Journal of Numerical Mathematics and Stochastics, **7** (1) : 30-47, 2015 © ***JNM@S***
http://www.jnmas.org/jnmas7-3.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

An Existence Result for Mild Solutions to Fractional Order Neutral Stochastic Integrodifferential Equations With Infinite Delay

N. AIT OUALI, and A. KANDOUCI

Laboratory of Stochastic Models, Statistics and Applications, Tahar Moulay University, POBox 138 En-Nasr, 20000 Saida, Algeria, E-mail:aitouali.nadia@gmail.com

Abstract. *The main purpose of this is paper to study the existence of mild solutions for a class of fractional neutral stochastic integrodifferential equations with infinite delay in Hilbert spaces. Using fractional calculus, Schaefer fixed point theorem and stochastic analysis techniques, under non-Lipschitz conditions, we obtain a sufficient condition for the existence result. An example is provided to illustrate the application of this result.*

Key words : Infinite Delay, Stochastic Fractional Integrodifferential Equations, Mild Solution, Fixed Point Theorem Method.

AMS Subject Classifications : 34K50, 60G22, 60H20

1. Introduction

It is well known that fractional calculus is a classical mathematical discipline, and encompasses a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. Nowadays, studying fractional-order calculus has become an active area of research([3], [9], [12], [22], [23], [26]). Much effort has been devoted to applying fractional calculus to networks control, e.g. the works of Chen et al [5], Delshad et al [6], and Wang and Zhang [21], who studied synchronization for fractional-order complex dynamical networks. Zhang et al [25] investigated a fractional order three-dimensional Hopfield neural network and pointed out that chaotic behaviors can emerge in a fractional network.

Fractional differential equations are in fact valuable tools in the modeling of many phenomena in various fields of science and engineering. Hence, they attracted the attention of many researchers (cf., e.g., [1] and [16] and references therein). On another note, integrodifferential equations arise in various applications such as viscoelasticity, heat conduction, and many other physical phenomena (cf., e.g., [13],[24] and references therein).

One of the emerging branches of this research is the theory of fractional evolution equations. These are evolution equations in which the integer derivative with respect to time is replaced by a derivative of fractional order. The increasing interest in this class of equations is motivated both by their application to problems ranging from fluid-dynamic traffic models, viscoelasticity, heat conduction in materials with memory, to electrodynamics with memory, and also because they can be employed to approach nonlinear conservation laws (see [20] and references therein). Moreover, neutral stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both science and engineering. For instance, in the theory developed by Gurtin and Pipkin [10] and Nunziato [18], for the description of heat conduction in materials with fading memory. It should be pointed out that the deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, there is a need to move from deterministic problems to stochastic ones. We mention here the recent papers , [7] ,[8] , concerning the existence of mild solutions to fractional stochastic systems.

The aim of this paper is to establish the existence of mild solutions to fractional order neutral stochastic integrodifferential equations with infinite delay of the form

$$\left.\begin{aligned} {}^cD_t^{\alpha}[x(t)+G(t,x_t)] &= -Ax(t)+f(t,x_t)+\int_{-\infty}^{t}\sigma(t,s,x_s)dW(s), && t\in J=[0,b] \\ x(t) &= \phi(t) && , \quad t\in(-\infty,0], \end{aligned}\right\} \tag{1}$$

where $0<\alpha<1$, ${}^cD^{\alpha}$ denotes the Caputo fractional derivative operator of order α. Here, $x(.)$ lives in a real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}$. The operator $-A:\mathcal{D}(-A)\subset\mathbb{H}\to\mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator $S(t), t\geq 0$, on $\mathbb{H}$. The history $x_t:(-\infty,0]\to\mathcal{C}_h$, $x_t=\{x(t+\theta),\theta\in(-\infty,0]\}$ belong to the phase space $\mathcal{C}_h$ which will be described axiomatically in Section 2.

Let $\mathbb{K}$ be another separable Hilbert space with inner product $(.,.)_{\mathbb{K}}$ and the norm $\|\cdot\|_{\mathbb{K}}$. Then suppose $\{W(t),t\geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q\geq 0$ defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$ which is generated by the Wiener process W. We are also employing the same notation $\|\cdot\|$ for the norm of $\mathcal{L}(\mathbb{K},\mathbb{H})$, where $\mathcal{L}(\mathbb{K},\mathbb{H})$ denotes the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$ .The initial data $\phi=\{\phi(t),t\in(-\infty,0]\}$ is an $\mathcal{F}_0$- measurable, $\mathcal{C}_h$-valued random variable independent of W with finite second moments, and $G:J\times\mathcal{C}_h\to\mathbb{H}$, $f:J\times\mathbb{H}\to\mathbb{H}$, $\sigma:J\times J\times\mathbb{H}\to\mathcal{L}_2^0(\mathbb{K},\mathbb{H})$ are appropriate functions where $\mathcal{L}_2^0(\mathbb{K},\mathbb{H})$ denotes the space of all Q-Hilbert Schmidt operators from $\mathbb{K}$ into $\mathbb{H}$.

The article is organized as follows. In section 2, for convenience of readers, we briefly present some basic notations and preliminaries. The existence of a mild solution to (1) by Schaefer fixed point theorem is proved in section 3. In the last section, an example is given to illustrate the obtained result.

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). An $\mathbb{H}$ valued random variable is an $\mathcal{F}$ measurable function $x(t) : \Omega \to \mathbb{H}$ and a collection of random variables $V = \{x(t,w) : \Omega \to \mathbb{H}, t \in J\}$ is called a stochastic process. Generally we just write $x(t)$ instead of $x(t,w)$ and $x(t) : J \to \mathbb{H}$ in the space of V. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of $\mathbb{K}$. Suppose that $\{W(t), t \geq 0\}$ is a cylindrical $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, and $Tr(Q) = \sum_{i=1}^{\infty} \lambda_i < \infty$, which satisfies that $Qe_i = \lambda_i e_i$, $i = 1, 2, \ldots$, and a sequence of independent Brownian motions $\{\beta_i\}_{i\geq 1}$ such that

$$(W(t), e)_{\mathbb{K}} = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\,(e_i, e)_{\mathbb{K}} \beta_i(t), \quad e \in \mathbb{K} \quad t \geq 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $\langle \varphi, \phi \rangle_{\mathcal{L}_2^0} = tr[\varphi Q \phi^*]$.

Throughout , $-A$ shall be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of uniformly bounded linear operators on $\mathbb{H}$. For the semigroup $S(t)$, there is an $M \geq 1$ such that $\|S(t)\| \leq M$. Note also that $0 \in \rho(-A)$, the resolvent set of $-A$. Then, for $\alpha \in (0,1]$, it is possible to define the fractional power operator A^α as a closed linear operator on its domain $\mathcal{D}(A^\alpha)$. Furthermore, the subspace $\mathcal{D}(A^\alpha)$ is dense in $\mathbb{H}$ and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in \mathcal{D}(A^\alpha),$$

defines a norm on $\mathbb{H}_\alpha = \mathcal{D}(A^\alpha)$. Next we list the following well known properties.

Lemma [19] **2.1**. *Under the preceding conditions,*
(i) If $0 < \beta < \alpha \leq 1$, then $H_\alpha \subset H_\beta$ and the embedding is compact whenever the resolvent operator of A is compact.
(ii) For every $\alpha \in (0,1]$, there exists a positive constant C_α such that

$$\|A^\alpha S(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad t > 0.$$

To present the abstract phase space $\mathcal{C}_h$, assume that $h : (-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^{0} h(t)dt < +\infty$, a continuous function. Recall that $\mathcal{C}_h$ is defined by $\mathcal{C}_h = \{\varphi : (-\infty, 0] \to \mathbb{H}$, for any $a > 0, (\mathbb{E}|\varphi(\theta)|^2)^{1/2}$ is bounded and measurable function on $[-a, 0]$ and $\int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}|\varphi(\theta)|^2)^{1/2} ds < +\infty\}$.

If $\mathcal{C}_h$ is endowed with the norm

$$\|\varphi\|_{\mathcal{C}_h} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}|\varphi(\theta)|^2)^{\frac{1}{2}} ds, \quad \varphi \in \mathcal{C}_h ,$$

then $(\mathcal{C}_h, \|.\|_{\mathcal{C}_h})$ is a Banach space [14].

Now, we consider the space,

$\mathcal{C}_h^{'} = \{x : (-\infty, b] \to \mathbb{H}, x_0 = \phi \in \mathcal{C}_h\}$,

to set $\|.\|_b$ as a seminorm defined by

$$\|x\|_b = \|x_0\|_{\mathcal{C}_h} + \sup_{s\in[0,b]} (E|x(s)|^2)^{\frac{1}{2}}, \; x \in \mathcal{C}_h^{'}.$$

This leads to the following useful lemma which appeared in [4] and [14].

Lemma [4] **2.2**. *Assume that* $x \in C_h^{'}$, *then for all* $t \in J$, $x_t \in C_h$,

$$l(E|x(t)|^2)^{\frac{1}{2}} \leq \|x_t\|_{\mathcal{C}_h} \leq l \sup_{s\in[0,t]} (E|x(s)|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{C}_h} ,$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$.

Let us now recall some basic definitions and results of fractional calculus.

Definition 2.1. The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^\alpha f = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds , \quad t > 0 , \quad \alpha > 0 ,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(.)$ is the gamma function.

Definition 2.2. The Caputo derivative of order α with the lower limit 0 for a function f can be written as

$$^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), t > 0, \; 0 \leq n-1 < \alpha < n .$$

If f is an abstract function with values in $\mathbb{H}$, then the integrals appearing in the above definitions are taken in Bochner's sense [15]. At the end of this section, we also recall the fixed point theorem of Schaefer, which is used to establish the existence of the mild solution to the system (1).

Lemma [11] **2.3**. *Let* $v(.), w(.) : [0,b] \to [0,\infty)$ *be continuous function. If* $w(.)$ *is nondecreasing and there exist two constants* $\theta \geq 0$ *and* $0 < \alpha < 1$ *such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \; t \in J ,$$

then

$$v(t) \leq e^{\theta^n(\Gamma(\alpha))^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha}\right)^j w(t),$$

for every $t \in [0,b]$ *and every* $n \in N$ *such that* $n\alpha > 1$ *and* $\Gamma(.)$ *is the Gamma function.*

Lemma 2.4. *Let X be a Banach space and* $\Phi : X \to X$ *be a completely continuous map. If the set*

$$U = \{x \in X : \lambda x = \Phi x \text{ for some } \lambda > 1\},$$

is bounded, then Φ *has a fixed point.*

3. Global Existence of a Mild Solution

Motivated by [8, 17], we give the following definition of a mild solution to the system (1).

Definition 3.1. An $\mathbb{H}$-valued stochastic process $\{x(t), t \in (-\infty, b]\}$ is said to be a mild solution of the system (1) if

- $x(t)$ is $\mathcal{F}_t$-adapted and measurable, $t \geq 0$.
- $x(t)$ is continuous on $[0,b]$ almost surely and for each $s \in [0,t)$, the function $(t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)$ is integrable in order that the following stochastic integral equation holds.

$$x(t) = S_\alpha(t)[\phi(0) + G(0,\phi)] - G(t,x_t) - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds$$
$$+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x_s)ds$$
$$+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau)dW(\tau)\right]ds$$

- $x(t) = \phi(t)$ on $(-\infty,0]$ satisfying $\|\phi\|^2_{\mathcal{C}_h} < \infty$, where

$$S_\alpha(t)x = \int_0^\infty \zeta_\alpha(\theta)\, S(t^\alpha\theta)xd\theta, \quad T_\alpha(t)x = \alpha\int_0^\infty \theta\zeta_\alpha(\theta)\, S(t^\alpha\theta)xd\theta\,,$$

and ζ_α is a probability density function defined on $(0,\infty)$.

The following properties of $S_\alpha(t)$ and $T_\alpha(t)$, which appeared in [26], are to be employed in this analysis.

Lemma 3.1. T*he operators* $S_\alpha(t)$ *and* $T_\alpha(t)$ *have the following properties.*

(i) For any fixed $t \geq 0$, $S_\alpha(t)$ *and* $T_\alpha(t)$ *are linear and bounded operators such that for any* $x \in H$,

$$\|S_\alpha(t)x\|_{\mathbb{H}} \leq M\|x\|_{\mathbb{H}} \ \text{and} \ \|T_\alpha(t)x\|_{\mathbb{H}} \leq \frac{M_\alpha}{\Gamma(1+\alpha)}\|x\|_{\mathbb{H}}\,.$$

(ii) $S_\alpha(t)$ *and* $T_\alpha(t)$ *are strongly continuous and compact.*

(iii) For any $x \in H$, $\beta \in (0,1)$ *and* $\eta \in (0,1]$ *we have*

$$AT_\alpha(t)x = A^{1-\beta}T_\alpha(t)A^\beta x \ \text{and} \ \|A^\eta T_\alpha(t)\| \leq \frac{\alpha C_\eta \Gamma(2-\eta)}{t^{\alpha\eta}\Gamma(1+\alpha(1-\eta))}, \quad t \in [0,b].$$

In order to obtain our existence result, we also need the following assumptions.
(H_0) : -A is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t)$ in $\mathbb{H}$, $0 \in \rho(-A)$, S(t) is compact for $t > 0$. And there exists a positive constant M such that $\|S(t)\| \leq M$.

(H_1) : The function $G : J \times \mathcal{C}_h \to \mathbb{H}$ is continuous and there exist some constants $L_G > 0$, $\beta \in (0,1)$, such that G is $\mathbb{H}_\beta$-valued and

$$E\|A^\beta G(t,x) - A^\beta G(t,y)\|^2_{\mathbb{H}} \leq L_G \|x-y\|^2_{\mathcal{C}_h}, x,y \in \mathcal{C}_h, \ t \in J ,$$

$$E\|A^\beta G(t,x)\|^2_{\mathbb{H}} \leq L_G(1 + \|x\|^2_{\mathcal{C}_h}) .$$

(H_2) : For each $\varphi \in \mathcal{C}_h$,

$$K(t) = \lim_{a\to\infty} \int_{0-a}^{0} \sigma(t,s,\varphi) dW(s)$$

exists and is continuous. Furthermore, there exists a positive constant M_k such that

$$E\|K(t)\|^2_{\mathbb{H}} \leq M_k .$$

(H_3) $f : J \times \mathcal{C}_h \to \mathbb{H}$ satisfies the following:
(i) $f(t,.) : \mathcal{C}_h \to \mathbb{H}$ is continuous for each $t \in J$ and for each $x \in \mathcal{C}_h$, $f(.,x) : J \to \mathbb{H}$ is strongly measurable,
(ii) there is a positive integrable function $P_f \in L^1([0,b])$ and a continuous nondecreasing function $\Omega_1 : [0,\infty) \to (0,\infty)$ such that for every $(t,x) \in J \times \mathcal{C}_h$, we have

$$E\|f(t,x)\|^2_{\mathbb{H}} \leq P_f(t)\Omega_1(\|x\|^2_{\mathcal{C}_h}), \quad \liminf_{r\to\infty} \frac{\Omega_1(r)}{r} ds = \Lambda < \infty .$$

(H_4) $\sigma : J \times J \times \mathcal{C}_h \to L(\mathbb{K},\mathbb{H})$ satisfies the following:
(i) for each $(t,s) \in D = J \times J \times, \sigma(t,s,.) : \mathcal{C}_h \to L(\mathbb{K},\mathbb{H})$ is continuous and for each $x \in \mathcal{C}_h$, $\sigma(.,.,x) : D \to L(\mathbb{K},\mathbb{H})$ is strongly measurable,
(ii) there is a positive integrable function $P_\sigma \in L^1([0,b])$ and a continuous nondecreasing function $\Omega_2 : [0,\infty) \to (0,\infty)$ such that for every $(t,s,x) \in J \times J \times \mathcal{C}_h$, we have

$$\int_0^t E\|\sigma(t,s,x)\|^2_{\mathcal{L}_2^0} ds \leq P_\sigma(t)\Omega_2(\|x\|^2_{\mathcal{C}_h}), \quad \liminf_{r\to\infty} \frac{\Omega_2(r)}{r} ds = \vartheta < \infty .$$

(H_5) :

$$Q_0 = 2l^2\{5\|A^{-B}\|^2 L_g\} \tag{2}$$

$$Q_1 = 2\|\phi\|^2_{\mathcal{C}_h} + 2l^2\bar{H} \tag{3}$$

$$Q_2 = 10l^2 \frac{\alpha^2 C^2_{1-\beta}\Gamma^2{}_{(1+\beta)}}{\Gamma^2{}_{(1+\alpha\beta)}} \frac{L_G b^{\alpha\beta}}{\alpha\beta} \tag{4}$$

$$Q_3 = 10bl^2\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2} \tag{5}$$

$$Q_4 = 20bl^2\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2}Tr(Q) \tag{6}$$

$$N_1 = \frac{Q_1}{1-Q_0}, \quad N_2 = \frac{Q_2}{1-Q_0}, \quad N_3 = \frac{Q_3}{1-Q_0}, \quad N_4 = \frac{Q_4}{1-Q_0} \tag{7}$$

$$\begin{aligned}\bar{H} &= 10M^2(C_1+C_2)+5\|A^{-B}\|^2L_g \\ &+ 5\frac{\alpha^2C^2_{1-\beta}\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{L_Gb^{2\alpha\beta}}{\alpha\beta^2}+10b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2}M_k\end{aligned} \tag{8}$$

(H_6) :

$$\int_0^b \pi(s)ds \leq \int_{C_0N_1}^{\infty}\frac{ds}{\Omega_1(s)+\Omega_2(s)},$$

where

$$\pi(t) = \max\{C_0N_2t^{\alpha-1}P_f(t), C_0N_3t^{\alpha-1}P_\sigma(t)\}.$$

The main objective of this paper is to explain and prove the following result.

Theorem 3.1. *Assume that the assumptions* $(H_0)-(H_5)$ *hold. Then there exists a mild solution to the system* (1).

Proof. We transform the problem (1) into a fixed point problem, starting with considering the map $\mathcal{D} : \mathcal{C}_h' \to \mathcal{C}_h'$ defined by

$$(\mathcal{D}x)(t) = \begin{cases}\phi(t), & t \in (-\infty,0] \\ S_\alpha(t)[\phi(0)+G(0,\phi)] - G(t,x_t) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x_s)ds & \\ \quad - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds & \\ \quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau)dW(\tau)\right]ds, & t \in J.\end{cases} \tag{9}$$

By virtue of lemma 3.1, it follows that

$$E\left\|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds\right\|_{\mathbb{H}}^2$$
$$\leq E\Big[\int_0^t \|(t-s)^{\alpha-1}A^{1-\beta}T_\alpha(t-s)A^\beta G(s,x_s)\|_{\mathbb{H}}ds\Big]^2$$
$$\leq \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} E\Big[\int_0^t \|(t-s)^{\alpha\beta-1}A^\beta G(s,x_s)\|_{\mathbb{H}}ds\Big]^2.$$

Next apply the Hölder inequality with assumption (H_1) to establish that

$$E\left\|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds\right\|_{\mathbb{H}}^2$$
$$\leq \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \int_0^t (t-s)^{\alpha\beta-1}ds \int_0^t (t-s)^{\alpha\beta-1}E\|A^\beta G(s,x_s)\|_{\mathbb{H}}^2 ds$$
$$\leq \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{b^{\alpha\beta}}{\alpha\beta} \int_0^t (t-s)^{\alpha\beta-1}E\|A^\beta G(s,x_s)\|_{\mathbb{H}}^2 ds$$
$$\leq \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{L_G b^{\alpha\beta}}{\alpha\beta} \int_0^t (t-s)^{\alpha\beta-1}(1+\|x_s\|_{\mathcal{C}_h}^2)ds\,,$$

which indicates that $(t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)$ is Bochner integrable on J (see [15] and lemma 2.2).

We shall show then that $\mathcal{D}$ has a fixed point, which is then a mild solution to the system (1). For $\phi\in\mathcal{C}_h$, define

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & , t\in(-\infty,0] \\ S_\alpha(t)\phi(0) & , t\in J\,. \end{cases} \tag{10}$$

Then $\tilde{\phi}\in\mathcal{C}_h'$. Next let $x(t)=\tilde{\phi}(t)+z(t)$, $-\infty<t\leq b$. It is easy to see that x satisfies (1) if and only if z satisfies $z_0=0$ and

$$z(t) = S_\alpha(t)G(0,\phi) - G(t,\tilde{\phi}_t+z_t) - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,\tilde{\phi}_s+z_s)ds$$
$$+\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s+z_s)ds$$
$$+\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\Big[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)\Big]ds\,.$$

If

$$\mathcal{C}_h'' = \{z\in\mathcal{C}_h',\ z_0=0\in\mathcal{C}_h\}\,,$$

then for any $z\in\mathcal{C}_h''$, we have

$$\|z\|_b = \|z_0\|_{\mathcal{C}_h} + \sup_{s\in[0,b]}(E\|z(s)\|^2)^{\frac{1}{2}} = \sup_{s\in[0,b]}(E\|z(s)\|^2)^{\frac{1}{2}}.$$

Thus $(\mathcal{C}_h'', \|.\|_b)$ is a Banach space. Now set

$$B_q = \{y \in \mathcal{C}_h'', \|y\|_b^2 \leq q\}, \text{ for some } q \geq 0,$$

to observe that $B_q \subset \mathcal{C}_h''$ is uniformly bounded.
Moreover, for $z \in B_q$, by lemma 3.1, we have

$$\begin{aligned}\|z_t + \tilde{\phi}_t\|_{\mathcal{C}_h}^2 &\leq 2(\|z_t\|_{\mathcal{C}_h}^2 + \|\tilde{\phi}_t\|_{\mathcal{C}_h}^2)\\ &\leq 4(l^2 \sup_{s\in[0,t]} E\|z(s)\|^2 + \|z_0\|_{\mathcal{C}_h}^2 + l^2 \sup_{s\in[0,t]} E\|\tilde{\phi}(s)\|^2 + \|\tilde{\phi}_0\|_{\mathcal{C}_h}^2)\\ &\leq 4l^2(q + M^2 E\|\phi(0)\|_{\mathbb{H}}^2) + 4\|\phi\|_{\mathcal{C}_h}^2\\ &= \grave{q}\,.\end{aligned}$$

Define the operator $\Pi : C_h'' \to C_h''$ by

$$(\Pi z)(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ S_\alpha(t)G(0,\phi) - G(t, \tilde{\phi}_t + z_t) - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) G(s, \tilde{\phi}_s + z_s) ds & \\ \quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds & \\ \quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau) dW(\tau) \right] ds, & t \in J\,. \end{cases}$$

And observe that Π is well defined on B_q for each $q > 0$.
Obviously, the operator $\mathcal{D}$ has a fixed point if and only if Π has a fixed point.

Then we shall prove, in 3 steps, that the operator Π is a completely continuous operator.

Step 1$\circ$

We first show that Π maps B_q into an equicontinuous family. Let $z \in B_q$ and $t_1, t_2 \in J$ and $\epsilon > 0$. Then if $0 < \epsilon < t_1 < t_2 < b$ and

$$\begin{aligned}&E\|(\Pi z)(t_1) - (\Pi z)(t_2)\|_{\mathbb{H}}^2\\ &\leq 5\|S_\alpha(t_1) - S_\alpha(t_2)\|_{\mathbb{H}}^2 E\|G(0,\phi)\|_{\mathbb{H}}^2\\ &+ 5E\|G(t_1, \tilde{\phi}_{t_1} + z_{t_1}) - G(t_2, \tilde{\phi}_{t_2} + z_{t_2})\|_{\mathbb{H}}^2\\ &+ 15E\left\| \int_0^{t_1-\epsilon} [(t_1-s)^{\alpha-1} A T_\alpha(t_1-s) - (t_2-s)^{\alpha-1} A T_\alpha(t_2-s)] G(s, \tilde{\phi}_s + z_s) ds \right\|_{\mathbb{H}}^2\\ &+ 15E\left\| \int_{t_1-\epsilon}^{t_1} [(t_1-s)^{\alpha-1} A T_\alpha(t_1-s) - (t_2-s)^{\alpha-1} A T_\alpha(t_2-s)] G(s, \tilde{\phi}_s + z_s) ds \right\|_{\mathbb{H}}^2\\ &+ 15E\left\| \int_{t_1}^{t_2} [(t_2-s)^{\alpha-1} A T_\alpha(t_2-s)] G(s, \tilde{\phi}_s + z_s) ds \right\|_{\mathbb{H}}^2\end{aligned}$$

$$+15E\left\|\int_0^{t_1-\epsilon}[(t_1-s)^{\alpha-1}T_\alpha(t_1-s)-(t_2-s)^{\alpha-1}T_\alpha(t_2-s)]f(s,\tilde{\phi}_s+z_s)ds\right\|^2_{\mathbb{H}}$$
$$+15E\left\|\int_{t_1-\epsilon}^{t_1}[(t_1-s)^{\alpha-1}T_\alpha(t_1-s)-(t_2-s)^{\alpha-1}T_\alpha(t_2-s)]f(s,\tilde{\phi}_s+z_s)ds\right\|^2_{\mathbb{H}}$$
$$+15E\left\|\int_{t_1}^{t_2}[(t_2-s)^{\alpha-1}T_\alpha(t_2-s)]f(s,\tilde{\phi}_s+z_s)ds\right\|^2_{\mathbb{H}}$$
$$+15E\left\|\int_0^{t_1-\epsilon}[(t_1-s)^{\alpha-1}T_\alpha(t_1-s)-(t_2-s)^{\alpha-1}T_\alpha(t_2-s)]\left[\int_{-\infty}^{s}\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)\right]ds\right\|^2_{\mathbb{H}}$$
$$+15E\left\|\int_{t_1-\epsilon}^{t_2}[(t_1-s)^{\alpha-1}T_\alpha(t_1-s)-(t_2-s)^{\alpha-1}T_\alpha(t_2-s)]\left[\int_{-\infty}^{s}\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)\right]ds\right\|^2_{\mathbb{H}}$$
$$+15E\left\|\int_{t_1}^{t_2}[(t_2-s)^{\alpha-1}T_\alpha(t_2-s)]\left[\int_{-\infty}^{s}\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)\right]ds\right\|^2_{\mathbb{H}}.$$

Therefore

$$E\|(\Pi z)(t_1)-(\Pi z)(t_2)\|^2_{\mathbb{H}}$$
$$\leq 5\|S_\alpha(t_1)-S_\alpha(t_2)\|^2_{\mathbb{H}}E\|G(0,\phi)\|^2_{\mathbb{H}}$$
$$+5\|A^{-B}\|^2L_G\|(z_{t_1}-z_{t_2})\|^2_{\mathcal{C}_h}$$
$$+15\int_0^{t_1-\epsilon}\|[t_1-s)^{\alpha-1}A^{1-B}T_\alpha(t_1-s)-(t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)]\|^2_{\mathbb{H}}L_G(1+\|\tilde{\phi}_s+z_s\|^2_{\mathcal{C}_h})ds$$
$$+15\int_{t_1-\epsilon}^{t_1}\|[t_1-s)^{\alpha-1}A^{1-B}T_\alpha(t_1-s)-(t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)]\|^2_{\mathbb{H}}L_G(1+\|\tilde{\phi}_s+z_s\|^2_{\mathcal{C}_h})ds$$
$$+15\int_{t_1}^{t_2}\|[(t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)]\|^2_{\mathbb{H}}L_G(1+\|\tilde{\phi}_s+z_s\|^2_{\mathcal{C}_h})ds$$
$$+15b\int_0^{t_1-\epsilon}\|[t_1-s)^{\alpha-1}T_\alpha(t_1-s)-(t_2-s)^{\alpha-1}T_\alpha(t_2-s)]\|^2_{\mathbb{H}}P_f(s)\Omega_1(\|\tilde{\phi}_s+z_s^q\|^2_{\mathcal{C}_h})ds$$
$$+15b\int_{t_1-\epsilon}^{t_1}\|[t_1-s)^{\alpha-1}T_\alpha(t_1-s)-(t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)]\|^2_{\mathbb{H}}P_f(s)\Omega_1(\|\tilde{\phi}_s+z_s^q\|^2_{\mathcal{C}_h})ds$$
$$+15b\int_{t_1}^{t_2}\|[(t_2-s)^{\alpha-1}T_\alpha(t_2-s)]\|^2_{\mathbb{H}}P_f(s)\Omega_1(\|\tilde{\phi}_s+z_s^q\|^2_{\mathcal{C}_h})ds$$
$$+15b\int_0^{t_1-\epsilon}\|[t_1-s)^{\alpha-1}T_\alpha(t_1-s)-(t_2-s)^{\alpha-1}T_\alpha(t_2-s)]\|^2_{\mathbb{H}}(2M_k+2Tr(Q)P_\sigma(s)$$
$$\Omega_2(\|\tilde{\phi}_s+z_s^q\|^2_{\mathcal{C}_h})ds$$
$$+15b\int_{t_1-\epsilon}^{t_1}\|[t_1-s)^{\alpha-1}AT_\alpha(t_1-s)-(t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)]\|^2_{\mathbb{H}}$$

$$(2M_k + 2Tr(Q)P_\sigma(s)\Omega_2(\|\tilde{\phi}_s + z_s^q\|^2_{\mathcal{C}_h})ds$$

$$+ 15b\int_{t_1}^{t_2}\|[(t_2-s)^{\alpha-1}AT_\alpha(t_2-s)]\|^2_{\mathbb{H}}(2M_k + 2Tr(Q)P_\sigma(s)\Omega_2(\|\tilde{\phi}_s + z_s^q\|^2_{\mathcal{C}_h})ds\,.$$

The right hand side is independent of $z \in B_q$ and tends to zero as $t_2 - t_1 \to 0$ and ϵ sufficiently small, since the compactness of $S_\alpha(t)$ and $T_\alpha(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

Thus Π maps B_q into an equicontinuous family of functions. Clearly the family B_q is uniformly bounded.

Step 2$\circ$

Next, we show that $\overline{\Pi B_q}$ is compact. Since we have shown that ΠB_q is an equicontinuous collection, it suffices by *Arzela – Ascoli* theorem to show that Π maps B_q into a precompact set in $\mathbb{H}$.

Let $0 \le t \le b$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_q$, we define

$$\begin{aligned}
(\Pi z_\epsilon)(t) &= S_\alpha(t)G(0,\phi) - G(t,\tilde{\phi}_t + z_t) - \int_0^{t-\epsilon}(t-s)^{\alpha-1}AT_\alpha(t-s)G(s,\tilde{\phi}_s + z_s)ds \\
&+ \int_0^{t-\epsilon}(t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s + z_s)ds \\
&+ \int_0^{t-\epsilon}(t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^{s}\sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)\right]ds \\
&= S_\alpha(t)G(0,\phi) - G(t,\tilde{\phi}_t + z_t) - T_\alpha(\epsilon)\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}AT_\alpha(t-s-\epsilon)G(s,\tilde{\phi}_s + z_s)ds \\
&+ T_\alpha(\epsilon)\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T_\alpha(t-s-\epsilon)f(s,\tilde{\phi}_s + z_s)ds \\
&+ T_\alpha(\epsilon)\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T_\alpha(t-s-\epsilon)\left[\int_{-\infty}^{s}\sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)\right]ds.
\end{aligned}$$

Since $S_\alpha(t)$ and $T_\alpha(t)$ are compact, the set $V_\epsilon(t) = \{(\Pi_\epsilon z)(t) : z \in B_q\}$ is precompact in $\mathbb{H}$, for every ϵ; $0 < \epsilon < t$. Moreover, for every $z \in B_q$ we have

$$\begin{aligned}
E\|(\Pi z)(t) - (\Pi_\epsilon)(t)z)\|^2 &\le 3\frac{\alpha^2 C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{L_G b^{\alpha\beta}}{\alpha\beta}\int_{t-\epsilon}^{t}(t-s)^{\alpha\beta-1}(1+\dot{q})ds \\
&+ 3b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}P_f(s)\Omega_1(\dot{q})ds \\
&+ 3b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}(2M_k + 2Tr(Q)P_\sigma(s)\Omega_2(\dot{q}))ds.
\end{aligned}$$

Therefore,

$$E\|(\Pi z)(t) - (\Pi_\epsilon z)(t))\|^2 \to 0\,,\ as\ \epsilon \to 0\,,$$

and there are precompact sets arbitrarily close to the set $\{(\Pi z)(t) : z \in B_q\}$ Thus, the set $\{(\Pi_\epsilon z)(t) : z \in B_q\}$ is precompact in $\mathbb{H}$.

Step 3∘

It remains to show that $\Pi : \mathcal{C}_h^{''} \to \mathcal{C}_h^{''}$ is continuous. Let $\{z^n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{C}_h^{''}$ such that $z^n \to z$ in $\mathcal{C}_h^{''}$. Then, there is a number $q \geq 0$ such that $|z^{(n)}(t)| \leq q$ for all n and a.e. $t \in J$, so $z^{(n)} \in B_q$ and $z \in B_q$. Clearly

$$A^\beta G(t, z_t^{(n)} + \tilde{\phi}_t) \to A^\beta G(t, z_t + \tilde{\phi}_t),$$

$$f(t, z_t^{(n)} + \tilde{\phi}_t) \to f(t, z_t + \tilde{\phi}_t),$$

$$\sigma(s,\tau, z_\tau^{(n)} + \tilde{\phi}_\tau) \to \int_0^t \sigma(s,\tau, z_\tau + \tilde{\phi}_\tau),$$

for $t \in J$, and since

$$E\left\| [A^B G(t, z_s^{(n)}) - A^B G(t, z_t)] \right\|^2 \leq 2\alpha_{q'}(t),$$

$$E\left\| [f(t, z_s^{(n)}) - f(t, z_t)] \right\|^2 \leq 2P_f(t)\Omega_1(q'),$$

$$E\left\| [\sigma(s,\tau, z_\tau^{(n)}) - \sigma(s,\tau, z_\tau^{(n)})] \right\|^2 \leq 2P_\sigma(t)\Omega_2(q'),$$

then by the dominated convergence theorem, it follows that

$$\begin{aligned}
E\left\| \Pi z_t^{(n)} - \Pi z_t \right\|^2 \leq{}& 4\sup_{t\in J} E\left\| [G(t,z_t) - G(t, z_t^{(n)})] \right\|^2 \\
&+ 4\sup_{t\in J} E\left\| \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s)[G(t,z_s) - G(t, z_s^{(n)})]ds \right\|^2 \\
&+ 4b\sup_{t\in J} E\left\| \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[f(t, z_s^{(n)})) - f(t,z_s)]ds \right\|^2 \\
&+ 4b\sup_{t\in J} E\left\| \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\left[\int_{-\infty}^s [\sigma(s,\tau, z_\tau^{(n)}) - \sigma(s,\tau,z_\tau) dw(\tau)]\right]ds \right\|^2 \\
&\leq 4\|A^{-B}\|^2 E\left\| [A^B G(t, z_s^{(n)}) - A^B G(t,z_t)] \right\|^2 \\
&+ 4\frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{b^{2\alpha\beta}}{(\alpha\beta)^2} \int_0^t E\left\| [A^B G(s, z_s^{(n)}) - A^B G(s,z_s)] \right\|^2 ds \\
&+ 4b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2} \int_0^t E\left\| [f(t, z_s^{(n)}) - f(t,z_s)] \right\|^2 ds \\
&+ 4b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2} \int_0^t E\left\| \left[\int_{-\infty}^s [\sigma(s,\tau, z_\tau^{(n)}) - \sigma(s,\tau,z_\tau)]\right] dw(\tau) \right\|^2 ds \\
&\to 0 \text{ as } n \to \infty .
\end{aligned}$$

Thus Π is continuous. This completes the proof that Π is completely continuous.

Now, we shall prove that the set

$U = \{x \in \mathcal{C}_h' : \lambda x = \Pi x \textit{ for some } \lambda > 1\}$

is bounded.

Let $x \in U$. Then $\lambda x = \Pi x$ for some $\lambda > 1$, and

$$\begin{aligned} x(t) &= \lambda^{-1}(S_\alpha(t)[\phi(0) + G(0,\phi)]) - \lambda^{-1}G(t,x_t) \\ &- \lambda^{-1}\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds \\ &+ \lambda^{-1}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x_s)ds \\ &+ \lambda^{-1}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau)dW(\tau)\right]ds \,. \end{aligned}$$

$$\begin{aligned} E\|x(t)\|^2 &\leq 5E\|S_\alpha(t)(\phi(0)+G(0,\phi))\|_{\mathbb{H}}^2 + 5\|G(t,x_t)\|_{\mathbb{H}}^2 \\ &+ 5E\left\|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds\right\|_{\mathbb{H}}^2 \\ &+ 5E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x_s)ds\right\|_{\mathbb{H}}^2 \\ &+ 5E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau)dW(\tau)\right]ds\right\|_{\mathbb{H}}^2 \\ &\leq 10M^2(C_1+C_2) + 5\|A^{-B}\|^2 L_g(\|x\|_{\mathcal{C}_h}^2 + 1) \\ &+ 5\frac{\alpha^2 C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{L_G b^{\alpha\beta}}{\alpha\beta}\int_0^t (t-s)^{\alpha\beta-1}(1+\|x_s\|_{\mathcal{C}_h}^2)ds \\ &+ 5b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}P_f(s)\Omega_1(\|x_s\|_{\mathcal{C}_h}^2)ds \\ &+ 5b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}(2M_k + 2Tr(Q)P_\sigma(s)\Omega_2(\|x_s\|_{\mathcal{C}_h}^2)ds \,. \end{aligned}$$

Observe then that the function μ, defined by

$\mu(t) = \sup\{E\|x(s)\|^2, 0 \leq s \leq t\}, 0 \leq t \leq b$,

is bounded.

From Lemma 2.2 and the above inequality we have

$$E\|x(t)\|^2 = 2\|\phi\|_{\mathcal{C}_h}^2 + 2l^2 \sup_{0\leq s\leq t}(E\|x(s)\|^2) \,.$$

Therefore

$$\mu(t) \leq 2\|\phi\|^2_{\mathcal{C}_h} + 2l^2\{\bar{H} + 5\|A^{-B}\|^2 L_g \mu(t) + 5\frac{\alpha^2 C^2_{1-\beta}\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{L_G b^{\alpha\beta}}{\alpha\beta}\int_0^t (t-s)^{\alpha\beta-1}\mu(s)ds$$
$$+ 5b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1} P_f(s)\Omega_1(\mu(s))ds$$
$$+ 10b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1} Tr(Q)P_\sigma(s)\Omega_2(\mu(s))ds\},$$

where $\bar{H}$ is defined in (8). Thus, we have

$$\mu(t) \leq N_1 + N_2\int_0^t \frac{\mu(s)}{(t-s)^{1-\alpha\beta}}ds + N_3\int_0^t P_f(s)\Omega_1(\mu(s))ds$$
$$+ N_4\int_0^t P_\sigma(s)\Omega_2(\mu(s))ds\ ,$$

where N_1, N_2, N_3, N_4 are given in (7). By lemma 2.3, we have

$$\mu(t) \leq C_0(N_1 + N_3\int_0^t P_f(s)\Omega_1(\mu(s))ds + N_4\int_0^t P_\sigma(s)\Omega_2(\mu(s))ds)\ ,$$

where

$$C_0 = e^{N_2^n(\Gamma(\alpha\beta))^n b^{n\alpha\beta}/\Gamma(n\alpha\beta)}\sum_{j=0}^{n-1}\left(\frac{N_2 b^{\alpha\beta}}{\alpha\beta}\right)^j .$$

Denote then by $v(t)$ the right hand side of the last inequality to write $v(0) = C_0 N_1$ and

$$\dot{v}(t) \leq C_0(N_3 P_f(t)\Omega_1(\mu(t)) + N_4 P_\sigma(t)\Omega_2(\mu(t)))\ ,$$
$$\dot{v}(t) \leq C_0(N_3 P_f(t)\Omega_1(v(t)) + N_4 P_\sigma(t)\Omega_2(v(t)))\ .$$

Or equivalently by (H_6), we have

$$\int_{v(0)}^{v(t)} \frac{ds}{\Omega_1(s)+\Omega_2(s)} \leq \int_0^b \pi(s)ds < \int_{C_0N_1}^{\infty} \frac{ds}{\Omega_1(s)+\Omega_2(s)}\ .$$

This inequality implies that there is a constant K such that $v(t) \leq K$, $t \in J$ and hence $\mu(t) \leq K$, $t \in J$. Furthermore, we get $\|x_t\|^2_{\mathcal{C}_h} \leq \mu(t) \leq v(t) \leq K$, $t \in J$.

As a consequence of lemma 2.4 we deduce that Π has a fixed point, which is a mild solution of (1). Here the proof ends. ■

4. Illustrative Example

Consider the following fractional neutral stochastic partial differential equation with infinite delay of the form:

$$\begin{cases} {}^{c}D_t^{\alpha}[u(t,x) - G(t,u(t-h,x))] = \dfrac{\partial^2}{\partial x^2}u(t,x) + f(t,u(t-h,x)) \\ + \displaystyle\int_{-\infty}^{t} \sigma(s,u(s-h,x))dW(s) \ , \ 0 \le x \le \pi, h > 0 \ , \ t \in J = [0,b], \\ u(t,0) = u(t,\pi) = 0, \qquad t \in [0,b], \\ u(t,x) = \phi(t,x), \qquad t \in (-\infty,0], \end{cases} \tag{11}$$

where $\alpha \in (0,1)$, and $W(t)$ is a standard cylindrical Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

To rewrite this system into the abstract form (1), let $\mathbb{H} = L^2([0,\pi])$ with the norm $\|.\|$, and define $A : \mathbb{H} \to \mathbb{H}$ by $A(t)Z = z''$ with the domain

$\mathcal{D}(A) = \left\{x(.) \ \in \mathbb{H} : x, x' \textit{are absolutely continuous}, x'' \in \mathbb{H}, x(0) = x(\pi) = 0\right\}$.

Then A generates a symmetric C_0-semigroup e^{-tA} in $\mathbb{H}\not\mathbb{H}$ and there exists a complete orthonormal set $\{z_n, n = 1,2,\dots\}$ of eigenvectors of A with

$z_n(s) = \sqrt{\frac{2}{\pi}}\, sin(ns), n = 1,2,\dots..$

Moreover, the operator $A^{-\frac{1}{2}}$ is given by

$$A^{-\frac{1}{2}}\zeta = \sum_{n=1}^{\infty} n\langle\zeta, z_n\rangle z_n$$

on the space $\mathcal{D}(A^{-\frac{1}{2}}) = \{\zeta(.) \ \in \mathbb{H} : \sum_{n=1}^{\infty} n\langle\zeta, z_n\rangle z_n \in \mathbb{H}\}$.

Now, we consider a special $\mathcal{C}_h$ space. Let $h(s) = e^{2s}, s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2}$. Also let

$$A\|\varphi\|_{\mathcal{C}_h} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} E\Big(\|\varphi(\theta)\|^2\Big)^{\frac{1}{2}} ds \ .$$

Then $(\mathcal{C}_h, \|.\|_{\mathcal{C}_h})$ is a Banach space.

Hence, for $(t,\varphi) \in [0,b] \times \mathcal{C}_h$, where $\varphi(\theta)(\zeta) = \phi(\theta,\zeta), (\theta,\zeta) \in (-\infty,0] \times [0,\pi]$. Next, we may set $u(t)(\zeta) = u(t,\zeta)$, and define the functions $G, f : J \times \mathcal{C}_h \to \mathbb{H}$, $\sigma : J \times \mathcal{C}_h \to \mathcal{L}_2^0(\mathbb{H},\mathbb{H})$ for the infinite delay as follows:

$$(-A)^{\frac{1}{2}} G(t,\varphi)(x) = \int_{-\infty}^{0} \mu_1(\theta)\varphi(\theta)(x)d\theta \ ,$$

$$f(t,\varphi)(x) = \int_{-\infty}^{0} \mu_2(t,x,\theta)G_1(\varphi(\theta)(x))d\theta \ ,$$

$$\sigma(t,\varphi)(x) = \int_{-\infty}^{0} \mu_3(t,x,\theta)G_2(\varphi(\theta)(x))d\theta \ .$$

At this point, we can possibly impose some hypotheses on $\mu_i, i = 1,2,3$, and G_k, k = 1, 2 (see [2]), to satisfy the assumptions stated in theorem 3.1 here. Thus, there exists a mild solution to the system (11).

References

[1] R. P. Agarwal, B. Andrade, and C. Cuevas, On type of periodicity and ergodicity to a class of fractional order differential equations, *Advances in Difference Equations* **2010** ,(2010), Article ID 179750, 25 pages.

[2] P. Balasubramaniam, and S. K. Ntouyas, Controllability for neutral stochastic functional differential inclusions with infinite delay in abstract space, *Journal of Mathematical Analysis and Applications* **324**, (2006), 161-76.

[3] M. Benchohra, J. Henderson, S. K. Ntouyas, and A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *Journal of Mathematical Analysis and Applications* **338**(**2**), (2008), 1340-1350.

[4] Y. K. Chang, Controllability of impulsive functional differential systems with infinite delay in Banach spaces, *Chaos Solitons and Fractals* **33**, 2007), 1601-1609.

[5] L. Chen, Y. Chai, R.Wu, J. Sun, and T. Ma, Cluster synchronization in fractional-order complex dynamical networks, *Physics Letters A* **376**, (2010), 2381-2388.

[6] S. Delshad, M. Asheghan, and M. Beheshti, Synchronization of N-coupled incommensurate fractional-order chaotic systems with ring connection, *Communications on Nonlinear Science and Numerical Simulation* **16**, (2011), 3815-3824.

[7] M. M. El Borai, K. E. El Nadi, O. L. Mostafa, and H. M. Ahmed,Volterra equations with fractional stochastic integrals, *Mathematical Problems in Engineering* **5**, (2004), 453-468.

[8] M. M. El Borai, On some stochastic fractional integro-differential equations, *Advances in Dynamical Systems and Applications* **1**, (2006), 49-57.

[9] C. S. Goodrich, Existence of a positive solution to systems of differential equations of fractional order, *Computers and Mathematics with Applications* **63**, (2011), 1251-1268.

[10] M. E. Gurtin, and A. C. Pipkin, A general theory of heat conduction with finite wave speed, *Archive for Rational Mechanics and Analysis* **31**, (1968), 113-126.

[11] E. Hernandez, Existence results for partial neutral functional integro-differential equations with unbounded delay, *Journal of Mathematical Analysis and Applications* **292** , (2004), 194-210.

[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional*

Differential Equations, North-Holland/Elsevier, Amsterdam, 2006.

[13] J. Liang, T. J. Xiao, and J. van Casteren, A note on semilinear abstract functional differential and integro-differential equations with infinite delay, *Applied Mathematics Letters* **17**(4) ,(2004), 473-477.

[14] Y. Li, and B. Liu, Existence of solution of nonlinear neutral stochastic differential inclusions with infinite delay, *Stochastic Analysis and Applications* **25**, (2007), 397-415.

[15] C. M. Marle, *Measures et Probabilités,* Paris, Hermann, 1974.

[16] G. M. Mophou, and G. M. N'Guérékata, Existence of the mild solution for some fractional differential equations with nonlocal conditions, *Semigroup Forum* **79**,(2009), 315-322.

[17] G. M. Mophou, and G. M. N'Guérékata, A note on a semilinear fractional differential equations of neutral type with infinite delay, *Advances in Difference Equations* **2010**, (2010), 674630.

[18] J. W. Nunziato, On heat conduction in materials with memory, *Quarterly of Applied Mathematics* **29**, (1971), 187-204.

[19] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Applied Mathematical Sciences **44**, Springer, New York,1983.

[20] K. Sobczyk, *Stochastic Differential Equations with Applications to Physics and Engineering,* Klüwer Academic Publishers, London, 1991.

[21] J. Wang, and Y. Zhang, Network synchronization in a population of star-coupled fractional nonlinear oscillators, *Physics Letters A* **374**, (2010), 1464-1468.

[22] J. Wang, and J. Zhou, A class of fractional evolution equations and optimal controls, *Nonlinear Analysis: Real World Applications* **12** ,(2011), 262-272.

[23] J. Wang, and J. Zhou, and M. Milan, On the solvability and optimal controls of fractional integro-differential evolution systems with infinite delay, *Journal of Optimization Theory and Applications* **152** , (2012), 31-50.

[24] T. J. Xiao, and J. Liang, Blow-up and global existence of solutions to integral equations with infinite delay in Banach spaces, *Nonlinear Analysis: Theory, Methods Applications* **71**(12), (2009), 1442– 1447.

[25] R. Zhang, D. Qi, and Y.Wang, Dynamics analysis of fractional order three- dimensional

Hopfield neural network, *Proceedings of the 6-th International Conference on Natural Computation*, (2010), 3037-3039.

[26] Y. Zhou, F. Jiao, and J. Li, Existence of mild solutions for fractional neutral evolution equations, *Computers and Mathematics with Applications* **59** , (2010), 1063-1077.

Article history: Submitted February, 23, 2015; Revised September, 20, 2015; Accepted November, 06, 2015.