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On L1 - Convergence of Some Sine and Cosine Modified Sums

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Abstract. *In this paper we define a new class of null-sequences of real numbers, named* $BV^{\log}$, *which tend to zero. We introduce the set of the following modified trigonometric sums*

$$\beta_n^{\sin}(x) = \sum_{k=1}^{n}\sum_{j=k}^{n}\triangle\left(\frac{a_j}{\log(j+1)}\right)\log(k+1)\sin kx\,,$$

and

$$\beta_n^{\cos}(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\triangle\left(\frac{a_j}{\log(j+1)}\right)\log(k+1)\cos kx\,,$$

and study their L^1*-convergence as* $n\to\infty$. *Finally, employing such a class of sequences we confirm the convergence in* L^1*-norm of a sine or cosine usual trigonometric series.*

Key words : L^1 – Convergence, Null Sequence, Trigonometric Series, Modified Trigonometric Sums.

AMS Subject Classifications : 42A20, 42A32

1. Introduction and Preliminaries

Let

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{1}$$

and

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{2}$$

be sine and cosine trigonometric series respectively, and let
$\|f\|_{L^1} = \|f\| = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|dx$
be the L^1-norm of a 2π-periodic function $f \in L^1$. Throughout this paper we will denote by $S_n(x)$ the partial sums of the series (1) or (2) and $\lim_{n\to\infty} S_n(x) = \beta(x)$.

The following problem has excited many researchers for more than 40 years and still receives considerable attention: If a trigonometric series converges in L^1-norm to a function $f \in L^1$, then it is the Fourier series of the function f. But what about the inverse statement? Riesz (see [7], Vol. II, Chap. VIII, § 22) gave a counter example showing that in the metric L^1 we can not expect the converse of the above mentioned statement to be true. This fact has motivated various authors to study the L^1-convergence of trigonometric series, by introducing the so-called modified cosine and sine sums. The reason being that these modified sums turn out to approximate their limits better than the classical trigonometric series in the sense that they converge in L^1-norm to the sum of the trigonometric series. Whereas the classical series itself may not. In this regard, C. S. Rees and Č. V. Stanojević introduced, for the first time in [3], the following type of modified cosine sums

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n}\triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}\triangle a_j \cos kx,$$

where $\triangle a_j = a_j - a_{j+1}$, and obtained a necessary and sufficient condition for the integrability of the limit of these sums.

Then several interesting properties (namely, integrability [2] or L^1-convergence [13]) of these sums were investigated. Properties, that impose several conditions on the coefficients a_k. The reader is referred here to some old papers like [1], [2], [8], [9] or to more recent papers like [5], [6], [12].

Motivated by the sums $f_n(x)$, S. Kumari and B. Ram [10] introduced the set of the sums

$$h_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\triangle\left(\frac{a_j}{j}\right)k\cos kx,$$

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and studied their L^1-convergence under the condition that the coefficients a_n belong in the class $\mathbb{R}$.

Definition [8]**1.1**. If $a_k \to 0$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty} k^2\left|\triangle^2\left(\frac{a_k}{k}\right)\right| < \infty,$$

then it is said that $\{a_k\}$ belongs in the class $\mathbb{R}$, where $\triangle^2 a_j = a_j - 2a_{j+1} + a_{j+2}$.

The above definition has been introduced in by T. Kano, who has verified, in [11], a result which we will formulate as follows.

Theorem [11] **1.1**. *If $\{a_k\} \in R$, then the series (1) and (2) are Fourier series or equivalently they represent integrable functions.*

Using this theorem, S. Kumari and B. Ram, amongst others, have proved the following result.

Theorem [10] **1.2**. *Let* $\{a_k\} \in R$. *Then for* $x \in (0,\pi]$

$$\lim_{n\to\infty} t_n(x) = t(x), \quad t \in L(0,\pi], \tag{3}$$

and

$$\|t_n - t\| \to 0, \text{ as } n \to \infty, \tag{4}$$

where $t_n(x)$ *represents either* $h_n(x)$ *or* $g_n(x)$.

Now we introduce the following set of new modified sums defined by

$$\beta_n^{\sin}(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{\log(j+1)}\right)\log(k+1)\sin kx\,,$$

and

$$\beta_n^{\cos}(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{\log(j+1)}\right)\log(k+1)\cos kx.$$

For the forthcoming analysis we also advance the following new class of null sequences of real numbers.

Definition 1.2. If $a_j \to 0$ as $j \to \infty$ and

$$\sum_{j=1}^{\infty} \log^2(j+1)\left|\frac{a_j}{\log(j+1)} - \frac{a_{j+1}}{\log(j+2)}\right| < \infty\,,$$

then we say that $\{a_j\}$ belongs to the class $BV^{\log}$.

In their joint paper [4], Č. V. Stanojević and V. B. Stanojević have introduced the so-called weakly even null-sequences and have denoted them by W. These motivate the definition that follows.

Definition 1.3. If $a_j \to 0$ as $j \to \infty$ and

$$\sum_{j=1}^{\infty} \log(j+1)|\triangle a_j| < \infty\,,$$

then we say that $\{a_j\}$ is weakly even, briefly denoted by $\{a_j\} \in W$.

In addition to other facts, [4] reports on a proof that $\lim_{n\to\infty} S_n(x) = \beta(x)$ exists and $\beta \in L^1$ (see Corollary 2.1, page 682 in [4]) under the conditions : $\{a_j\} \in W$ and existence of a positive monotone sequence $\{A_j\}$, such that $|\triangle a_j| \le A_j$ for all j. These facts turn out to hold in the present work only when $\{a_j\} \in BV^{\log}$ (see theorem 2.1, blow). More generally, however, the next lemma shows that the class $BV^{\log}$ is a wider class of sequences, and could be more useful in applications than the class W.

Lemma 1.1. *If the implication* $\{a_j\} \in W \Rightarrow \{a_j\} \in BV^{\log}$ *holds true, then* $W \subseteq BV^{\log}$.

Proof. Let $\{a_j\} \in W$. After some elementary calculations we have

$$\frac{a_j}{\log(j+1)} - \frac{a_{j+1}}{\log(j+2)} = \frac{a_j - a_{j+1}}{\log(j+1)} + \frac{a_{j+1}\log\left(1+\frac{1}{j+1}\right)}{\log(j+1)\log(j+2)}.$$

Hence,

$$\log^2(j+1)\left|\frac{a_j}{\log(j+1)} - \frac{a_{j+1}}{\log(j+2)}\right| \le \log(j+1)|\triangle a_j| + \frac{|a_{j+1}|\log(j+1)}{(j+1)\log(j+2)}$$
$$\le \log(j+1)|\triangle a_j| + 2\frac{|a_{j+1}|}{j+1}.$$

Thus,

$$\sum_{j=1}^{\infty}\log^2(j+1)\left|\frac{a_j}{\log(j+1)} - \frac{a_{j+1}}{\log(j+2)}\right|$$
$$\le \sum_{j=1}^{\infty}\log(j+1)|\triangle a_j| + 2\sum_{j=1}^{\infty}\frac{|a_{j+1}|}{j+1}$$
$$\le \sum_{j=1}^{\infty}\log(j+1)|\triangle a_j| + 2\sum_{j=1}^{\infty}\frac{1}{j+1}\sum_{i=j+1}^{\infty}|\triangle a_i|$$
$$\le \sum_{j=1}^{\infty}\log(j+1)|\triangle a_j| + 2\sum_{i=1}^{\infty}|\triangle a_i|\sum_{j=1}^{i+1}\frac{1}{j}$$
$$\le \mathcal{O}\left(\sum_{j=1}^{\infty}\log(j+1)|\triangle a_j|\right) < +\infty,$$

which clearly implies that $\{a_j\} \in BV^{\log}$. ■

The aim of this paper is to prove an analogue of theorem 1.2 to address the $\beta_n^{\sin}(x)$ and $\beta_n^{\cos}(x)$ sums, under the condition that the coefficients a_k belong in the class $BV^{\log}$. In this regard, the following lemma is required for the proof of the main result.

Lemma 1.2. *Let*

$$\widetilde{D}_n^{\log}(x) = \sum_{j=1}^{n}\log(j+1)\sin(jx).$$

and

$$D_n^{\log}(x) = \sum_{j=1}^{n}\log(j+1)\cos(jx),$$

Then the following estimates:

$$\widetilde{D}_n^{\log}(x)| = O\left(\frac{\log(n+1)}{x}\right), \quad 0 < x \le \pi,$$

and

$$|D_n^{\log}(x)| = O\left(\left(\frac{\pi}{x} + \frac{1}{2}\right)\log(n+1)\right), \quad 0 < x \le \pi,$$

hold true.

Proof. Applying Abel's transformation, when $0 < x \le \pi$, we obtain

$$\begin{aligned}\widetilde{D}_n^{\log}(x) &= \sum_{j=1}^{n} \log(j+1)\sin(jx)\\ &= \sum_{j=1}^{n-1} \triangle(\log(j+1)) \sum_{s=1}^{j} \sin(sx) + \log(n+1)\sum_{s=1}^{n}\sin(sx)\\ &= -\sum_{j=1}^{n-1} \log\left(1+\frac{1}{j+1}\right)\frac{\cos\frac{x}{2}-\cos(j+\frac{1}{2})x}{2\sin\frac{x}{2}} + \log(n+1)\frac{\cos\frac{x}{2}-\cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}},\end{aligned}$$

Thus, using Young's inequality $\frac{\sin\beta}{\beta} \geq \frac{2}{\pi}$ for $\beta \in [0, \pi/2]$, we have

$$|\widetilde{D}_n^{\log}(x)| \leq \sum_{j=1}^{n-1} \frac{1}{j+1}\frac{\pi}{x} + \log(n+1)\frac{\pi}{x} = \mathcal{O}\left(\frac{\log(n+1)}{x}\right), \quad 0 < x \leq \pi.$$

Similarly, we have

$$\begin{aligned}D_n^{\log}(x) &= \sum_{j=1}^{n} \log(j+1)\cos(jx)\\ &= \sum_{j=1}^{n-1} \triangle(\log(j+1)) \sum_{s=1}^{j} \cos(sx) + \log(n+1)\sum_{s=1}^{n}\cos(sx)\\ &= -\sum_{j=1}^{n-1} \log\left(1+\frac{1}{j+1}\right)\left[\frac{\sin(j+\frac{1}{2})x}{2\sin\frac{x}{2}} - \frac{1}{2}\right] + \log(n+1)\left[\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} - \frac{1}{2}\right],\end{aligned}$$

for $0 < x \leq \pi$. Then reinvoking again Young's inequality leads to

$$|D_n^{\log}(x)| \leq \sum_{j=1}^{n-1} \frac{1}{j+1}\left(\frac{\pi}{x}+\frac{1}{2}\right) + \log(n+1)\left(\frac{\pi}{x}+\frac{1}{2}\right) = \mathcal{O}\left(\left(\frac{\pi}{x}+\frac{1}{2}\right)\log(n+1)\right).$$

Here the proof ends. ■

2. Main Results

The following theorem and its corollary represent the main results of this work.

Theorem 2.1. *Let $\{a_k\}$ be a sequence that belongs to the class $BV^{\log}$, then*
(i) $\beta_n(x)$ converges point-wise to $\beta(x)$ for $\delta \leq x \leq \pi$, $\delta > 0$,
(ii) $\beta_n(x)$ converges to $\beta(x)$ in the L^1-norm, and
(iii) $\beta(x)$ is an integrable function i.e. $\beta \in L^1$,
where $\beta_n(x)$ represents either $\beta_n^{\cos}(x)$ or $\beta_n^{\sin}(x)$.

Proof. As for i), we consider only the case of the $\beta_n^{\sin}(x)$ sums ; since the case of cosine sums, $\beta_n^{\cos}(x)$, can be treated in a similar way.

Obviously

$$\beta_n^{\sin}(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{\log(j+1)}\right)\log(k+1)\sin kx$$
$$= \sum_{k=1}^{n}\left[\triangle\left(\frac{a_k}{\log(k+1)}\right) + \cdots + \triangle\left(\frac{a_n}{\log(n+1)}\right)\right]\log(k+1)\sin kx$$
$$= \sum_{k=1}^{n}\left(\frac{a_k}{\log(k+1)} - \frac{a_{n+1}}{\log(n+2)}\right)\log(k+1)\sin kx$$
$$= \sum_{k=1}^{n} a_k \sin kx - \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x). \tag{5}$$

Since the sequence $\{a_k\}$ tends to zero, then the second term in (5) tends to zero as well. Next, using lemma 1.2 for $0 < \delta \le x \le \pi$, we find that

$$\left|\frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x)\right| = \frac{a_{n+1}}{\log(n+2)}\left|\widetilde{D}_n^{\log}(x)\right|$$
$$= \frac{a_{n+1}}{\log(n+2)}\mathcal{O}\left(\frac{\log(n+1)}{x}\right)$$
$$= \frac{1}{\delta}\mathcal{O}\left(\frac{a_{n+1}\log(n+1)}{\log(n+2)}\right)$$
$$= \frac{1}{\delta}\mathcal{O}(a_{n+1}) = o(1), \quad \text{as} \quad n \to \infty.$$

The last line follows in view of the inequality $\frac{\log(n+1)}{\log(n+2)} \le 1$ for all $n \in \mathbb{N}$.

We may conclude therefore that

$$\lim_{n\to\infty} \beta_n^{\sin}(x) = \lim_{n\to\infty} S_n(x) = \beta(x), \quad \text{for} \quad 0 < x \le \pi.$$

To prove ii), we invoke (5) to write

$$\beta(x) - \beta_n^{\sin}(x) = \sum_{k=n+1}^{\infty} a_k \sin kx + \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x)$$
$$= \lim_{p\to\infty}\sum_{k=n+1}^{p}\frac{a_k}{\log(k+1)}\log(k+1)\sin kx + \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x).$$

Apply then summation by parts to the above relation to write

$$\beta(x) - \beta_n^{\sin}(x) = \lim_{p\to\infty}\left[\sum_{k=n+1}^{p-1}\triangle\left(\frac{a_k}{\log(k+1)}\right)\widetilde{D}_k^{\log}(x) + \frac{a_p}{\log(p+1)}\widetilde{D}_p^{\log}(x) - \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x)\right] + \frac{a_{n+1}}{\log(n+2)}\widetilde{D}_n^{\log}(x).$$

In a similar fashion, as in the proof of i), one can show that the second term, in brackets of the above equality, tends to zero, and hence,

$$\beta(x) - \beta_n^{\sin}(x) = \sum_{k=n+1}^{\infty}\triangle\left(\frac{a_k}{\log(k+1)}\right)\widetilde{D}_k^{\log}(x).$$

Since

$$\sum_{j=1}^{k}\log(j+1)\sin(jx) = \sum_{j=1}^{k-1}\triangle(\log(j+1))\sum_{s=1}^{j}\sin(sx)+\log(k+1)\sum_{s=1}^{k}\sin(sx),$$

we can write

$$\begin{aligned}\int_0^\pi|\widetilde{D}_k^{\log}(x)|dx &\le \sum_{j=1}^{k-1}|\triangle(\log(j+1))|\int_0^\pi\left|\sum_{s=1}^{j}\sin(sx)\right|dx+\log(k+1)\int_0^\pi\left|\sum_{s=1}^{k}\sin(sx)\right|dx\\ &= \sum_{j=1}^{k-1}\log\left(1+\frac{1}{j+1}\right)|\mathcal{O}(\log j)+\mathcal{O}(\log^2 k)\\ &= \mathcal{O}(\log k)\sum_{j=1}^{k}\frac{1}{j+1}+\mathcal{O}(\log^2(k+1)) = \mathcal{O}(\log^2(k+1)).\end{aligned}$$

Subsequently, since $\{a_k\}\in BV^{\log}$ we get

$$\begin{aligned}\|\beta(x)-\beta_n^{\sin}(x)\| &\le \sum_{k=n+1}^{\infty}\left|\triangle\ \right|\int_0^\pi|\widetilde{D}_k^{\log}(x)|dx\\ &= \mathcal{O}\left(\sum_{k=n+1}^{\infty}\log^2(k+1)\left|\triangle\left(\frac{a_k}{\log(k+1)}\right)\right|\right) = o(1)\quad\text{as}\quad n\to\infty,\end{aligned}$$

which obviously means that $\beta_n^{\sin}(x)\to\beta(x)$ in the L^1-norm.

Finally regarding iii), Since $\beta_n^{\sin}(x)$ is a polynomial, then the obtained relation $\|\beta(x)-\beta_n^{\sin}(x)\| = o(1)$ as $n\to\infty$ in ii), clearly implies that $\beta\in L^1$. This completes the proof.■

Now we shall derive a sufficient condition for the L^1-convergence of the sine series (1). Accordingly, let $g(x)$ be the sum function of the series (1) and $\mathcal{S}_n(f;x)$ its partial sums.

Corollary 2.1. *Let $\{a_k\}\in BV^{\log}$, then $\|g-S_n(g)\| = o(1)$ as $n\to\infty$, i.e. the series (1) is the Fourier series of the function g.*

Proof. By (5), theorem 2.1, ii), and

$$\int_0^\pi|\widetilde{D}_n^{\log}(x)|dx = \mathcal{O}(\log^2(n+1)),$$

we have

$$\begin{aligned}\int_0^\pi|g(x)-\mathcal{S}_n(x)|dx &\le \int_0^\pi|f(x)-\beta_n^{\sin}(x)|dx+\int_0^\pi|\beta_n^{\sin}(x)-\mathcal{S}_n(x)|dx\\ &= o(1)+\frac{|a_{n+1}|}{\log(n+1)}\int_0^\pi|\widetilde{D}_n^{\log}(x)|dx\\ &= o(1)+\mathcal{O}(|a_{n+1}|\log(n+1)).\end{aligned}$$

Since

$$0 \le |a_{n+1}| \log(n+1) \le \sum_{j=n}^{\infty} \log^2(j+1) \left| \frac{a_j}{\log(j+1)} - \frac{a_{j+1}}{\log(j+2)} \right| \to 0,$$

as $n \to \infty$. Subsequently,
$\int_0^{\pi} |g(x) - \mathcal{S}_n(x)| dx \to 0 \quad \text{as} \quad n \to \infty.$
Here the proof ends. ■

Remark 2.1. Reasoning in the same manner, one can prove that corollary 2.1 should also hold true for the cosine series.

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