical comparisons are reported to demonstrate the good performance of the new iterative method.

2. Preliminaries

In order to discuss the order of convergence of iterative methods, some definitions need to be stated.

Definition [5] **2**. **1**. Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence p is given by

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^p} = \zeta \neq 0, \tag{1}$$

where $p \in \mathbb{R}^+$ and ζ is the asymptotic error constant. Moreover, if $e_n = x_n - \alpha$ is the error in the n th iteration, then the relation

$$e_{n+1} = \zeta e_n^p + \mathrm{O}\left(e_k^{p+1}\right), \tag{2}$$

is the error equation. If the error equation holds then p is the order of convergence of the iterative method, [2,4,5,10].

Definition [5] **2**. **2**. Let r be the number of function evaluations of the iterative method. The efficiency of the iterative method is measured via the concept of efficiency index

$$\sqrt[r]{p}\,, \tag{3}$$

where p is the order of the method.

Definition [3] **2**. **3**. (Kung and Traub conjecture) Let $x_{n+1} = g(x_n)$ define an iterative function without memory with k-evaluations. Then

$$p(g) \leq p_{opt} = 2^{k-1}, \tag{4}$$

where p_{opt} is the maximum order of convergence.

Definition [8] **2**. **4**. Suppose that $x_{n-1},\ x_n$ and x_{n+1} are three successive iterations close to the root α of (1). Then the computational order of convergence may be approximated by

$$\text{COC} \approx \frac{ln|\delta_n \div \delta_{n-1}|}{ln|\delta_{n-1} \div \delta_{n-2}|}, \tag{5}$$

where $\delta_i = f(x_i) \div f'(x_i)$.

3. Convergence Analysis

As mentioned in the introduction, in this section we define a new two-point ninth-order method for finding simple roots of a nonlinear equation. In fact, the new iterative method is an improvement of the one-point third-order Newton-type method, introduced in [9]. The one-point third-order method is used as our first step, and by simply repeating the iterative process, we achieve a ninth-order convergence. Accordingly, the two-point ninth-order Newton-type method is expressed by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - (2x_n)^{-1}\left[x_n^2 - \left(\frac{f(x_n)}{f'(x_n)}\right)^2\right], \tag{6}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - (2y_n)^{-1}\left[y_n^2 - \left(\frac{f(y_n)}{f'(y_n)}\right)^2\right], \tag{7}$$

where x_0 is the initial guess. Obviously, it is assumed that the denominators of (6)-(7) are not equal to zero, and the first-step (6) is one-point third-order method [9] mentioned earlier. Then by repeating the process at an improved point, the second step is the ninth-order method.

Now, we shall verify the convergence property of the new two-point ninth-order iterative method (7).

Theorem 3. 1. *Let $\alpha \in D$ be a simple zero of a sufficiently smooth function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D. If the initial guess x_0 is sufficiently close to α, then the convergence order of the new two-point iterative method defined by (7) is nine.*

Proof. Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ with $f'(\alpha) \neq 0$, and let the errors be expressed as $e = x - \alpha$ and $\hat{e} = y - \alpha$.
Using the Taylor series expansion, we have

$$f(x_n) = f(\alpha) + f'(\alpha)e_n + 2^{-1}f''(\alpha)e_n^2 + 6^{-1}f'''(\alpha)e_n^3 + 24^{-1}f^{iv}(\alpha)e_n^4 + \cdots. \tag{8}$$

Taking $f(\alpha) = 0$ into account and by further simplification we obtain

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots], \tag{9}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots], \tag{10}$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{k!\,f'(\alpha)} k \geq 2. \tag{11}$$

Division of (9) by (10) leads to

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + \cdots, \tag{12}$$

and

$$\left(\frac{f(x_n)}{f'(x_n)}\right)^2 = e_n^2 - 2c_2e_n^3 + (5c_2^2 - 4c_3)e_n^4 + \cdots. \tag{13}$$

Substitution of (12) and (13) in (6) results with

$$y_n - \alpha = e_n - \frac{f(x_n)}{f'(x_n)} - (2e_n)^{-1}\left[e_n^2 - \left(\frac{f(x_n)}{f'(x_n)}\right)^2\right]. \tag{14}$$

This readily reduces to a third-order error equation

$$\hat{e}_n = y_n - \alpha = 2^{-1}c_2^2e_n^3 - 2c_2(c_2^2 - c_3)e_n^4 + \cdots. \tag{15}$$

Now, Expansion of $f(y_n)$ about α yields

$$f(y_n) = f'(\alpha)[\hat{e}_n + c_2\hat{e}_n^2 + c_3\hat{e}_n^3 + \cdots], \tag{16}$$

which in view of (15) is the same as

$$f(y_n) = f'(\alpha)[2^{-1}c_2^2e_n^3 - 2c_2(c_2^2 - c_3)e_n^4 + \cdots]. \tag{17}$$

Furthermore, by expanding $f'(y_n)$ about α we obtain

$$f'(y_n) = f'(\alpha)[1 + 2c_2\hat{e}_n + 3c_3\hat{e}_n^2 + \cdots]. \tag{18}$$

Similarly, we get

$$f'(y_n) = f'(\alpha)[1 + 2c_2^3e_n^3 - 4c_2^2(c_2^2 - c_3)e_n^4 + \cdots\cdots]. \tag{19}$$

Division of (17) by (19) yields

$$\frac{f(y_n)}{f'(y_n)} = 2^{-1}c_2^2e_n^3 - 2c_2(c_2^2 - c_3)e_n^4 + \cdots. \tag{20}$$

with

$$\left(\frac{f(y_n)}{f'(y_n)}\right)^2 = 2^{-2}c_2^4e_n^6 - 2c_2^3(c_2^2 - c_3)e_n^7 + \cdots. \tag{21}$$

Substitution of (20) and (21) in (7) leads to

$$e_{n+1} = e_n - \frac{f(y_n)}{f'(y_n)} - (2e_n)^{-1}\left[e_n^2 - \left(\frac{f(y_n)}{f'(y_n)}\right)^2\right], \tag{22}$$

which in view of (13) becomes

$$e_{n+1} = 2^{-4}c_2^8e_n^9 + \mathrm{O}(e_n^{10}), \tag{23}$$

indicating that the order of convergence of our Newton-type method defined by (7) is at least nine. This completes the proof. ■

4. Alternative Established Methods

For the purpose of comparison, a two-point fourth-order Newton method and four three-point eighth-order methods presented in [2,5,7,10] are revisited. Since these methods are well established, we state the essential formulas used in them to calculate the simple root of nonlinear equations in order to compare their effectiveness with the effectiveness of the new one-point ninth-order method.

The classical two-point fourth-order Newton method is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{24}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \tag{25}$$

Dzunic et al. [1] developed the family of eighth-order Newton-type method, given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{26}$$

$$z_n = y_n - (1-2t)^{-1}\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{27}$$

$$x_{n+1} = z_n - \left[\frac{(1+v)(1+2w)}{(1-2t-t^2)}\right]\left(\frac{f(z_n)}{f(x_n)}\right), \tag{28}$$

where

$$t = \frac{f(y_n)}{f(x_n)}, \quad v = \frac{f(z_n)}{f(y_n)}, \quad w = tv = \frac{f(z_n)}{f(x_n)}. \tag{29}$$

Sharma et al. [6] developed the family of eighth-order variants of the Ostrowski-type methods. The particular form we however consider in this paper is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{30}$$

$$z_n = y_n - (1-2t)^{-1}\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{31}$$

$$x_{n+1} = z_n - [1 + w + \beta w^2]\left(\frac{f[x_n, y_n]\, f(z_n)}{f[y_n, z_n]\, f[x_n, z_n]}\right), \tag{32}$$

$$f[x_n, y_n] = \left(\frac{f(x_n) - f(y_n)}{x_n - y_n}\right) \triangleq f'(y_n),$$

where $\beta \in \mathbb{R}, t,$ and w are given in (29). The computational results reported in the next section are based on the particular value of $\beta = 0$.

Another variant of Ostrowski-type method was considered by Wang et al. [11] and is represented by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{33}$$

$$z_n = y_n - (1-2t)^{-1}\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{34}$$

$$x_{n+1} = z_n - \{2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + (y_n - z_n) f[y_n, x_n, x_n]\}^{-1} f(z_n). \tag{35}$$

where t is as in (29).

Finally Thukral has developed in [8] the eighth-order Newton-type iterative process

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{36}$$

$$z_n = y_n - K(t)\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{37}$$

$$x_{n+1} = z_n - \left(2^{-2}[1 + K(t)]^2 + \frac{v}{1-bv} + 4\frac{w}{1-cw} + pt^4\right)\left(\frac{f(z_n)}{f'(x_n)}\right). \tag{38}$$

where

$$K(t) = \left(\frac{1+at}{1+(a-2)t}\right), \tag{39}$$

$a, b, c, p \in \mathbb{R}$ while $t, v,$ and w are as given by (29). The associated results reported in the next section are based on $a = b = c = 0$ and $p = 3$.

5. Numerical Examples

The present two-point ninth-order method, represented by (7), is employed to solve nonlinear equations with simple roots. To demonstrate the performance of this method, ten distinct nonlinear equations are considered. These ten test functions are displayed in Table 1. The difference between the simple root α and the approximation x_n for test functions with initial guess x_0 are displayed in Table 2. In fact, x_n is computed by using the same total number of function evaluations for all methods. While the estimates of the solutions produced by all the methods are given in Table 2, the corresponding errors are listed in Table 3. In fact, the errors displayed are of absolute value, and insignificant figures in these results have been omitted in these tables. The new two-point ninth-order method happens to require four function evaluations and has an order of convergence nine. To determine the efficiency index of the new method, definition 2.2 has been used. Hence, the efficiency index of the new iterative method given by (7) is $\sqrt[4]{9} \approx 1.7132$. The efficiency index of the three-point eighth-order methods considered in this paper, described by (25), (28), (32), (35), is $\sqrt[4]{8} \approx 1.682$ and the efficiency index of the two-point fourth-order Newton method, given by (22), is $\sqrt[4]{4} \approx 1.4142$.

Table 1 : Test functions with simple root $\alpha = 0$ and x_0 initial guess

		$f(x)$	x_0
$f_1(x)$	=	$e^x \sin(x) + \ln(1 + x^2)$	1/2
$f_2(x)$	=	$\cos(x) \ln(1 + x^3) - e^{-x} \sin(x)$	–1/3
$f_3(x)$	=	$\exp[\sin(x^2)] - x/25 - 1$	–1/10
$f_4(x)$	=	$1 + x^2 \exp[\cos(x/2)] - (x + 1) \exp[\sin(x/2)]$	1/5
$f_5(x)$	=	$1 - \cos(3x) + \tan(2x) + \sin(4x)$	1/4
$f_6(x)$	=	$\cos(x^2) - e^{-x}$	–1/9
$f_7(x)$	=	$\exp[2x^3 - 3x^2] \sin(x) + \ln(1 + x^3)$	–1/2
$f_8(x)$	=	$\ln(1 + x^2) + \sin(x) \cos(x)$	1/7
$f_9(x)$	=	$\sin(x) - x^3/2$	1/3
$f_{10}(x)$	=	$(x - 2)^{10} - 2^{10}$	–1/8

It is clear that the efficiency index of the new two-point ninth-order method is much better than that of other similar methods. Furthermore, we display the computational order of convergence approximations in table 3. From the tables we observe that the COC perfectly coincides with the theoretical result.

Table 2 : Comparison of solutions by various iterative methods

f_i	**(25)**	**(38)**	**(28)**	**(32)**	**(35)**	**(7)**
f_1	0.394e-19	0.394e-111	0.137e-126	0.280e-131	0.623e-155	0.877e-349
f_2	0.827e-44	0.417e-248	0.635e-263	0.504e-277	0.220e-304	0.520e-648
f_3	0.201e-10	0.914e-51	0.101e-59	0.213e-62	0.112e-75	0.378e-223
f_4	0.363e-27	0.336e-215	0.220e-249	0.163e-180	0.420e-224	0.143e-351
f_5	0.112e-73	0.129e-343	0.234e-348	0.117e-400	0.134e-421	0.473e-1054
f_6	0.887e-54	0.162e-368	0.864e-401	0.109e-403	0.990e-438	0.107e-734
f_7	0.607e-71	0.399e-170	0.319e-190	0.130e-222	0.187e-181	0.854e-2481
f_8	0.337e-47	0.199e-273	0.104e-313	0.148e-311	0.366e-361	0.114e-611
f_9	0.125e-275	0.586e-315	0.246e-338	0.206e-546	0.818e-790	0.115e-6981
f_{10}	0.114e-38	0.440e-297	0.140e-368	0.846e-335	0.170e-339	0942e-557

Table 3 : COC of various iterative methods

f_i	**(25)**	**(38)**	**(28)**	**(32)**	**(35)**	**(7)**
f_1	4.0007	7.9998	8.0005	8.0001	8.0001	9.0000
f_2	3.9998	8.0000	8.0000	8.0000	8.0000	9.0000
f_3	3.9304	7.9726	7.9964	7.9909	7.9980	8.9998
f_4	4.0001	8.0000	8.0000	8.0000	8.0000	9.0000
f_5	4.0000	8.0000	8.0000	8.0000	8.0000	9.0000
f_6	4.0000	8.0000	8.0000	8.0000	8.0000	9.0000
f_7	9.0067	8.9487	8.9586	9.9546	9.9305	25.000
f_8	3.9999	8.0000	8.0000	8.0000	8.0000	9.0000
f_9	9.0000	8.9847	8.9867	10.991	11.000	25.000
f_{10}	4.0006	8.0000	8.0000	8.0000	8.0000	9.0000

6. Conclusion

The good performance of the new two-point ninth-order Newton-type method has been demonstrated. The effectiveness of this method has been examined by displaying the accuracy of the simple root of some test nonlinear equations. It has been verified numerically that the new Newton-type method has a convergence order of nine. The major advantages of the new method are: (i) it is not limited by the Kung and Traub conjecture, (ii) its very high computational efficiency, and (iii) better efficiency index. Indeed, the two-point iteration produces a better approximation of the simple root than the three point method and is very competitive with the three-point eighth-order methods. Finally, we note that the new two-point ninth-order method is only effective when the simple root is zero. This is apparently a limitation that calls for a further future special investigation.

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