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Colored-Noise-Like Multiple Itô Stochastic Integrals: Algorithms and Numerics

M. ZAHRI

Department of Mathematics, College of Sciences, Taibah University, P.O 30002, Madinah, Saudi Arabia,
E-mail:mzahri@taibahu.edu.sa

Abstract. *Mixed multiple stochastic integrals for independent Brownian motions, can not be explicitly approximated. However, integrating a time dependent process in the stochastic sense, namely with respect to the associated Brownian motion, leads to interesting analytical and numerical facts and studies. The main concern of this paper is to provide a recurrence formula (theorem 3.5) for simulating a class of multiple Itô stochastic integrals, which possess a behavior similar to the Gaussian colored noise. Moreover, it contains a numerical analysis, in a review style, of the time-integral and time-differential, in the distributional sense, of the non-differentiable time dependent Brownian motion. All Matlab codes used in the numerical algorithms are also listed.*

Key words : Brownian Motion, Multidimensional Itô Formula, Multiple Stochastic Integrals, Colored Noise, White Noise.

AMS Subject Classifications : 35R60, 35K57, 60H15, 65L06

1. Introduction

Early in 1944 and 1951, K. Itô published the first meaningful analysis for Wiener multiple stochastic integrals [9, 10]. Later the works of Wong and Zakai [18, 22] gave a more explicit analysis to these topics. These are necessary tools for solving either stochastic differential equations (SDEs) (or systems of these equations) [5, 6, 19, 23] or evolutionary partial differential equations with uncertainties [16, 15, 17, 20, 21], especially in Finance, Physics, Biology, etc...[12, 14, 2, 11, 1, 13].

The purpose of this work is to provide an introduction to computational stochastics for numerical integration and simulation of a class of multiple Itô integrals. Instead of attempting to describe the largest possible class of stochastic integrals, we shall only single out a class of these processes. Namely, we shall illustrate some of their graphical similarities to the Gaussian colored noises. Moreover, because the aim is the application of such integrals, much more emphasis is directed at analysis of the theoretical and computational properties of multiple stochastic integrals with respect to a Brownian motion. In this respect, we present interesting technics to be used and developed by graduate students and junior researchers. From a pedagogical point of view, the purpose of these notes is to provide an intuitive understanding of the nature of the multiple stochastic integral, and to clearly outline the difficulties in this type of calculus. For a rigorous analytical theory, we would refer the reader to the books of Karatzas and Shreve (1991), Kloeden and Platen (1992) [5] and Øksendal (1985, 2003) [8]. The present work in meant, however, to combine the interests of Finance and Mathematics graduate students and to jointly introduce them to the subject of Computational Stochastics.

This paper is structured as follows. The second section consists of a numerical construction of normally distributed random numbers using the famous method of Box Müller. We also state the computational aspect of the Brownian motion and some related processes. In the third section, we will prove the main theorem (theorem 3.5 and corollary 3.1) for the construction the colored-noise-like multiple stochastic integrals. With some critical remarks and open questions, we end this paper. We note here that this work paves the way towards a future similar work on SDEs and stochastic partial differential equations (SPDEs).

2. Numerical Simulation of the Brownian Motion

Stochastic calculus is in general based on the Brownian motion process. This was first discovered by the Scottish botanist Robert Brown in 1827. The notion that the increments of the Brownian motion are normally distributed is the source of immense scientific results, either in stochastic analysis or in the interpretation of physical, biological, econometric models. In the following analysis, we will focus on the behavior of some derived processes, namely the time-integral and time differential of the Brownian motion in the distributional sense. For more properties of the Brownian motion, we refer the reader to [3].

2.1. **The Brownian motion**

Definition 3.1. A one-dimensional Brownian motion (also called standard Wiener process) is a real-valued stochastic process $\{W_t\}_{t\geq 0}$ indexed by nonnegative real numbers t with the following properties:

1) $W_0 = 0$.
2) With probability 1, the function $t \mapsto W_t$ is continuous in t.
3) The process $\{W_t\}_{t>0}$ has stationary, independent increments.
4) The increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$ i.e.

$W_t - W_s \sim \sqrt{t-s}\,\mathcal{N}(0,1), \quad \text{for all} \quad t > s.$

A Wiener process with initial value $W_0 = x_0$ is achieved by adding x_0 to a standard Wiener process. The term independent increments means that for every choice of nonnegative real numbers $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_n < t_n < \infty$, the random variables (Wiener increments)

$W_{t_1} - W_{s_1}, W_{t_2} - W_{s_2}, \ldots, W_{t_n} - W_{s_n}$

are pairwise independent. The stationarity of increments means that the distribution of the increment $W_{t+s} - W_s$ has the same distribution as $W_t - W_0 = W_t$, for any $0 < s,t < 1$

In general, a stochastic process with stationary, independent increments is called a Levy process. Moreover, It should not be obvious that properties 1) - 4) in the definition of a standard Brownian motion are mutually consistent, so it is not a priory clear that a standard Brownian motion exists. That it does exist was first proved by N. Wiener in about 1920. His proof was simplified by P. Levy. The compatibility of the properties 3. and 4. follows directly from elementary properties of the normal distributions: If X and Y are independent, normally distributed random variables with means μ_X ; μ_Y and variances σ_X^2 ; σ_Y^2, then the random variable $X+Y$ is normally distributed with mean $\mu_X+\mu_Y$ and variance $\sigma_X^2+\sigma_Y^2$.

The random function $W:[0,1]\to\mathbb{R}$ is continuous but nowhere differentiable (almost surely). Its proof was early given by Paley, Wiener and Zygmund in 1933. This is particularly interesting, as it is not easy to construct a continuous, nowhere differentiable function without the aid of randomness.

One of the interesting interpretations of the Brownian motion is the relationship to the random walk, namely W_t could be interpreted as a limit of symmetric random walks. Let us consider a subdivision of the interval $[0,\infty)$ into subintervals of length δ. Each subinterval corresponds to a time slot of length δ. Thus, the intervals are $(0,\delta],(\delta,2\delta],(2\delta,3\delta],\ldots$. where the k^{th} subinterval is $((k-1)\delta,k\delta]$. Furthermore, we define the symmetric random variables X_i, for $i\in\mathbb{N}$ as

$P\left(X_i=\sqrt{\delta}\right)=P\left(X_i=-\sqrt{\delta}\right)=\frac{1}{2}$.

It is easy to see that X_i's are independent and $E[X_i]=0$; $Var(X_i)=\delta$. We may define then the process W_t as follows: Set $W_0=0$ and at time $t=n\delta$ to define the value of W_t by $W_t=W_{n\delta}=\sum_{i=1}^{n}X_i$. Since W_t is the sum of n $i.i.d.$ random variables, $E[W_t]=0$ and $Var(W_t)=t$. Then, for any $t\in(0,\infty)$, by the passage to the limit for large n, δ tends to zero and by using the central limit theorem, W_t will be a normally distributed random variable with mean 0 and variance t. Moreover, Since X_i are $i.i.d$, we conclude that W_t has independent stationary increments. And by this way, the above method leads to the construction of a process with continuous sample paths, i.e. W_t is a continuous function of t , nowhere differentiable. These are called a standard Brownian motion or a standard Wiener process. Moreover, even if the differentiability is not satisfied, one of the most interesting processes is the Gaussian white noise $\xi(t)=dW_t/dt$, defined as the time-derivative in the distributional sense of the Brownian motion.

2.2. Construction of normally distributed numbers

One of the most useful methods for generating random numbers with a normal distribution is the Box-Müller transform, which was suggested by George Edward Pelham Box and Mervin Edgar Müller (1958). Altogether, the Box-Müller method takes independent standard uniform random variables U_1 and U_2 and produces independent standard normals X_1 and X_2 using the formulas:

$$\theta=2\pi U_1,\quad R=\sqrt{-2\ln(U_2)},\quad X_1=R\cos(\theta),\quad X_2=R\sin(\theta).\tag{1}$$

In other words, from two random numbers u_1 $,u_2\in(0,1]$ (generated by a uniform distribution), we produce two independent standard normally distributed numbers n_1 and n_2, namely

$$n_1=\sqrt{-2\ln(u_1)}\,\cos(2\pi u_2),\qquad n_2=\sqrt{-2\ln(u_1)}\,\sin(2\pi u_2).\tag{2}$$

It has been proven that the random variables X_1 and X_2 are independent, given that they incorporate the same R and θ. Here the independence property is analytically and computationally satisfied.

The Box-Müller Matlab code is given by:

```
function x=boxm();
%return a uniform normally distributed number x
u1=rand;
u2=rand;
x=sqrt(-2*log(u1))*cos(2*pi*u2);
```

Code 1: boxm.m

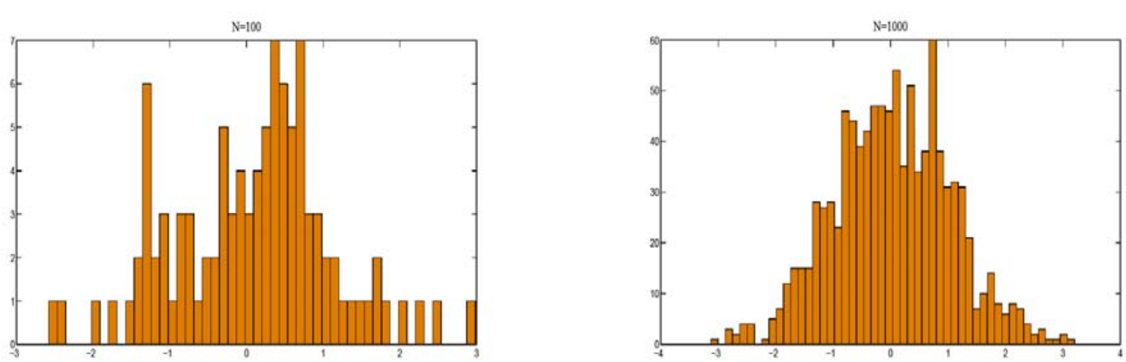
To generate the histograms above, use the following code:

```
function H=NormalDist(n);
% return a histogram of a uniform normal distribution
% n is the number of ND random numbers
X=zeros(1,n);
for i=1:n
u1=rand;
u2=rand;
X(i)=sqrt(-2*log(u1))*cos(2*pi*u2);
end
hist(X,50);
```

Code 2: NormalDist.m

2.3. **Simulation of the Brownian motion**

Consider the upper time bound $T \in \mathbb{R}^+$ and let $0 = t_0 < t_1 < \ldots < t_N = T$ be an equidistant discretization of the time Interval $[t_0, T]$, i.e. $t_k = k\Delta$ with $\Delta = \frac{T}{N}$. Per definition of the Brownian motion, the increments are $i.i.d.$ and normally distributed. Moreover, it yields



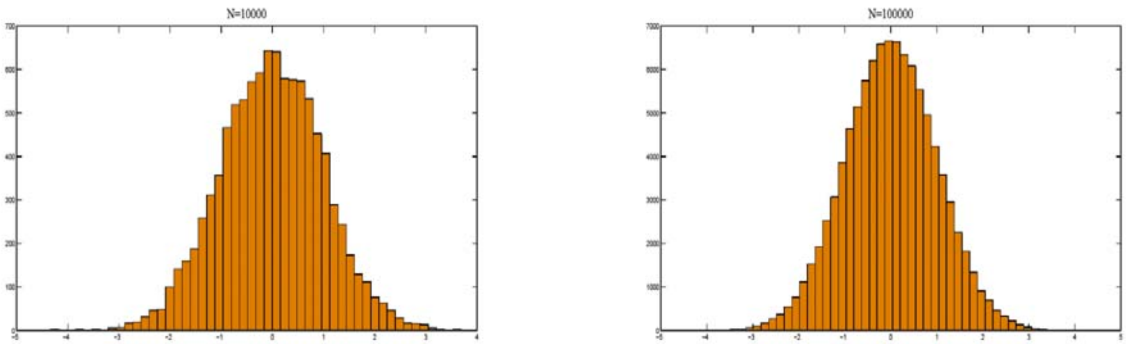


Figure 1: Histogram of the random numbers generated by the Box-Muller method.

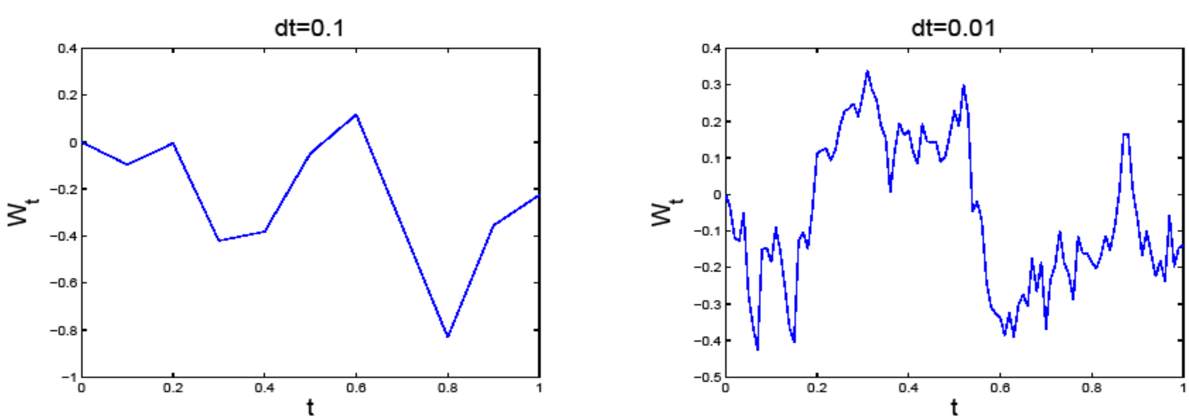
$$\frac{(W_{t_{k+1}} - W_{t_k})}{\sqrt{\Delta t}} \sim \mathcal{N}(0,1).$$

To simulate the paths of Brownian motion, the values W_{t_k} $\forall k = 0, 1, \ldots, N$ are per recursion obtained, and by using linear interpolation one can compute the value of W_t for all $t \in (t_k, t_{k+1})$.

The Matlab code for generating the path of a Brownian motion is:

```
function W=BrownianMotion(dt);
% this code generates a Brownian motion path
%  dt time step size
% the path of the BM will be showed in the time interval [0,1]
N=round(1/dt);
W = zeros(1,N);
T = zeros(1,N);
W(1)=0;
T(1)=0;
for j=1:N
T(j+1)=j*dt;
W(j+1)=W(j)+sqrt(dt)*boxm();
    end
plot(T,W);
```

Code 3: BrownianMotion.m



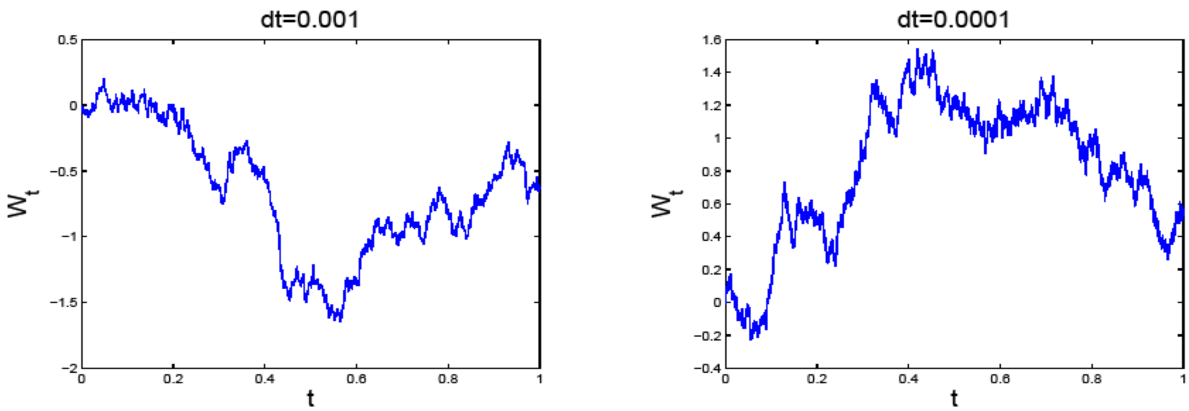


Figure 2: Brownian motion for different time-steps on the time inerval[0,1].

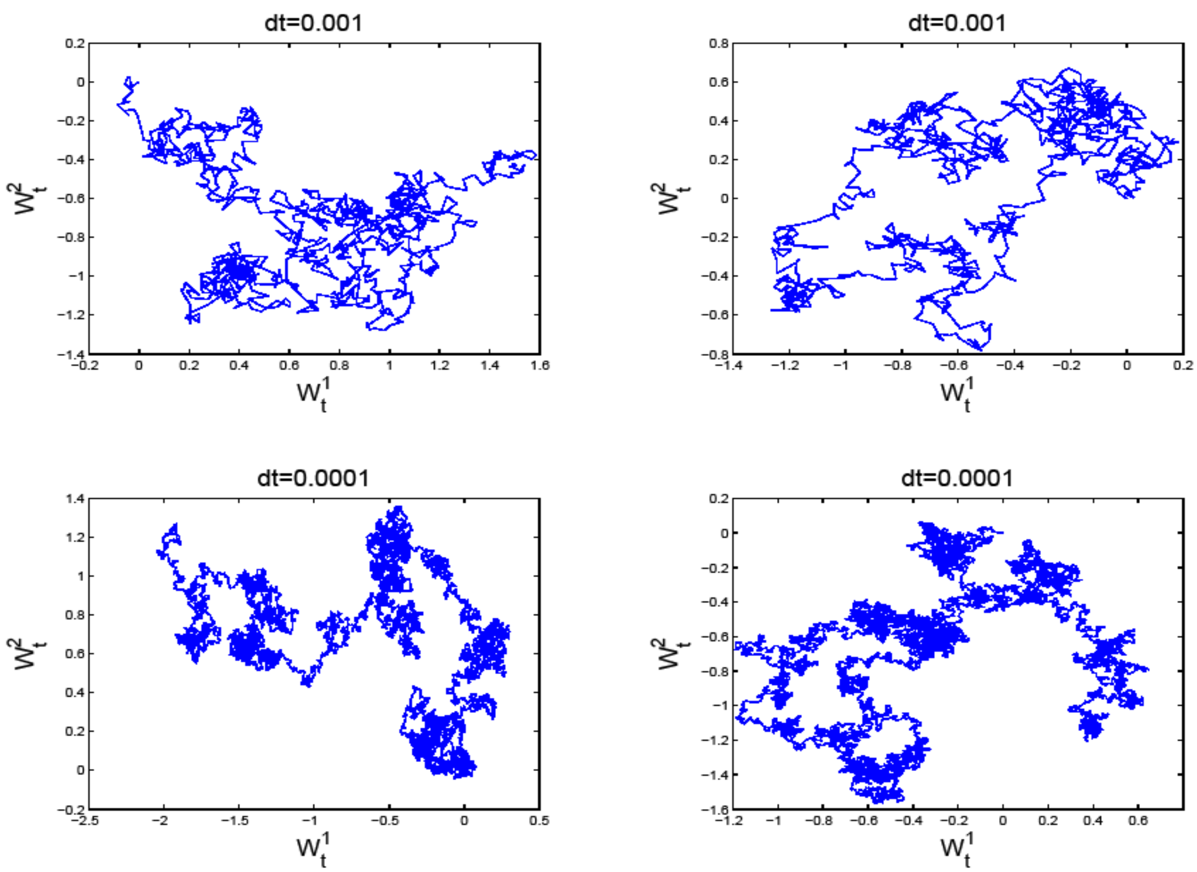


Figure 3: Planar Brownian motion for different time-steps on the time interval[0,1].

3. Simulation of a Class of Multiple Stochastic Integrals

The main concern in this section is to define and simulate a class of multiple stochastic integrals in the sense of Itô. In this way, we deal with a continuous (time-parameter) stochastic process $Z_t(\bullet)$ adapted to a filtration $\mathfrak{F}_t$ progressively measurable, i.e. $Z_t(w)$ is $B_t \times \mathfrak{F}_t$ measurable for all Borel σ-field B_t on $[0,t]$. For instance, all processes with continuous sample paths are progressively measurable.

3.1. Itô integral

Let us consider $T \in \mathbb{T}$ and $(\Omega, \mathfrak{A}, P)$ a probability space with a Filtration $\mathfrak{F} = (\mathfrak{F}_t)_{t\in[0,T]}$. Define then the set Λ_T of square-integrable $\mathfrak{F}_t$-adapted process (SIASP). Throughout this paper, the filtration $\mathfrak{F}_0$ contains the sets with measure zero and $\mathfrak{F}_t$ is right continuous in time. In this case, $\mathfrak{F}_t$ will be called a right continuous augmented filtration. In the following, we will use the norm $\|Z_t\|_{\Lambda_T} = \left(\mathbb{E}\left[\int_0^T Z_s^2 ds\right]\right)^{\frac{1}{2}}$.

Definition 3.1. For $Z \in \Lambda_T$, the the Itô integral with respect to the Brownian motion is defined as

$$I[Z]_{0,t} := \int_0^t Z_s dW_s = \lim_{N\to\infty} I^{(N)}[Z]_{0,t}\ , \tag{3}$$

where

$$I^{(N)}[Z]_{0,t} = \sum_{k=1}^{N} Z_{t_{k-1}^{N,l}} (W_{t_k^{N,l}} - W_{t_{k-1}^{N,l}}), \tag{4}$$

and $\tau_N^l = \{t_k^{N,l} : k = 0, \ldots, N \text{ and } l \in \mathbb{N}\}$ is a sequence of discretizations of the time interval $[0,t]$. The limit (3) is a mean square limit of random variables, i.e., it should satisfy

$$\lim_{N\to\infty} \mathbb{E}(I[Z]_{0,t} - I^{(N)}[Z]_{0,t})^2 = 0. \tag{5}$$

Theorem [8] **3.1**. *The limit (5) exists in $L^2(P)$ and is unique for all $t \in [0,T]$.*

The Itô integral satisfies the properties that follow.

Lemma 3.1. (Linearity) *Consider $(Z_t^{(1)})_{t\in\mathbb{T}}, (Z_t^{(2)})_{t\in\mathbb{T}} \in \Lambda_T$ and $K_1, K_2 \in R$. For*

$$I[Z^{(1)}]_{0,t} = \int_0^t Z_s^{(1)} dW_s \quad \textit{and} \quad I[Z^{(2)}]_{0,t} = \int_0^t Z_s^{(2)} dW_s,$$

the relation

$$I[K_1 Z^{(1)} + K_2 Z^{(2)}]_{0,t} = \int_0^t (K_1 Z_s^{(1)} + K_2 Z_s^{(2)}) dW_s = K_1 I[Z^{(1)}]_{0,t} + K_2 I[Z^{(2)}]_{0,t}\ , \tag{6}$$

must *hold.*

Proof. The proof of (6) follows directly from the definition 3.1. ■

Remark 3.1. It is important to note that, the linearity discussed in lemma 3.1, require the integration with respect to the same Brownian motion W_t. Therefore, for

$$I[Z^{(1)}]_{0,t} = \int_0^t Z_s^{(1)} dW_s^1 \quad \text{and} \quad I[Z^{(2)}]_{0,t} = \int_0^t Z_s^{(2)} dW_s^2, \tag{7}$$

the linearity property of the Itô integral is not true.

Theorem [8] **3.2**. *For $(Z_t)_{t\in\mathbb{T}} \in \Lambda_T$, $0 < s < t$ the following properties all hold.*

i) (Martingale)

$$\mathbb{E}(I[Z]_{0,t}|F_s) = I[Z]_{0,s}. \quad (8)$$

ii) (Itô Isometry)

$$\mathbb{E}[(I[Z]_{0,t})^2] = E\int_0^t Z_u^2 du. \quad (9)$$

iii) (Continuity of I_t)

There exists a continuous process h_t such that

$$P(h_t = I[Z]_{0,t}) = 1, \quad \forall t, 0 \le t \le T. \quad (10)$$

3.2. **Higher order Itô formula**

One of the main concerns of Stochastic Calculus is the new concept of differentiability. For instance, we know that the path of a Brownian motion is continuous but nowhere differentiable and in order to define a stochastic differential equation and integrals, we have to introduce the notion of stochastic differentiability. The central result is the Itô-Formula, which leads to a new definition of differential equations and to a new concept of Taylor expansion. A process satisfying a stochastic differential equation (SDE) in the sense of Itô, will be called an Itô process.

Definition 3.2. Let $(W_t)_{t\in\mathbb{T}}$ be an m-dimensional Brownian motion, defined on a $(\Omega, \mathfrak{A})^m$, with right continuous augmented filtration $\mathfrak{F} = (\mathfrak{F}_t)_{t\in\mathbb{T}}$. The process $(X_t^1 \ldots, X_t^d)$ is called an Itô Processes, if and only if it has the following form

$$X_t^i = X_{t_0}^i + \int_{t_0}^t a_s^i ds + \sum_{j=1}^m \int_{t_0}^t b_s^{i,j} dW_s^j; \qquad i = 1, \ldots, d; \quad j = 1, \ldots, m, \quad (11)$$

where for all i,j; $(a_t^i)_{t\in\mathbb{T}}, (b_t^{i,j})_{t\in\mathbb{T}}$ are $\mathfrak{F}_t$ adapted, $\int_{t_0}^T a_s^i ds < \infty$ and $\int_{t_0}^T (b_s^{i,j})^2 ds < \infty$ a.s.

Lemma 3.2. *Consider a one dimensional Brownian motion and a non-necessary uniform time discretization $t_k = k\frac{T-t_0}{2^n}$ of the interval $[t_0, T]$. Then we have,*

i) $$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\Delta t_k)^2 = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Delta t_k \Delta W_{t_k} = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Delta W_{t_k} \Delta t_k = 0.$$

ii) $$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\Delta W_{t_k})^2 = \int_{t_0}^T ds = (T - t_0), \ (\textit{Convergence in } L^2).$$

where $\Delta t_k = t_{k+1} - t_k$ and $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$.

Proof. i) follows from the construction

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\Delta t_k)^2 \le \lim_{n\to\infty} \max_k (\Delta t_k) \sum_{k=0}^{2^n-1} \Delta t_k = \lim_{n\to\infty} \max_k (\Delta t_k) \int_{t_0}^T dt = 0.$$

For part ii), with the Brownian motion, we have

$$0 = \lim_{n\to\infty} \min_k (\Delta t_k) \int_{t_0}^T dW_s \le \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Delta t_k \Delta W_{t_k} \le \lim_{n\to\infty} \max_k (\Delta t_k) \int_{t_0}^T dW_s = 0.$$

Since ΔW_{t_k} are i.i.d. and normally distributed with mean zero and variance Δt_k , and by using the strong law of large numbers the following convergence in L^2 is true.

$$\lim_{n\to\infty}\sum_{k=0}^{2^n-1}(\Delta W_{t_k})^2 = \left(\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\Delta t_k\right) = \int_{t_0}^{T} ds = (T-t_0). \qquad \blacksquare$$

Lemma 3.3. *Let us consider the functional* $f:[t_0,T]\times\mathbb{R}^d\to\mathbb{R}$ *with continuous partial derivatives* $\frac{\partial f}{\partial t},\frac{\partial f}{\partial x^i}$ *and* $\frac{\partial^2 f}{\partial x^i\partial x^j}$ *for* $i=1,\dots,d$ *and a one dimensional Itô Process* $(X_t)_{t\in\mathbb{T}}$. *For any time discretization* $t_k = k\frac{T-t_0}{2^n}$ *of the interval* $[t_0,T]$ *, then it follows that*

i) $\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\frac{\partial f}{\partial t}\Delta t_k = \int_{t_0}^{t}\frac{\partial f}{\partial t}ds.$

ii) $\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\frac{\partial f}{\partial x}\Delta X_{t_k} = \int_{t_0}^{t}\frac{\partial f}{\partial x}dX_s = \int_{t_0}^{t}\frac{\partial f}{\partial x}a_s ds + \int_{t_0}^{t}\frac{\partial f}{\partial x}b_s dW_s.$

iii) $\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\frac{\partial^2 f}{\partial t^2}(\Delta t_k)^2 = 0\cdot\int_{t_0}^{t}\frac{\partial^2 f}{\partial t^2}ds = 0.$

iv) $\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\frac{\partial^2 f}{\partial x^2}(\Delta X_{t_k})^2 = \int_{t_0}^{t}\frac{\partial^2 f}{\partial x^2}b_s^2 ds.$

where $\Delta t_k = t_{k+1}-t_k$ *and* $\Delta X_{t_k} = X_{t_{k+1}} - X_{t_k}$.

Proof. The result i) is trivial, while the proof for ii) is

$$\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\frac{\partial f}{\partial x}\Delta X_{t_k} = \int_{t_0}^{t}\frac{\partial f}{\partial x}dX_s = \int_{t_0}^{t}\frac{\partial f}{\partial x}(a_s ds + b_s dW_s) = \int_{t_0}^{t}\frac{\partial f}{\partial x}a_s ds + \int_{t_0}^{t}\frac{\partial f}{\partial x}b_s dW_s.$$

As for iii), consider a uniform time discretization Δt of the interval $[t_0,T]$, then we have

$$\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\frac{\partial^2 f}{\partial t^2}(\Delta t_k)^2 = \lim_{n\to\infty}\underbrace{(\Delta t)}_{\to 0}\cdot\underbrace{\int_{t_0}^{t}\frac{\partial^2 f}{\partial t^2}ds}_{bounded} = 0.$$

Finally to prove *iv)*, we write

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\frac{\partial^2 f}{\partial x^2}(\Delta X_{t_k})^2 &= \lim_{n\to\infty}\sum_{k=0}^{2^n-1}\frac{\partial^2 f}{\partial x^2}b_{t_k}^2\Delta W_{t_k}^2\\ &+ \lim_{n\to\infty}\underbrace{\sum_{k=0}^{2^n-1}\frac{\partial^2 f}{\partial x^2}a_{t_k}^2\Delta t_k{}^2}_{\to 0\ (lemma\ 3.2)}\\ &+ \lim_{n\to\infty}\underbrace{2\sum_{k=0}^{2^n-1}\frac{\partial^2 f}{\partial x^2}a_{t_k}b_{t_k}^2\Delta t_k\Delta W_{t_k}}_{\to 0}\quad \text{(applying Itô isometry in } L^2)\\ &= \int_{t_0}^{t}\frac{\partial^2 f}{\partial x^2}b_s^2 ds.\ (\text{in } L^2). \qquad \blacksquare\end{aligned}$$

Lemma 3.4. *Under the assumption of the lemmas above, the one dimensional case* $d=m=1$ *of the Itô-Formula can be written as*

$$f(t,X_t) = f(t_0,X_{t_0}) + \int_{t_0}^{t} \left\{ \frac{\partial f}{\partial s}(s,X_s) + a_s \frac{\partial f}{\partial x}(s,X_s) + \frac{1}{2} b_s^2 \frac{\partial^2 f}{\partial x^2}(s,X_s) \right\} ds + \int_{t_0}^{t} b_s \frac{\partial f}{\partial x}(s,X_s) dW_s. \tag{12}$$

Proof. For a given discretization of the time interval $[t_0,T]$ by $t_k = k\frac{(T-t_0)}{2^n}$, define $\Delta t_k = t_{k+1} - t_k$; $\Delta X_{t_k} = X_{t_{k+1}} - X_{t_k}$ and $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$. By using the Taylor expansion of order two, we have

$$f(t,X_t) = f(t_0,X_{t_0}) + \sum_{k=0}^{2^n-1} \Delta f(t_k,X_{t_k}) \tag{13}$$

$$= f(t_0,X_{t_0}) + \sum_{k=0}^{2^n-1} \frac{\partial f}{\partial t}\Delta t_k + \sum_{k=0}^{2^n-1} \frac{\partial f}{\partial x}\Delta X_{t_k} + \frac{1}{2}\sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial x^2}(\Delta X_{t_k})^2$$

$$+ \sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t \partial x}\Delta t_k \Delta X_{t_k} + \frac{1}{2}\sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t^2}(\Delta t_k)^2 + \sum_{k=0}^{2^n-1} R_k, \tag{14}$$

where R_k consists of sums of higher order partial derivatives of f , as a factor of $(\Delta t)^2$, $\Delta W_{t_k}(\Delta t)^2$, $\Delta(W_{t_k})^2\Delta t$ and $\Delta W_{t_k}\Delta t$. Using the results of lemma 3.2, we conclude that $R_k = O((\Delta t)^2)$ and therefore the remainder term vanish in L^2. Also using the results of lemma 3.2, all terms with $(\Delta t)^2$ vanish (at least in L^2 if the increment of the Brownian motion appears.) Similar construction could be done for the mixed partial derivatives, which are in general factors either of $(\Delta t)^2$ or $\Delta W_{t_k}\Delta t$. Thus, all terms in (14) vanish in L^2,

$$\lim_{n\to\infty} \left[\sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t \partial x}\Delta t_k \Delta X_{t_k} + \frac{1}{2}\sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t^2}(\Delta t_k)^2 + \sum_{k=0}^{2^n-1} R_k \right] = 0.$$

The passage to the limit in (13), leads to

$$f(t,X_t) = f(t_0,X_{t_0}) + \lim_{n\to\infty} \left[\sum_{k=0}^{2^n-1} \frac{\partial f}{\partial t}\Delta t_k + \sum_{k=0}^{2^n-1} \frac{\partial f}{\partial x}\Delta X_{t_k} + \frac{1}{2}\sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial x^2}(\Delta X_{t_k})^2 \right].$$

Since $dX_t = a_t dt + b_t dW_t$, and using the results of lemma 3.3, the one dimensional Itô formula is proved. ■

Example 3.1. For $f(t,x) = \frac{1}{2}x^2$ with $X_t = W_t$ and $a_t = 0, b_t = 1$. By applying Itô's formula, we have:

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + a_t\frac{\partial f}{\partial x}dt + b_t\frac{\partial f}{\partial x}dW + \frac{1}{2}b_t^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + (1)\frac{\partial f}{\partial x}dW + \frac{1}{2}(1)^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dW + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial x}dW + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt. \end{aligned}$$

Hence,

$$\frac{1}{2}dW_t^2 = W_t dW_t + \frac{1}{2}dt,$$

and

$\frac{1}{2}\int dW_s^2 = \int W_s dW_s + \frac{1}{2}\int dt$.

Thus,

$$I_t = \int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t). \tag{15}$$

Note that W_t^2, represents the square of the end value of the Brownian motion. Thus I_t will be considered as a time process if we change the upper bound of the integration interval.

Example 3.2. Consider $n > 1$ in $f(t,x) = x^{n+1}$. Then apply Itô's formula for $X_t = W_t$, to obtain $d(W_t^{n+1}) = (n+1)W_t^n dW_t + \frac{n(n+1)}{2} W_t^{n-1} dt.$

Hence,

$$\int_0^t dW_s^n = \frac{1}{n+1} W_t^{n+1} - \frac{n}{2}\int_0^t W_s^{n-1} ds.$$

Simulation of the sample path of the Itô integral (15).

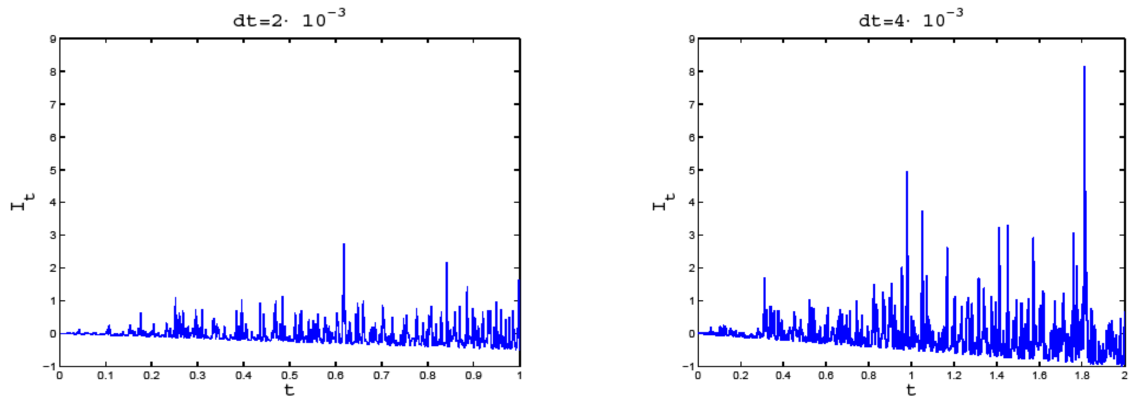


Figure 4: Simulation sample path of the Itô integral with the same number of steps N=500.

It is important to note that the integral of a Brownian motion path with respect to time, represented by $I(W_t) = \int_0^t W_s\ ds$ is not a stochastic integral. It represents the area under Brownian motion path, $I(W_t)$ is a normal random variable with mean 0 and variance $\frac{t^3}{3}$; i.e. $I(W_t) \sim N(0, \frac{t^3}{3})$. The proof is similar to the constructions done in lemma 3.2.

The function codes used for generating the process I_t are

```
%intWdW Approximate stochastic integrals
function ito=IntWdW(t);
N = 500; dt = t/N;
R=zeros(1,N);
for j=1:N
R(j)=boxm();
end
dW = sqrt(dt)*R;            % increments
W = cumsum(dW);             % cumulative sum
ito =0.5*(W(end)^2-t);
```

Code 4: IntWdW.m

The function 'IntWdWprocess' recall the previous one .

```
function Wp=IntWdWprocess(T,N);
 return  the process int WdW on [0,T]
 N is the number of subdivisions
 the process will be ploted
dt=T/N;
Wp = zeros(1,N);
T = zeros(1,N);
for j=1:N
T(j)=j*dt;
Wp(j)=IntWdW(T(j));
    end
plot(T,Wp);
```

Code 5: IntWdWprocess.m

Theorem 3.3. *Consider the functional* $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ *with continuous partial derivatives* $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x^i}$ *and* $\frac{\partial^2 f}{\partial x^i \partial x^j}$ *for* $i = 1, \ldots, d$. *Moreover, consider a d-dimensional Itô-Process* $(X_t)_{t\in\mathbb{T}}$, *then the following must be satisfied.*

$$\begin{aligned} f(t, X_t^1, \ldots, X_t^d) &= f(t_0, X_{t_0}^1, \ldots, X_{t_0}^d) + \int_{t_0}^t \frac{\partial f}{\partial s}(s, X_s^1, \ldots, X_s^d) ds \\ &\quad + \sum_{i=1}^d \int_{t_0}^t \frac{\partial f}{\partial x^i}(s, X_s^1, \ldots, X_s^d) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_{t_0}^t \frac{\partial^2 f}{\partial x^i \partial x^j}(s, X_s^1, \ldots, X_s^d) d < X^i, X^j >_s , \end{aligned} \tag{16}$$

where

$$dX_t^i = a^i(s, X_s) ds + \sum_{j=1}^m b^{i,j}(s, X_s) dW_s^j,$$

and

$$d < X^i, X^j >_s = \sum_{k=1}^m b^{i,k}(s, X_s) b^{j,k}(s, X_s) ds,$$

with $dW_i dW_j = \delta_{ij} dt, dW_i dt = dt dW_i = dt dt = 0$.

Proof. Similar to the one-dimensional case, only with some more complexity. ■

Theorem [7] **3.4**. (Partial integration) *For the following two one-dimensional Itô processes* $(X_t)_{t\in\mathbb{T}}$ *and* $(Y_t)_{t\in\mathbb{T}}$, *defined on the same probability space,*

$$X_t = X_0 + \int_0^t a_s^1 ds + \int_0^t b_s^1 dW_s, \qquad Y_t = Y_0 + \int_0^t a_s^2 ds + \int_0^t b_s^2 dW_s,$$

the stochastic partial integration formula is given by

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t b_s^1 b_s^2 ds. \qquad (17)$$

Example 3.3. For $X_t = Y_t = W_t$ and $a_t = 0, b_t = 1$. By applying the stochastic partial integration, we get

$d(W_t W_t) = W_s dW_s + W_s dW_s + (1)(1)ds,$

$d(W_t^2) = 2W_s dW_s + ds,$

$W_t^2 = 2\int_0^t W_s dW_s + \int_0^t 1ds.$

Thus,

$\int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t).$

Remark 3.2. Regarding the behavior of time-integral and time-differential of a Brownian motion, since this is nowhere differentiable, we use it for the time derivative, in the distributional sense, of their paths. Thus, we get in both cases a Gaussian stochastic processes. Explicitly, we may consider a finite difference approximation of ξ_t using a time interval of width t,

$\xi_{\Delta t}(t) := \frac{W_{t+\Delta t} - W_t}{\Delta t},$

then the time integral

$Z_t := \int_0^t W_s ds,$

represents the area under the path of the Brownian motion $\{W_s\}_{0 \le s \le t}$.

We may summarize the previous relationships in the diagram that follows.

$\frac{1}{2}(W_t^2 - t) = \int_0^t W_s dW_s$ $\qquad$ $\frac{d}{dt} W_t = \xi_t$ Gaussian (White Noise)

$\nwarrow \quad \nearrow$ $\qquad$ $\xi_t \sim \frac{1}{\sqrt{\Delta t}} \mathcal{N}(\prime\Leftrightarrow\infty)$

W_t

$\swarrow \quad \searrow$

BM increment $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$ $\qquad$ $\int_0^t W_s ds = Z_t$ Gaussian (Random Variable)

$\Delta W_{t_i} \sim \mathcal{N}(\prime\Leftrightarrow \sqrt{\Delta\sqcup_\rangle})$ $\qquad$ $Z_t \sim \sqrt{\frac{t^3}{3}} \mathcal{N}(\prime\Leftrightarrow\infty)$

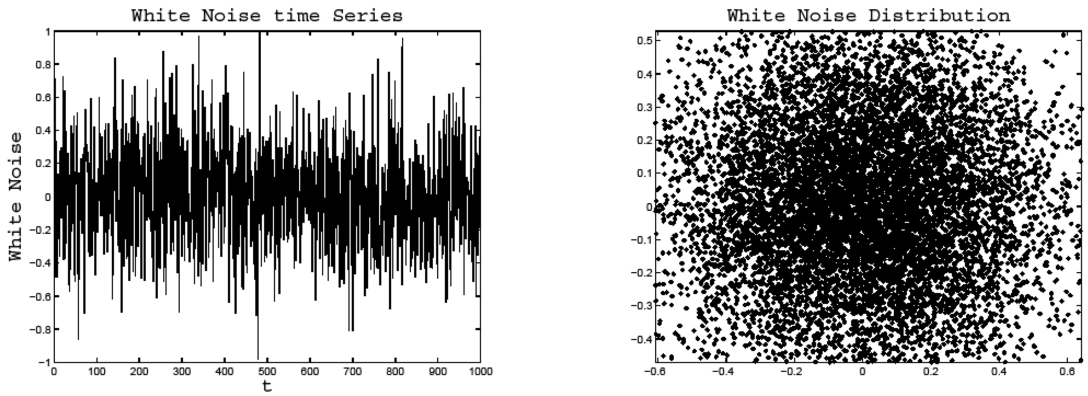


Figure 5: Example of white noise.

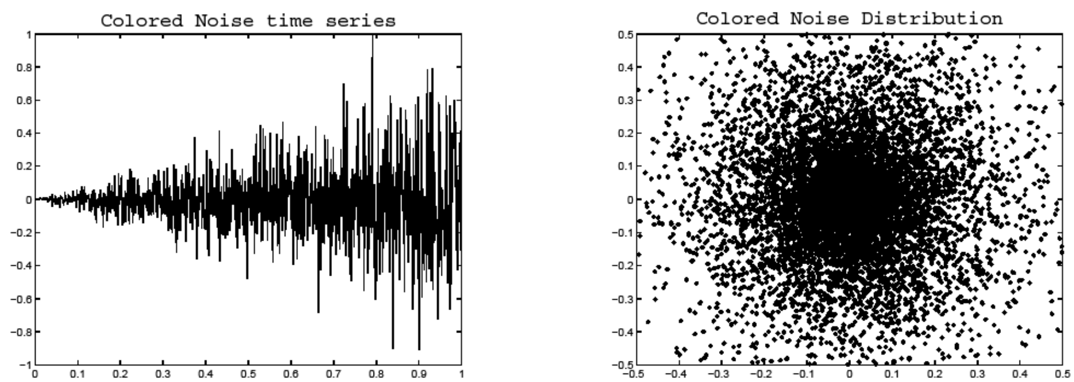


Figure 6: Example of colored noise.

The white noise as a stationary process has the following properties:
$\mathbb{E}(\xi_{\Delta t}) = 0; \quad Var(\xi_{\Delta t}) = \frac{1}{\Delta t}; \; Cov(\xi_{\Delta t}(t), \xi_{\Delta s}(s)) = 0; \quad \text{if } t \neq s,$
where $\delta_{\Delta t}(t)$) is an approximation of the following δ –function. This noise is called white whenever one talks about uncorrelated (or independent) noise at each pixel. White noise is the noise signal whose power spectrum is flat (the Fourier transform of its covariance). Otherwise the noise is called colored noise.

3.3. **Multi-indices**

In order to be able to define the multiple stochastic integrals, we introduce the following set of multi-indices. Let us consider $m \in \mathbb{N}$ and $F = \{0, 1, \ldots, m\}$. A multi-index α refers to a row vector with components in F such as $\alpha = (j_1, \ldots, j_l)$ where $j_i \in F$, for $1 \leq i \leq l$.
We denotes the size of α by $l(\alpha) := l$ and by $n(\alpha)$ the number of zero components of α. The set of all multi-indices with respect to F is represented by

$$\mathcal{M} = \bigcup_{l=1}^{\infty} F^l \cup \{v\}, \tag{18}$$

where v refers to the empty multi-index with size zero. The following example gives more sense for the definition above.

Example 3.4. For $\alpha = (1,0,2)$, $l(\alpha) = 3$ and $n(\alpha) = 1.$ While for $\alpha = (1,0,0,2,3,1,0,0), l(\alpha) = 8$ and $n(\alpha) = 4.$

Next for $l,k \in \mathbb{N}$, we define the following operations on the multi-index set.

Definition 3.3. ("$-$" operator). For $\alpha \in \mathcal{M}$ with $\alpha = (j_1,j_2,\ldots,j_l)$. For $l \geq 1$, we define $\alpha-$ and $-\alpha$ as follow:
$\alpha - := (j_1,j_2,\ldots,j_{l-1})$ and $-\alpha := (j_2,\ldots,j_l)$.
If $l(\alpha) = l > 1$ then, it implies that $l(-\alpha) = l(\alpha-) = l-1$.
If $l(\alpha) = l = 1$ then, it implies that $-\alpha = \alpha - = v$ and $l(-\alpha) = l(\alpha-) = 0$.

Definition 3.4.($\star$ operator). Let us consider $\alpha = (j_1,j_2,\ldots,j_l), \beta = (i_1,i_2,\ldots,i_k) \in \mathcal{M}$.
The operator $\star$ is defined via
$\alpha * \beta := (j_1,j_2,\ldots,j_l,i_1,i_2,\ldots,i_k)$ and $\beta * \alpha := (i_1,i_2,\ldots,i_k,j_1,j_2,\ldots,j_l)$.

Definition 3.5. ("-[i]" operator). For $\alpha = (j_1,j_2,\ldots,j_l)$ and $i \in \mathbb{N}$, the Operation "$-[i]$" represents the "i"-times application of "$-$", where the last i components should be deleted.

$$\alpha - [i] := \begin{cases} (j_1,j_2,\ldots,j_{l-i}), & \text{if } i < l, \\ v, & \text{if } i \geq l. \end{cases}$$

This yields $\alpha - [i] - [j] = \alpha - [i+j]$ for $i,j \in \mathbb{N}$.

Example 3.5. If $\alpha = (1,0,2)$, $\beta = (0,3,1)$, then we have
1) $-\alpha = (0,2)$ and $\alpha - = (1,0)$,
2) $\alpha \star \beta = (1,0,2,0,3,1)$ and $\beta \star \alpha = (0,3,1,1,0,2)$,
3) $\alpha - [1] = (1,0)$, $\alpha - [1] - [1] = \alpha - [2] = (1)$ and $(1,0,2) - [i] = v, \forall i \geq 3$.

3.4. **Multiple Itô-integrals**

Throughout the following section, all stochastic processes are defined on a probability space $(\Omega,\mathfrak{A},P)$ with right continuous augmented filtration $\mathfrak{F} = (\mathfrak{F}_t)_{t\in\mathbb{T}}$.

Definition 3.6. Define the set H of stochastic processes $(f_t)_{t\geq 0}$, which are progressively adapted to the associated filtration $\{\mathfrak{F}_t\}_{t\geq 0})$, right continuous with a left limit. Conceptively we may define the sets $H_v, H_{(0)}, H_{(1)}$ as follow
1) $H_v := \left\{ f \in H : \forall\ t \geq t_0 \quad |f(t,w)| < \infty \quad \text{a.s.} \right\}$,
2) $H_{(0)} := \left\{ f \in H : \forall\ t \geq t_0 \quad \int_{t_0}^{t} |f(s,w)| ds < \infty \quad \text{a.s.} \right\}$,
3) $H_{(1)} := \left\{ f \in H : \forall\ t \geq t_0 \quad \int_{t_0}^{t} |f(s,w)|^2 ds < \infty \quad \text{a.s.} \right\}$.

For $j \in F\backslash\{0\}$ one can set $H_{(j)} = H_{(1)}$.

Definition 3.7. Let us consider $\alpha = (j_1,j_2,\ldots,j_l)$, a multi-index and $(W_t)_{t\geq 0}$ an m-dimensional Brownian motion. For $f \in H_{(\alpha)}$, multiple Itô-Integrals are defined per recursion as follows:

$$I_\alpha[f(.)]_{t_0,t} := \begin{cases} f(t), & \text{if } l = 0 \\ \int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s}ds, & \text{if } l \geq 1 \text{ and } j_l = 0 \\ \int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s}dW_s^{j_l}, & \text{if } l \geq 1 \text{ and } j_l \geq 1. \end{cases}$$

Here $H_{(\alpha)}$ is defined per recursion as

$$H_{(\alpha)} := \{f \in H : \quad I_{(\alpha-)}[f(.)]_{t_0,.} \in H_{(j_l)}\}, \tag{19}$$

for $j_l = 0, 1, \ldots, m$ and $l \geq 2$.

Example 3.6. Clearly

$$I_{(1,2)}[f(.)]_{t_0,t} = \int_{t_0}^{t}\int_{t_0}^{s} f(z)dW_z^1 dW_s^2,$$

$$I_{(1,2,0)}[f(.)]_{t_0,t} = \int_{t_0}^{t} I_{(1,2)}[f(.)]_{0,s}ds = \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} f(s_2)dW_{s_2}^1 dW_{s_1}^2 ds.$$

In what follows, and for the sake of simplicity of notation, we shall use $I_{\alpha,t} = I_\alpha[1]_{0,t}$ and $W_t^0 = t$,
and recall the Kronecker symbol δ for $j_{i_1}, j_{i_2} = 0, 1, \ldots, l,$ satisfying

$$\delta_{j_{i_1},j_{i_2}} = \begin{cases} 1 & ,\text{if} \quad j_{i_1} = j_{i_2}, \\ 0 & ,\text{otherwise.} \end{cases}$$

Theorem 3.5. *For $l \in N$, $\alpha = (j_1, \ldots, j_l) \in M$, and $t \geq 0$, there holds*

$$I_{(j),t}I_{(\alpha),t} = \sum_{i=0}^{l} I_{(\alpha-[l-i])*(j,j_{i+1},\ldots,j_l),t} + \sum_{i=1}^{l} B_{jj_i}I_{(\alpha-[l-i+1])*(0,j_{i+1},\ldots,j_l),t}\,, \tag{20}$$

where $B_{jj_i} = \delta_{j,j_i}(1 - \delta_{0,j})$.

Proof. By using partial integration, we get

$$\begin{aligned} d(I_{(j),t}I_{\alpha,t}) &= I_{(j),t}d(I_{\alpha,t}) + I_{\alpha,t}d(I_{(j),t}) + (1 - \delta_{0,j})I_{\alpha-}dW_t^j dW_t^{j_l} \\ &= I_{(j),t}d(I_{\alpha,t}) + I_{\alpha,t}d(I_{(j),t}) + (1 - \delta_{0,j})\delta_{jj_l}I_{\alpha-,t}dt \\ &= I_{(j),t}d(I_{\alpha,t}) + I_{\alpha,t}d(I_{(j),t}) + (1 - \delta_{0,j})\delta_{jj_l}I_{\alpha-,t}dt \\ &= I_{(j),t)}I_{\alpha-,t}dW_t^{j_l} + I_{\alpha,t}d(I_{(j),t}) + B_{jj_l}I_{\alpha-,t}dt. \end{aligned}$$

For the sake of simplicity, let us define the terms $A_{\alpha,t}^j = I_{(j),t}I_{\alpha,t}$ for $\alpha \in \mathcal{M}$, to obtain

$$\begin{aligned} A_{\alpha,t}^j &= \int_0^t I_{\alpha,s}dI_{(j),s} + \int_0^t I_{(j),s}I_{\alpha-,s}dW_s^{j_l} + B_{jj_l}\int_0^t I_{\alpha-,s}ds \\ &= \int_0^t I_{\alpha,s}dW_s^j + \int_0^t A_{(\alpha-[1]),s}^j dW_s^j + B_{jj_l}I_{(\alpha-[1])*(0),t} \\ &= I_{\alpha*(j),t} + \int_0^t A_{(\alpha-[1]),s}^j dW_s^j + B_{jj_l}I_{(\alpha-[1])*(0),t}. \end{aligned}$$

By induction over l in α inside $A_{\alpha-[1],t}^j$, we may write

$$\begin{aligned} A_{\alpha,t}^j &= I_{\alpha*(j),t} + \int_0^t I_{(\alpha-[1])*(j),s_l}dW_{s_l}^{j_l} + \int_0^t\int_0^{s_l} A_{(\alpha-[2]),s_{l-1}}^j dW_{s_{l-1}}^{j_{l-1}}dW_{s_l}^{j_l} \\ &\quad + B_{jj_{l-1}}\int_0^t I_{\alpha-[2]*(0),t}dW_{s_l}^{j_l} + B_{jj_l}I_{(\alpha-[1])*(0),t} \\ &= I_{\alpha*(j),t} + I_{(\alpha-[1])*(j,j_l),t} + \int_0^t\int_0^{s_l} A_{(\alpha-[2]),s_{l-1}^j} dW_{s_{l-1}}^{j_{l-1}}dW_{s_l}^{j_l} \end{aligned}$$

$$+ B_{jj_{l-1}} I_{(\alpha-[2])*(0,j_l),t} + B_{jj_l} I_{(\alpha-[1])*(0),t}\ .$$

Apply then the same procedure with $A_{(\alpha-[2]),s^j_{l-1}}$, to obtain

$$A^j_{\alpha,t} = \sum_{i=1}^{l} I_{(\alpha-[l-i])*(j,j_{i+1},\cdots,j_l),t} + \int_0^t\int_0^{s_l}\cdots\int_0^{s_2} A^j_{(\alpha-[l]),s_1} dW^{j_1}_{s_1}\cdots dW^{j_l}_{s_l}$$
$$+\sum_{i=1}^{l} B_{jj_l} I_{(\alpha-[l-i+1])*(0,j_{i+1},\cdots,j_l),t}.$$

Note that

$$A^j_{(\alpha-[l]),s_1} = I_{(j),s_1} I_{(\alpha-[l]),j_l} = I_{(j),s_1} I_{v,s_1} = I_{(j),s_l} = \int_0^{s_1} dW^j_s. \tag{21}$$

Hence, we have

$$I_{(\alpha-[l])*(j,j_1,\cdots,j_l),t} = \int_0^t\int_0^{s_l}\cdots\int_0^{s_2} A^j_{(\alpha-[l]),s_1} dW^{j_1}_{s_1}\cdots dW^{j_l}_{l}$$
$$= \int_0^t\int_0^{s_l}\cdots\int_0^{s_2}\int_0^{s_1} dW^j_s dW^{j_1}_{s_1}\cdots dW^{j_l}_{l}. \tag{22}$$

By replacing (21) and (22) in (21), we obtain (20). Here the proof completes. ■

The following corollary gives a clear idea about an interesting class of multiple stochastic integrals

Corollary 3.1. *If $l,j \in N$ and $\alpha = (j,j,\dots,j)$ with $l(\alpha) = l$, then*

$$I_{\alpha,t}=\begin{cases} \frac{t^l}{l!}, & \text{for } j = 0,\\ \frac{1}{l}(W^j_t I_{\alpha-,t} - tI_{\alpha-[2],t}), & \text{for } j \geq 1.\end{cases}$$

Proof. From theorem 3.5 ($B_{0,0} = 0$) it follows that

$$tI_{\alpha,t} = I_{(0),t} I_{\alpha,t} = \sum_{i=0}^{l} I_{(\alpha-[l-i])*(j,j_{i+1},\cdots,j_l)} \tag{23}$$
$$= \sum_{i=0}^{l} I_{\underbrace{(0,0,\dots,0)}_{(l+1)-times}} = (l+1)\frac{t^{l+1}}{(l+1)!}.$$

The length of the multi-index $((\alpha-[l-i]) * (j,j_{i+1},\cdots,j_l))$ is determined by:

$$\begin{aligned} l((\alpha-[l-i]) * (j,j_{i+1},\cdots,j_l)) &= l(\alpha-[l-i]) + l((j,j_{i+1},\cdots,j_l))\\ &= l(\alpha-[l-i]) + l((j)) + l(j_{i+1},\cdots,j_l))\\ &= l-(l-i)+1+(l-i)\\ &= l+1.\end{aligned}$$

From (23), we get $I_{\alpha,t} = \frac{t^l}{l!}$ and for $j \geq 1$ it yields $B_{jj} = 1$. Moreover,

$$I_{(j),t} I_{\alpha-,t} = \sum_{i=0}^{l-1} I_{\underbrace{(j,\dots,j)}_{l-times}} + \sum_{i=1}^{l-1} I_{((\alpha)-[1]-[l-i+1])*(0,j_{i+1},\cdots,j_l)}$$
$$= lI_{\underbrace{(j,\dots,j)}_{l-times}} + \sum_{i=1}^{l-1} I_{\underbrace{((\alpha)-[1]-[l-i+1])*(0,j,\cdots,j)}_{size=(l-1)}}$$
$$= lI_{\underbrace{(j,\dots,j)}_{l-times}} + \sum_{i=1}^{l-1} I_{\underbrace{((\alpha)-[2]-[l-i])*(0,j,\cdots,j)}_{size=(l-1)}}. \tag{24}$$

Then invoke theorem 3.5, for $j = 0$, to write

$$I_{(0),t}I_{\alpha-[2],t} = tI_{\alpha-[2],t}$$
$$= \sum_{i=1}^{l-1} I_{\underbrace{((\alpha)-[2]-[l-i])*(0,j,\cdots,j)}_{size=(l-1)}}. \quad (25)$$

From (24) and (25) it follows that

$$I_{(j),t}I_{\alpha-,t} = lI_{\underbrace{(j,\ldots,j)}_{(l)-times}} + tI_{\alpha-[2],t}.$$

Thus

$$I_{\underbrace{(j,\ldots,j)}_{l-times}} = \frac{1}{l}(I_{(j),t}I_{\alpha-,t} - tI_{\alpha-[2],t}),$$

which ends the proof. ■

Lemma 3.5. *The multiple stochastic integrals for the special case* $\alpha = (j, j, \ldots, j) \in M$ *are*

$$I_{(j,j,j),t} = \frac{1}{3}\Big[I_{(j),t}\frac{1}{2}(I^2_{(j),t} - t) - tI_{(j),t}\Big] = \frac{1}{3!}\Big[I^3_{(j),t} - 3tI_{(j),t}\Big], \quad (26)$$
$$I_{(j,j,j,j),t} = \frac{1}{4!}\Big[I^4_{(j),t} - 6tI^2_{(j),t} + 3t^2\Big], \quad (27)$$
$$I_{(j,j,j,j,j),t} = \frac{1}{5!}\Big[I^5_{(j),t} - 10tI^3_{(j),t} + 15t^2I_{(j),t}\Big], \quad (28)$$
$$I_{(j,j,j,j,j,j),t} = \frac{1}{6!}\Big[{}^6_{(j),t} - 15tI^4_{(j),t} + 45t^2I^2_{(j),t} - 15t^3\Big], \quad (29)$$
$$I_{(j,j,j,j,j,j,j),t} = \frac{1}{7!}\Big[I^7_{(j),t} - 21tI^5_{(j),t} + 105t^2I^3_{(j),t} - 105t^3I_{(j),t}\Big]. \quad (30)$$

Proof. Note that, since $(j, j) - [2] = v$, we have

$$I_{(j,j),t} = \frac{1}{2}(I^2_{(j),t} - t) = \frac{1}{2}\Big[(W^j_t)^2 - t\Big]. \quad (31)$$

and the proof of the other multiple integrals may be achieved by invoking corollary3.1. ■

Remark 3.3. The following stochastic integral

$$I_{(j_1,j_2)t_0,t} = \int_{t_0}^{t}\int_{t_0}^{s_1} dW^{j_1}_s dW^{j_2}_{s_1},$$

where $W^{j_1}_t$ and $W^{j_2}_t$ are two independent Brownian motions can not be evaluated exactly. So approximations must be used for its estimation. Some proposed approximations may be found in literature, for instance in [4]. Here use is made of a direct expansion for the variation of the double integral. Whereas in [5], use is made of the periodicity concept of a Brownian Bridge; and Fourier series gives another method of approximation. Another useful approximation is a direct evaluation of the variation via

$$\hat{I}_{(j_1,j_2)t_0,t} = \sum_{i=0}^{m-1}\int_{t_0}^{t_{i+1}}\int_{t_0}^{t_i} dW^{j_1}_s dW^{j_2}_{s_1} = \sum_{i=0}^{m-1}(W^{j_1}_{t_i} - W^{j_1}_{t_0})(W^{j_2}_{t_{i+1}} - W^{j_2}_{t_i}), \quad (32)$$

where t^N_i is a time discretization of the time interval $[t_0, t]$.

One of the interesting open questions to pose itself here is : Can we find a more accurate formula for approximating $I_{(1,2),t}$ by using the class of multiple stochastic integrals $I_{(1,1),t}$ and $I_{(2,2),t}$? More difficult types of multiple stochastic integrals are $I_{\alpha,t}$, where at least tow components of α, j_1 and j_2 are disjunct. These integrals can not be analyzed in a similar way as for α with the same components. However, it is more difficult and interesting to establish an explicit formula for approximating their numerical behaviors.

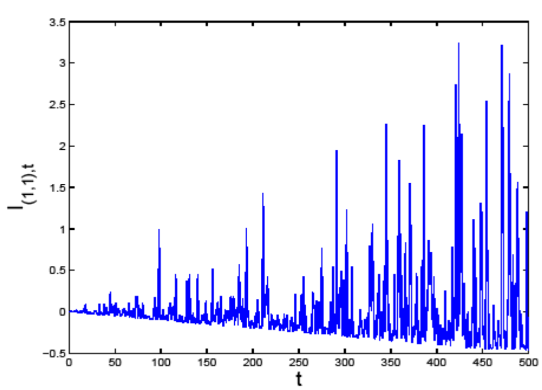
3.5. **Simulation of a class of multiple Itô-integrals**

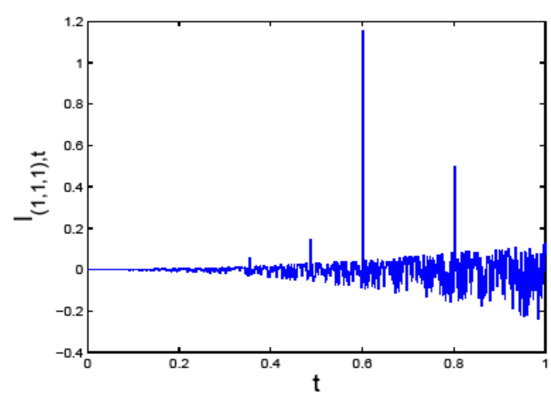
Let α be chosen according to corollary 3.1. Consider an equidistant discretization of the interval $[0,1]$. The path of the stochastic integrals are generated on t_n, and by using linear interpolation, the generated trajectory is a continuous one. The stochastic paths (26)-(30) show simulation of multiple Itô-Integrals, with step size $\Delta = \frac{1}{500}$. Obviously, the time series behavior of the multiple stochastic integrals, with $\alpha = (1,\ldots,1) \in \mathcal{M}$ have the same behavior. Furthermore, according to our simulations of these stochastic processes, they cannot have similar behavior to the Gaussian. One of the reasons for this dilemma is that for larger size of the multi-index, i.e. $l(\alpha_1) > l(\alpha_2)$, the amplitude of the corresponding time series has smaller values. This is apparent in figure 7. In all figures bellow, we observe a colored-noise behavior of the stochastic integrals I_α. Note also here the increasing time dependent amplitude.

The following code generates one value of $I_{(1,1,1),t}$ using 500 –time iterations

```
% int3dW.m Approximate stochastic integrals
% return the value of the multiple stoch. int. for (1,1,1) on [0,t]
% enter the integral upper bound t
function x=Int3dW(t);
N = 500; dt = t/N;
R=zeros(1,N);
for j=1:N
R(j)=boxm();
end
dW = sqrt(dt)*R;
W = cumsum(dW);
% compute the valute of the integral
% based on the upper bound and the end value of the BM.
r=W(end);
x =(r^3-3*t*r)/6;
```

Code 6: Int3dW.m





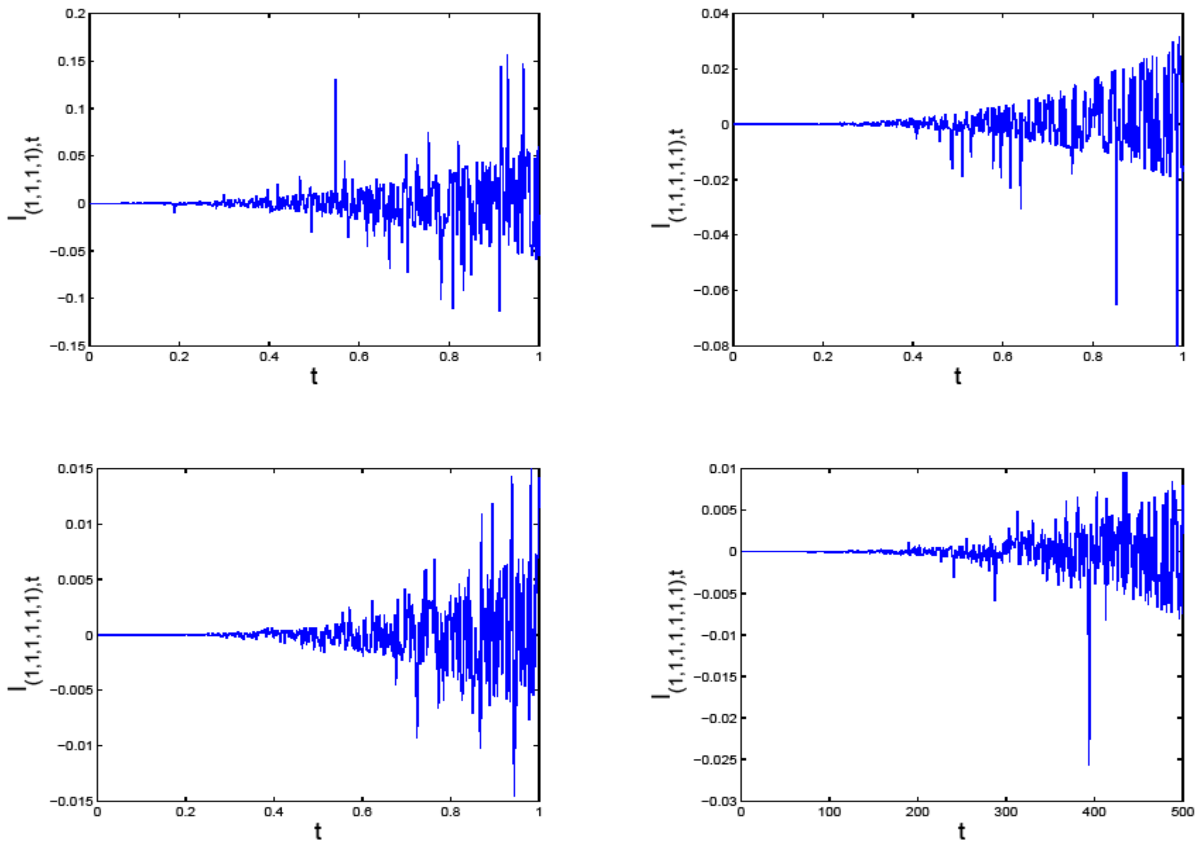


Figure7: Paths of multiple integral processes for different $\alpha \in \mathcal{M}$.

The following code generates the process $I_{(1,1,1),t}$ in N –time iterations

```
function Wp=Int3dWprocess(T,N);
dt=T/N;
Wp = zeros(1,N);
T = zeros(1,N);
for j=1:N
T(j)=j*dt;
Wp(j)=Int3dW(T(j));
    end
plot(T,Wp);
```

Code 7: Int3dWprocess.m

4. Concluding Remarks

In this contribution to the subject of computational stochastics, we have proved the recurrence relationship for the class of stochastic integrals, where the multi-index has the same components. In several examples, we have shown the graphical behavior of such processes.

Furthermore, we have introduced the reader to many techniques and open questions. The numerical approach presented here, could be employed for the treatment of many processes derived from the Brownian motion. In particular, we have simulated the time-integral and the time-differential of the Brownian motion. Consequently, we have clearly illustrated the difference between the time-behavior of the area under the path of Brownian motion and the behavior of white noise.

This paper is a semi-review of the stochastic integration, which is intended to motivate graduate students and also junior researchers in some topics of computational stochastics, which includes interesting Matlab codes. It is our intention for the future to finish similar work on SDEs and SPDEs.

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