

Existence and Uniqueness of Mild Solutions to Neutral SFDEs Driven by a Fractional Brownian Motion With Non-Lipschitz Coefficients

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Abstract. *The article presents results on existence and uniqueness of mild solutions to a class of non linear neutral stochastic functional differential equations (NSFDEs) driven by Fractional Brownian motion in a Hilbert space with non-Lipschitzian coefficients. The results are obtained by using the method of Picard approximation and generalize the results that were reported by [4].*

Key words : Mild Solutions, Semigroup of Bounded Linear Operator, Fractional Powers of Closed Operators, Fractional Brownian Motion, Wiener Integral.

AMS Subject Classifications : 93E03, 34K50, 60H15, 60G22

1. Introduction

Stochastic partial differential equations (SPDEs) have been studied by many authors (see, for example, [7]) where the random disturbances are described by stochastic integrals with respect to semimartingales, especially by Wiener processes. The Wiener process is however not suitable to represent a noise process if long-range dependence is modelled (see [17]). It is then desirable to replace the Wiener process by fractional Brownian motions (fbm). Over the last years some new techniques have been developed in order to define stochastic integrals with respect to fBm. The study of solutions of stochastic equations in infinite-dimensional space with a (cylindrical) fractional Brownian motion (for example, stochastic partial differential equations) has relatively been limited. Linear and semilinear equations with additive fractional noise (the formal derivative of a fBm) are considered in [10] , [6] and the same type of equation has been studied recently in [5].

Let us now say a few words on stochastic functional differential equations (SFDEs) driven by a fBm which arise in many areas of applied mathematics. For this reason, the study of this type of equations has been receiving increased attention in the last few years. In [8], the authors studied the existence and regularity of the density by using the Skorokhod integral based on the Malliavin calculus. The authors of [14] studied this problem by using rough path analysis, while [9] reports on the existence and convergence when the delay goes to zero by using the Riemann-Stieltjes integral. Using also the Riemann-Stieltjes integral, the authors of [3] proved the existence and uniqueness of a mild solution and studied the dependence of the solution on the initial condition in finite and infinite dimensional space.

However, in some cases, many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays (see, e.g., [11] and [12]). Neutral stochastic differential equations with delays are often used to describe such systems. To the best of our knowledge, there is only a little systematic investigation of mild solutions to neutral SPDEs with delays (see, e.g., [4] and references therein).

In this paper, motivated by the previous references, we are concerned with the existence and uniqueness of mild solutions for a class of neutral functional stochastic differential equations (FSDEs) given in the form

$$\begin{aligned} d[x(t) + g(t, x(\rho(t)))] &= [A x(t) + f(t, x(\rho(t)))]dt + \sigma(t)dB_Q^H(t), \quad 0 \leq t \leq T, \\ x(t) &= \varphi(t), \quad -r \leq t \leq 0, \end{aligned} \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup, $(T(t))_{t \geq 0}$, of bounded linear operators in a separable Hilbert space X ; B_Q^H is a fractional Brownian motion on a Hilbert space Y (see section 2 below); f , g and σ are given functions to be specified later, $\rho : [0, +\infty) \rightarrow [-r, +\infty)$ is a suitable delay function, $\varphi : [-r, 0] \times \Omega \rightarrow X$ is the initial value. The goal of this work is to establish an existence and uniqueness result for a mild solution of equation (1). The results are obtained by imposing a condition on the nonlinearities, which is weaker than the classical Lipschitz condition and generalize the results that were reported in [4]. Our approach is similar to the one in [13] and [1] in the case of Wiener process. The rest of this paper is organized as follows. In Section 2 we give a brief review and preliminaries needed to establish our results. Section 3 is devoted to the study of existence and uniqueness of a mild solution of (1) by using a Picard type iteration.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary results which we require to establish the existence and uniqueness of a solution to equation (1).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t), t \in [0, T]\}$ the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(s, t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Moreover β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s) d\beta(s), \quad (2)$$

where $\beta = \{\beta(t) : t \in [0, T]\}$ is a Wiener process, and $K_H(t,s)$ is the kernel given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(\cdot, \cdot)$ denotes the Beta function. We put $K_H(t,s) = 0$ if $t \leq s$.

We will denote by \mathcal{H} the reproducing kernel Hilbert space of the fBm. In fact \mathcal{H} is the closure of set of indicator functions $\{1_{[0;t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0;t]}, 1_{[0;s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $1_{[0;t]} \rightarrow \beta^H(t)$ can be extended to an isometry between \mathcal{H} and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry.

We recall that for $\psi, \varphi \in \mathcal{H}$ their scalar product in \mathcal{H} is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s)\varphi(t) |t-s|^{2H-2} ds dt.$$

Let us consider the operator K_H^* from \mathcal{H} to $L^2([0, T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r,s) dr.$$

We refer to [15] for the proof of the fact that K_H^* is an isometry between \mathcal{H} and $L^2([0, T])$. Moreover for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T (K_H^* \varphi)(t) d\beta(t).$$

It follows from [15] that the elements of \mathcal{H} may be not functions but distributions of negative order. In order to obtain a space of functions contained in \mathcal{H} , we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|_{|\mathcal{H}|}^2 := \alpha_H \int_0^T \int_0^T |\psi(s)\psi(t)| |s-t|^{2H-2} ds dt < \infty,$$

where $\alpha_H = H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the inclusion that follows.

Lemma [15] 2.1.

$$\mathbb{L}^2([0, T]) \subseteq \mathbb{L}^{1/H}([0, T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any $\varphi \in L^2([0, T])$, we have

$$\|\varphi\|_{|\mathcal{H}|}^2 \leq 2HT^{2H-1} \int_0^T |\varphi(s)|^2 ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y, X)$ be the space of bounded linear operator from Y to X . For the sake of convenience, we shall use the same notation to

denote the norms in X, Y and $\mathcal{L}(Y, X)$. Let $Q \in \mathcal{L}(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\text{tr}Q = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \geq 0$ ($n = 1, 2, \dots$) are non-negative real numbers and $\{e_n\}$ ($n = 1, 2, \dots$) is a complete orthonormal basis in Y . Let $B^H = (B^H(t))$ be Y -valued fBm on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where β_n^H are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$\mathbb{E}\langle B^H(t), x \rangle \langle B^H(s), y \rangle = R(s, t) \langle Q(x), y \rangle \quad \forall x, y \in Y \text{ and } t, s \in [0, T].$$

In order to define Wiener integrals with respect to the Q -fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$ of all Q -Hilbert-Schmidt operators $\psi : Y \rightarrow X$. We recall that $\psi \in \mathcal{L}(Y, X)$ is called a Q -Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty,$$

and that the space \mathcal{L}_2^0 equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Now, let $\phi(s); s \in [0, T]$ be a function with values in $\mathcal{L}_2^0(Y, X)$. The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} (K_H^*(\phi e_n)(s) d\beta_n(s), \quad (3)$$

where β_n is the standard Brownian motion used to present β_n^H as in (2).

Now, we end this subsection by stating the following lemma which is fundamental for proving our result.

Lemma 2.2. *If $\psi : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$ satisfies $\int_0^T \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$. Then the above sum in (3) is well defined as a X -valued random variable and we have*

$$\mathbb{E} \left\| \int_0^t \psi(s) dB^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds.$$

This lemma can be proved by similar arguments to those used to prove Lemma 2 in [5].

Let $A : D(A) \rightarrow X$ be the infinitesimal generator of an analytic semigroup, $(T(t))_{t \geq 0}$, of bounded linear operators on X . For the theory of strongly continuous semigroup, we refer to [16]. We will point out here some notations and properties that will be used in this work. Hence, for convenience, we suppose that $\|T(t)\| \leq M$ for $t \geq 0$, and $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A , then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X , and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|,$$

defines a norm in $D(-A)^\alpha$. If H_α represents the space $D(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, then the following well known properties should hold.

Lemma [16] 2. 3. *Suppose that the preceding conditions are satisfied.*

- (1) *Let $0 < \alpha \leq 1$. Then H_α is a Banach space.*
- (2) *If $0 < \beta \leq \alpha$ then the injection $H_\alpha \hookrightarrow H_\beta$ is continuous.*
- (3) *For every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that*

$$\|(-A)^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq T.$$

Finally, we remark that for the proof of our result we need the following Bihari's inequality.

Lemma [2] 2.4. *Let $\rho : R^+ \rightarrow R^+$ be a continuous and non-decreasing function and let g, h, λ be non-negative functions on R^+ such that*

$$g(t) \leq h(t) + \int_0^t \lambda(s)\rho(g(s))ds, \quad t \geq 0,$$

then

$$g(t) \leq G^{-1}\left(G(h^*(t)) + \int_0^t \lambda(s)ds\right),$$

where $G(x) := \int_{x_0}^x \frac{1}{\rho(y)}dy$ is well defined for some $x_0 > 0$, G^{-1} is the inverse function of G and $h^*(t) := \sup_{s \leq t} h(s)$.

In particular, we have the Gronwall-Bellman lemma:

If

$$g(t) \leq h(t) + \int_0^t \lambda(s)g(s)ds,$$

then

$$g(t) \leq h^*(t) \exp\left(\int_0^t \lambda(s)ds\right).$$

3. Main Result

In this section we study the existence and uniqueness of mild solution of equation (1). Henceforth we will assume that A is the infinitesimal generator of an analytic semigroup, $(T(t))_{t \geq 0}$, of bounded linear operators on X . Further, to avoid unnecessary notations, we suppose that $0 \in \rho(A)$ and that (see lemma 2.3),

$$\|T(t)\| \leq M \quad \text{and} \quad \|(-A)^{1-\beta}T(t)\| \leq \frac{C_{1-\beta}}{t^{1-\beta}},$$

for some constants M , $M_{1-\beta}$ and every $t \in [0, T]$.

Similar to the deterministic situation, we give the following definition of mild solutions for equation (1).

Definition 3.1. An X -valued process $\{x(t), t \in [-r, T]\}$, is called a mild solution of equation (1) if

i) $x(\cdot) \in C([-r, T], L^2(\Omega, X))$,

ii) $x(t) = \varphi(t)$, $-r \leq t \leq 0$.

iii) For arbitrary $t \in [0, T]$, we have

$$\begin{aligned} x(t) &= T(t)(\varphi(0) + g(0, \varphi(\rho(0)))) - g(t, x(\rho(t))) \\ &\quad - \int_0^t AT(t-s)g(s, x(\rho(s)))ds + \int_0^t T(t-s)f(s, x(\rho(s)))ds \\ &\quad + \int_0^t T(t-s)\sigma(s)dB^H(s), \quad \mathbb{P} - a.s. \end{aligned}$$

In order to demonstrate the existence and the uniqueness of a mild solution to equation (1), the following weaker conditions (instead of the global Lipschitz condition and linear growth) are listed.

($\mathcal{H}.1$) $f : [0, T] \times X \rightarrow X$ and $\sigma : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$ satisfying the following conditions: there exists a function $K : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ such that

(1a) $\forall t K(t, \cdot)$ is continuous non-decreasing, concave, and for each fixed $x \in \mathbb{R}_+$, $\int_0^T K(s, x)ds < \infty$.

(1b) For any fixed $t \in [0, T]$ and $x \in X$

$$\|f(t, x)\|^2 \leq K(t, \|x\|^2) \text{ and } \int_0^T \|\sigma(t)\|_{\mathcal{L}_2^0}^2 dt < \infty.$$

(1c) For any constant $\alpha > 0, u_0 \geq 0$, the integral equation

$$u(t) = u_0 + \alpha \int_0^t K(s, u(s))ds, \tag{4}$$

has a global solution on $[0, T]$.

($\mathcal{H}.2$) There exists a function $G : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$:

(2a) $\forall t \in [0, T]$, $G(t, \cdot)$ is continuous non-decreasing and concave with $G(t, 0) = 0$, and for each fixed $x \in \mathbb{R}_+$, $\int_0^T G(s, x)ds < +\infty$

(2b) For any $t \in [0, T]$ and $x, y \in X$

$$\|f(t, x) - f(t, y)\|^2 \leq G(t, \|x - y\|^2),$$

(2c) For any constant $D > 0$; if a non negative function $z(t), t \in [0, T]$ satisfies $z(0) = 0$ and $z(t) \leq D \int_0^t G(s, z(s))ds$, then $z(t) = 0$ for all $t \in [0, T]$.

($\mathcal{H}.3$) There exist constants $\frac{1}{2} < \beta < 1$, l, M_g such that the function g is H_β -valued, $(-A)^\beta g : [0, T] \times X \rightarrow X$ is continuous and satisfies

(3a) For all $t \in [0, T]$ and $x \in X$,

$$\|(-A)^\beta g(t, x)\|^2 \leq l(\|x\|^2 + 1).$$

(3b) For all $t \in [0, T]$ and $x, y \in X$

$$\|(-A)^\beta g(t, x) - (-A)^\beta g(t, y)\| \leq M_g \|x - y\|.$$

(3c) The constants M_g, l and β satisfy the following inequalities

$$3\|(-A)^{-\beta}\|^2 M_g^2 < 1, \quad 5\|(-A)^{-\beta}\|^2 l < 1.$$

(H.4) $\rho : [0, \infty] \rightarrow \mathbb{R}$ is a continuous function satisfying the condition that

$$-r \leq \rho(t) \leq t, \quad \forall t \in [0, T].$$

Moreover, we assume that $\varphi \in \mathcal{C}([-r, 0], \mathbb{L}^2(\Omega, X))$.

The main result of this paper is given in the next theorem.

Theorem 3.1. *Suppose that (H.1)-(H.4) hold. Then, for all $T > 0$, equation (1) has a unique mild solution on $[-r, T]$.*

For the proof of this theorem, we will need the following lemmas.

Lemma 3.1. *Let $\tilde{f} \in L^2([0, T], X)$, $\tilde{\sigma} \in L^2([0, T], L^0)$, and consider the equation*

$$\begin{aligned} d[x(t) + g(t, x(\rho(t)))] &= [A x(t) + \tilde{f}(t)]dt + \tilde{\sigma}(t)dB^H(t), \quad 0 \leq t \leq T, \\ x(t) &= \varphi(t), \quad -r \leq t \leq 0. \end{aligned} \tag{5}$$

Under the conditions (H.3) and (H.4), equation (5) has a unique mild solution on $[-r, T]$.

Proof. Fix $T > 0$ and let $B_T := \mathcal{C}([-r, T], \mathbb{L}^2(\Omega, X))$ be the Banach space of all continuous functions from $[-r, T]$ into $\mathbb{L}^2(\Omega, X)$, equipped with the supremum norm

$$\|x\|_{B_T}^2 = \sup_{-r \leq t \leq T} \mathbb{E}\|x(t, \omega)\|^2.$$

Let us consider the set

$$S_T = \{x \in B_T : x(s) = \varphi(s), \text{ for } s \in [-r, 0]\}.$$

Clearly S_T is a closed subset of B_T provided with the norm $\|\cdot\|_{B_T}$.

Let ψ be the function defined on S_T by $\psi(x)(t) = \varphi(t)$ for $t \in [-r, 0]$ and for $t \in [0, T]$ let

$$\begin{aligned} \psi(x)(t) &= T(t)(\varphi(0) + g(0, \varphi(\rho(0)))) - g(t, x(\rho(t))) - \int_0^t AT(t-s)g(s, x(\rho(s)))ds \\ &\quad + \int_0^t T(t-s)\tilde{f}(s)ds + \int_0^t T(t-s)\tilde{\sigma}(s)dB^H(s) = \sum_{i=1}^5 I_i(t). \end{aligned}$$

We are going to show that each function $t \rightarrow I_i(t)$ is continuous on $[0, T]$ in the $\mathbb{L}^2(\Omega, X)$ -sense.

The continuity of I_1 follows directly from the continuity of $t \rightarrow T(t)h$. By (H.3), the function $(-A)^\beta g$ is continuous and since the operator $(-A)^{-\beta}$ is bounded then $t \rightarrow g(t, x(\rho(t)))$ is continuous on $[0, T]$.

For the third term $I_3(t) = \int_0^t AT(t-s)g(s, x(\rho(s)))ds$, we have

$$\begin{aligned} \|I_3(t+h) - I_3(t)\| &\leq \left| \int_0^t (T(h) - I)(-A)^{1-\beta}T(t-s)(-A)^\beta g(s, x(\rho(s)))ds \right| \\ &\quad + \left| \int_t^{t+h} (-A)^{1-\beta}T(t+h-s)(-A)^\beta g(s, x(\rho(s)))ds \right| \leq I_{31}(h) + I_{32}(h). \end{aligned}$$

By the strong continuity of $T(t)$, we have for each $s \in [0, T]$,

$$\lim_{h \rightarrow 0} (T(h) - I)(-A)^{1-\beta}T(t-s)(-A)^\beta g(s, x(\rho(s))) = 0,$$

and since

$$\|(T(h) - I)(-A)^{1-\beta}T(t-s)(-A)^\beta g(s, x(\rho(s)))\| \leq (M+1) \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \|(-A)^\beta g(s, x(\rho(s)))\|,$$

we conclude by the Lebesgue dominated theorem that

$$\lim_{h \rightarrow 0} \mathbb{E} \|I_{31}(h)\|^2 = 0.$$

On another hand,

$$\|I_{32}(h)\|^2 \leq C \cdot |h|^\beta \int_0^T (l \|x(\rho(s))\|^2 + l) ds.$$

Then

$$\lim_{h \rightarrow 0} \mathbb{E} \|I_3(t+h) - I_3(t)\|^2 = 0.$$

Standard computations can be used to show the continuity of I_4 . As for the term $I_5(h)$, we have

$$\begin{aligned} I_5(h) &\leq \left\| \int_0^t (T(h) - I)T(t-s)\tilde{\sigma}(s)dB^H(s) \right\| \\ &\quad + \left\| \int_t^{t+h} T(t+h-s)\tilde{\sigma}(s)dB^H(s) \right\| \leq I_{51}(h) + I_{52}(h). \end{aligned}$$

By lemma 2.2, it follows that

$$\begin{aligned}\mathbb{E}\|I_{51}(h)\|^2 &\leq 2Ht^{2H-1} \int_0^t \|(T(h) - I)T(t-s)\tilde{\sigma}(s)\|_{\mathcal{L}_2^0}^2 ds \\ &\leq 2HT^{2H-1}M^2 \int_0^T \|(T(h) - I)\tilde{\sigma}(s)\|_{\mathcal{L}_2^0}^2 ds.\end{aligned}$$

Since $\lim_{h \rightarrow 0} \|(T(h) - I)\tilde{\sigma}(s)\|_{\mathcal{L}_2^0}^2 = 0$ and

$$\|(T(h) - I)\tilde{\sigma}(s)\|_{\mathcal{L}_2^0}^2 \leq (M+1)^2 \|\tilde{\sigma}(s)\|_{\mathcal{L}_2^0}^2 \in \mathbb{L}^1([0, T], ds),$$

it follows, by the dominated convergence theorem, that

$$\lim_{h \rightarrow 0} \mathbb{E}\|I_{51}(h)\|^2 = 0.$$

Again lemma 2.2 leads to

$$\mathbb{E}\|I_{52}(h)\|^2 \leq 2Hh^{2H-1}M^2 \int_t^{t+h} \|\tilde{\sigma}(s)\|_{\mathcal{L}_2^0}^2 ds \rightarrow 0.$$

The above arguments show that $\lim_{h \rightarrow 0} \mathbb{E}\|\psi(x)(t+h) - \psi(x)(t)\|^2 = 0$. Hence, we conclude that the function $t \rightarrow \psi(x)(t)$ is continuous on $[0, T]$ in the \mathbb{L}^2 -sense.

Next, to see that $\psi(S_T) \subset S_T$, let $x \in S_T$ and $t \in [0, T]$, we consider

$$\begin{aligned}\|\psi(x)(t)\|^2 &\leq 5 \|T(t)(\varphi(0) + g(0, \varphi(\rho(0)))\|^2 + 5 \|g(t, x(\rho(t)))\|^2 \\ &\quad + 5 \left\| \int_0^t A T(t-s)g(s, x(\rho(s)))ds \right\|^2 + 5 \left\| \int_0^t T(t-s)\tilde{f}(s)ds \right\|^2 \\ &\quad + 5 \left\| \int_0^t T(t-s)\tilde{\sigma}(s)dB^H(s) \right\|^2 = 5 \sum_{i=1}^5 J_i(t).\end{aligned}$$

Standard computation yields

$$\sup_{0 \leq t \leq T} \mathbb{E}J_1(t) \leq M^2 \mathbb{E}\|\varphi(0) + g(0, \varphi(\rho(0)))\|^2.$$

And by using condition (3a) and Hölder's inequality, we have

$$\mathbb{E}J_2(t) \leq \|(-A)^{-\beta}\|^2 [l \mathbb{E}\|x(\rho(t))\|^2 + l],$$

and hence,

$$\sup_{0 \leq t \leq T} \mathbb{E}J_2(t) \leq \|(-A)^{-\beta}\|^2 \left[l \sup_{0 \leq t \leq T} \mathbb{E}\|x(\rho(t))\|^2 + l \right].$$

Using again condition (3a) and Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}J_3(t) &\leq \int_0^t \|(-A)^{1-\beta}T(t-s)\|^2 ds \int_0^t \mathbb{E}\|(-A)^\beta g(s, x(s))\|^2 ds \\
&\leq \int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}} \int_0^t (l \mathbb{E}\|x(\rho(s))\|^2 + l) ds. \\
&\leq C_{1-\beta}^2 \left(\frac{T^{2\beta-1}}{2\beta-1} \right) \int_0^T (l \mathbb{E}\|x(\rho(s))\|^2 + l) ds.
\end{aligned}$$

Also a standard computation yields

$$\sup_{0 \leq t \leq T} \mathbb{E}J_4(t) \leq M^2 T \int_0^T \mathbb{E}\|\tilde{f}(s)\|^2 ds.$$

Then by using lemma 2.2 we arrive at

$$\sup_{0 \leq t \leq T} \mathbb{E}J_5(t) \leq 2HT^{2H-1} \int_0^T \|\tilde{\sigma}(s)\|_{\mathcal{L}_2^0}^2 ds.$$

Since $\psi(x)(t) = \varphi(t)$ on $[-r, 0]$, the inequalities together imply that

$$\sup_{-r \leq t \leq T} \mathbb{E}\|\psi(x)(t)\|^2 < \infty.$$

Hence, we conclude that ψ is well defined.

Now, we are going to show that ψ is a contraction mapping in S_{T_1} with some $T_1 \leq T$ to be specified later. Let $x, y \in S_T$ and $t \in [0, T]$, to write

$$\begin{aligned}
\|\psi(x)(t) - \psi(y)(t)\|^2 &\leq 2 \|g(t, x(\rho(t))) - g(t, y(\rho(t)))\|^2 \\
&\quad + 2 \left\| \int_0^t A T(t-s) (g(s, x(\rho(s))) - g(s, y(\rho(s)))) ds \right\|^2 \\
&\leq 2 \|(-A)^{-\beta}\|^2 \|(-A)^\beta g(t, x(\rho(t))) - (-A)^\beta g(t, y(\rho(t)))\|^2 \\
&\quad + 2 \left\| \int_0^t (-A)^{1-\beta} T(t-s) (-A)^\beta (g(s, x(\rho(s))) - g(s, y(\rho(s)))) ds \right\|^2.
\end{aligned}$$

By condition (3b), lemma 2.3 and Hölder's inequality, we have

$$\begin{aligned}
\|\psi(x)(t) - \psi(y)(t)\|^2 &\leq 2 \|(-A)^{-\beta}\|^2 M_g^2 \|x(\rho(t)) - y(\rho(t))\|^2 \\
&\quad + 2 M_g^2 C_{1-\beta}^2 \left(\frac{T^{2\beta-1}}{2\beta-1} \right) \int_0^t \|x(\rho(s)) - y(\rho(s))\|^2 ds.
\end{aligned}$$

Hence

$$\sup_{s \in [-r, t]} \mathbb{E}\|\psi(x)(s) - \psi(y)(s)\|^2 \leq \gamma(t) \sup_{s \in [-r, t]} \mathbb{E}\|x(s) - y(s)\|^2,$$

where

$$\gamma(t) = 2M_g^2 \left\{ \|(-A)^{-\beta}\|^2 + C_{1-\beta}^2 \left(\frac{T^{2\beta-1}}{2\beta-1} \right) t \right\}.$$

By condition (3c), we have $\gamma(0) = 2\|(-A)^{-\beta}\|^2 M_g^2 < 1$. Then there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$ and ψ is a contraction mapping on S_{T_1} , and therefore has a unique fixed point, which is a mild solution to equation (5) on $[0, T_1]$. This procedure can be repeated in

order to extend the solution to the entire interval $[-r, T]$ in finitely many steps. ■

We now construct a successive approximation sequence, using a Picard type iteration, with the help of lemma 3.1. Let x^0 be a solution of equation (5) with $\tilde{f} = 0$, $\tilde{\sigma} = 0$. For $n \geq 0$, let x^{n+1} be the solution of equation (5) on $[-r, T]$ with $f(t) = f(t, x^n(\rho(t)))$, and $\tilde{\sigma}(t) = \sigma(t)$, i.e.

$$\begin{aligned} x^{n+1}(t) &= \varphi(t), \text{ if } t \in [-r, 0] \\ x^{n+1}(t) &= T(t)(\varphi(0) + g(0, \varphi(0))) - g(t, x^{n+1}(\rho(t))) - \int_0^t AT(t-s)g(s, x^{n+1}(\rho(s)))ds \\ &\quad + \int_0^t T(t-s)f(s, x^n(\rho(s)))ds + \int_0^t T(t-s)\sigma(s)dB^H(s), \text{ if } t \in [0, T]. \end{aligned} \quad (6)$$

Now, we prove the existence of a solution to problem (1). The proof starts by stating the following lemmas.

Lemma 3.2. *Under conditions (H.1) – (H.4), the sequence $\{x^n, n \geq 0\}$ is well defined and there exist positive constants M_1, M_2, D_0 such that for all $m, n \in \mathbb{N}$ and $t \in [0, T]$,*

$$(1) \quad \sup_{-r \leq s \leq t} \mathbb{E} \|x^{m+1}(s) - x^{n+1}(s)\|^2 \leq M_1 \int_0^t G(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^m(\theta) - x^n(\theta)\|^2) ds, \quad (7)$$

$$(2) \quad \sup_{-r \leq s \leq t} \mathbb{E} \|x^{n+1}(s)\|^2 \leq D_0 + M_2 \int_0^t K(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^n(\theta)\|^2) ds. \quad (8)$$

Proof. • 1: For $m, n \in \mathbb{N}$ and $t \in [0, T]$ we have

$$\|x^{m+1}(t) - x^{n+1}(t)\|^2 \leq 3(I_1(t) + I_2(t) + I_3(t)),$$

where

$$\begin{aligned} I_1(t) &:= \|g(t, x^{m+1}(\rho(t))) - g(t, x^{n+1}(\rho(t)))\|^2, \\ I_2(t) &:= \left\| \int_0^t AT(t-s)(g(s, x^{m+1}(\rho(s))) - g(s, x^{n+1}(\rho(s))))ds \right\|^2, \\ I_3(t) &:= \left\| \int_0^t T(t-s)(f(s, x^m(\rho(s))) - f(s, x^n(\rho(s))))ds \right\|^2. \end{aligned}$$

By using condition (3b) for the terms I_1 and I_2 , we obtain

$$\begin{aligned} I_1(t) &\leq \|(-A)^{-\beta}\|^2 M_g^2 \|x^{m+1}(\rho(t)) - x^{n+1}(\rho(t))\|^2, \\ I_2(t) &\leq \frac{C_{1-\beta}^2}{2\beta-1} T^{2\beta-1} \int_0^t M_g^2 \|x^{m+1}(\rho(s)) - x^{n+1}(\rho(s))\|^2 ds. \end{aligned}$$

Then the condition (2b) for the term I_3 leads to

$$\sup_{0 \leq s \leq t} \mathbb{E} I_3(s) \leq C \int_0^t G(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^m(\theta) - x^n(\theta)\|^2) ds.$$

Consideration of the fact that $3\|(-A)^{-\beta}\|^2 M_g^2 < 1$ and the above inequalities result with

$$\begin{aligned} \sup_{-r \leq s \leq t} \mathbb{E} \|x^{m+1}(s) - x^{n+1}(s)\|^2 &\leq C \int_0^t \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^{m+1}(\theta) - x^{n+1}(\theta)\|^2 ds \\ &\quad + C \int_0^t G(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^m(\theta) - x^n(\theta)\|^2) ds. \end{aligned}$$

Finally by lemma 2.4 we obtain

$$\sup_{-r \leq s \leq t} \mathbb{E} \|x^{m+1}(s) - x^{n+1}(s)\|^2 \leq C \int_0^t G(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^m(\theta) - x^n(\theta)\|^2) ds.$$

• 2: Following the same method as in the proof of assertion (1) leads to

$$\sup_{-r \leq s \leq t} \mathbb{E} \|x^{m+1}(s)\|^2 \leq C + C \int_0^t \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^{m+1}(\theta)\|^2 ds + C \int_0^t K(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^m(\theta)\|^2) ds.$$

Also by lemma 2.4 we obtain

$$\sup_{-r \leq s \leq t} \mathbb{E} \|x^{m+1}(s)\|^2 \leq C + C \int_0^t K(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^m(\theta)\|^2) ds.$$

Here the proof ends. ■

Lemma 3.3. *Under conditions (H.1) – (H.4), there exists a $u(t)$ satisfying*

$$u(t) = u_0 + D \int_0^t K(u(s)) ds,$$

for some $u_0 \geq 0$, $D > 0$, and the sequence $\{x^n, n \geq 0\}$, for all $n \in \mathbb{N}$ and $t \in [0, T]$, satisfies

$$\sup_{-r \leq s \leq t} \mathbb{E} \|x^n(s)\|^2 \leq u(t). \quad (9)$$

Proof. Let $u : [0, T] \rightarrow \mathbb{R}$ be a global solution of the integral equation (4) with an initial condition $u_0 = D_0 \vee \sup_{-r \leq t \leq T} \mathbb{E} \|x^0(t)\|^2$ and with $\alpha = M_2$, where D_0, M_2 are the same constants as in lemma 3.2. We prove inequality (9) by mathematical induction. For $n = 0$, the inequality (9) holds by the definition of u_0 . Furthermore, let us assume that $\sup_{-r \leq s \leq t} \mathbb{E} \|x^n(s)\|^2 \leq u(t)$. Then by (8), we obtain

$$\begin{aligned} \sup_{-r \leq s \leq t} \mathbb{E} \|x^{n+1}(s)\|^2 &\leq D_0 + M_2 \int_0^t K(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^n(\theta)\|^2) ds \\ &\leq u_0 + M_2 \int_0^t K(s, u(s)) ds = u(t). \end{aligned}$$

This completes the proof. ■

3.1. Proof of theorem 3.1

The proof shall address the issues of existence and uniqueness of a mild solution to (1).

• Existence : For $t \in [0, T]$, by lemma 3.2 we note that

$$\sup_{-r \leq s \leq t} \mathbb{E} \|x^{m+1}(s) - x^{n+1}(s)\|^2 \leq M_1 \int_0^t G(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^m(\theta) - x^n(\theta)\|^2) ds. \quad (10)$$

Moreover, by lemma 3.3 and the Fatou Lemma it follows that

$$\limsup_{m, n \rightarrow \infty} \left(\sup_{-r \leq s \leq t} \mathbb{E} \|x^m(s) - x^n(s)\|^2 \right) \leq M_1 \int_0^t G(s, \limsup_{m, n \rightarrow \infty} \sup_{-r \leq \theta \leq s} \mathbb{E} \|x^m(\theta) - x^n(\theta)\|^2) ds.$$

Also by condition (2c), we have

$$\lim_{m, n \rightarrow \infty} \sup_{-r \leq s \leq T} \mathbb{E} \|x^m(s) - x^n(s)\|^2 = 0.$$

This implies that $(x^n, n \geq 1)$ is a Cauchy sequence in B_T . Therefore, the completeness of B_T guarantees the existence of a process $x \in B_T$ such that

$$\lim_{m, n \rightarrow \infty} \sup_{-r \leq s \leq T} \mathbb{E} \|x^n(s) - x(s)\|^2 = 0.$$

By letting $n \rightarrow +\infty$ in (6), it is seen that x is a mild solution to equation (1) on $[-r, T]$.

• **Uniqueness** : Let x and y be two mild solutions of equation (1) on $[-r, T]$, then

$$\sup_{-r \leq s \leq t} \mathbb{E} \|x(s) - y(s)\|^2 \leq M_1 \int_0^t G(s, \sup_{-r \leq \theta \leq s} \mathbb{E} \|x(\theta) - y(\theta)\|^2) ds.$$

By condition (2c), it follows that $\sup_{-r \leq s \leq T} \mathbb{E} \|x(s) - y(s)\|^2 = 0$. Consequently $x = y$, which implies uniqueness. Thus the proof of the theorem is complete.

Remark 3.1. If $\forall t \in [0, T]$, we have $G(t, u) = Lu, u \geq 0, L > 0$, the condition (\mathcal{H}_2) implies a global Lipschitz condition. Clearly, the Lipschitz condition is a special case of the proposed conditions.

Corollary 3.1. Suppose that (H.3) – (H.4) are satisfied. Further we suppose that for each fixed $t \in [0, T]$ and $x, y \in X$, the following conditions are satisfied,

a.1 $\|f(t, x) - f(t, y)\|^2 \leq \alpha(t)\lambda(\|x - y\|^2)$.

a.2 $\|f(t, 0)\|, \|\sigma(t)\|_{\mathcal{L}_2^q} \in L^2([0, T]; \mathbb{R}^+)$ for all $t \in [0, T]$,

where $\alpha(t) \geq 0$ is such that $\int_0^T \alpha(s) ds < +\infty$, and $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous concave

nondecreasing function such that $\lambda(0) = 0, \lambda(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{1}{\lambda(x)} = +\infty$. Then

equation (1) has a unique solution.

Remark 3.2. As a concrete example of the function $\lambda(\cdot)$. Let $L > 0$ and $\delta \in (0, 1)$ be sufficiently small. Define

$$\lambda_1(u) = Lu, u \geq 0,$$

$$\lambda_2(u) = \begin{cases} u \ln(u^{-1}), & 0 \leq u \leq \delta, \\ \delta \ln(\delta^{-1}) + \lambda_2'(\delta_-)(u - \delta), & u > \delta. \end{cases}$$

where λ_2' denotes the derivative of function λ_2 . They are all concave nondecreasing functions satisfying $\int_{0^+} \frac{1}{\lambda_i(x)} = +\infty$, ($i = 1, 2$).

4. Illustrative Example

In this section, an example is provided to illustrate our result. Consider the following neutral stochastic partial differential equation with finite delays driven by a fractional Brownian motion of the form:

$$\begin{aligned} d[x(t, \xi) + \frac{\alpha_1}{M_{1/2} \|(-A)^{\frac{1}{2}}\|} x(\rho(t), \xi)] &= \left[\frac{\partial^2}{\partial^2 \xi} x(t, \xi) + \alpha_2 x(\rho(t), \xi) \right] dt \\ &+ \sigma(t) dB^H(t), \quad t \in [0, T], \\ x(t, 0) = x(t, \pi) &= 0, \quad t \in [0, T], \quad \alpha_i > 0, \quad i = 1, 2, \\ x(s, \xi) = \varphi(s, \xi), \quad \varphi(s, \cdot) &\in L^2([0, \pi]) ; \quad -r \leq s \leq 0 \quad a.s., \end{aligned} \quad (11)$$

where $M_{1/2}$ is the corresponding constant in lemma 2.3, $B^H(t)$ is a standard one-dimensional fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, and $\rho : [0, \infty] \rightarrow \mathbb{R}$ is a continuous function satisfying the condition that

$$-r \leq \rho(t) \leq t, \quad \forall t \in [0, T].$$

For the sake of simplicity of notation in the following text, the variable ξ of $x(t, \xi)$ shall be omitted.

Let us conceive (11) in the abstract form of (1), with $X = L^2([0, \pi])$. Define then the operator $A : D(A) \subset X \rightarrow X$, given by $A = \frac{\partial^2}{\partial^2 \xi}$ with

$$D(A) = \{y \in X : y' \text{ is absolutely continuous, } y'' \in X, \quad y(0) = y(\pi) = 0\},$$

to obtain

$$Ax = \sum_{n=1}^{\infty} n^2 \langle x, e_n \rangle_X e_n, \quad x \in D(A),$$

where $e_n := \sqrt{\frac{2}{\pi}} \sin nx$, $n = 1, 2, \dots$ is an orthogonal set of eigenvector of $-A$. The bounded linear operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}} x = \sum_{n=1}^{\infty} n \langle x, e_n \rangle_X e_n,$$

with domain

$$D((-A)^{\frac{1}{2}}) = \left\{ x \in X, \sum_{n=1}^{\infty} n \langle x, e_n \rangle e_n \in X \right\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t \geq 0}$ in X , and is given by (see [16])

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n,$$

for $x \in X$ and $t \geq 0$, that satisfies $\|S(t)\| \leq e^{-\pi^2 t}$ for every $t \geq 0$.

We may respectively define

$$g(t, x(\rho(t))) = \frac{\alpha_1}{M_{1/2} \|(-A)^{\frac{1}{2}}\|} x(\rho(t)),$$

and

$$f(t, x(\rho(t))) = \alpha_2 x(\rho(t)).$$

It is obvious that all the underlying assumptions are satisfied with $\beta = \frac{1}{2}$, $M = 1$,

$$M_g = \frac{\alpha_1}{M_{1/2}}, \quad l = \frac{\alpha_1^2}{M_{1/2}^2} \quad \text{and} \quad \|(-A)^{-\frac{1}{2}}\| \leq \frac{1}{\Gamma(1/2)} \int_0^{\infty} t^{-\frac{1}{2}} \|S(t)\| dt \leq \frac{1}{\pi}.$$

Thus, by theorem 3.1, there exists a unique mild solution to (11), provided that $5 \frac{\alpha_1^2}{M_{1/2}^2 \pi^2} < 1$.

Acknowledgements

The authors would like to thank the referee and the editor for comments and valuable suggestions on this work.

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