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A Note on the Existence and Uniqueness for Neutral Stochastic Differential Equations With Infinite Delays and Poisson Jumps

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Abstract. *This paper studies the existence and uniqueness of a mild solution, satisfying Caratheodory conditions, for a neutral stochastic partial functional differential equation with infinite delays and Poisson jumps.*

Key words : Neutral Stochastic Partial Differential Equations, Mild Solution, Infinite Delay, Poisson Jumps.

AMS Subject Classifications : 93E15, 34K50, 60H15, 60J75

1. Introduction

In this paper, a neutral stochastic partial functional differential equation with delays and Poisson jumps is considered in a real separable Hilbert space of the form

$$\begin{aligned} d[X(t)+a(t,X(t))] &= [A\,X(t)+f(t,X(t-\tau(t)))]dt+g(t,X(t-\delta(t)))dW(t) \\ &+ \int_z h(t,X(t-\alpha(t)),y)\widetilde{N}(dt,dy), \quad t \geq 0, \\ X_0(\cdot) &= \xi \in D^b_{\mathcal{F}_0}([m(0),0],H), \end{aligned} \tag{1}$$

where $t-\tau(t), t-\delta(t), t-\alpha(t) \to \infty$ with delays $\tau(t), \delta(t), \alpha(t) \to \infty, \quad t \to \infty.$

The existence and uniqueness with delays has been considered by many authors. Under the global Lipschitz and linear growth condition, Taniguchi, Liu and Truman [15] considered the existence and uniqueness of mild solutions to stochastic neutral partial functional differential equations by the well-known Banach fixed point theorem and strong approximating system, respectively, Govindan [5], showed by the stochastic convolution the existence, uniqueness and almost sure exponential stability of stochastic neutral partial functional differential equations under the global Lipschitz and linear growth condition. While, by the comparison principle, in Govindan [6], the existence and uniqueness of mild solution to stochastic evolution equations with variable delays was investigated under a less restrictive hypothesis than the Lipschitz condition on the nonlinear terms. For many practical situations, the nonlinear terms do not obey the global Lipschitz and linear growth condition, even the local Lipschitz condition, and the readers can refer to Govindan [6], Rodkina [13], Taniguchi [14], [15], He [9], Yamada [17]. Subsequently, Luo [9] and Taniguchi [15] applied this valuable method into dealing with the asymptotical stability in mean square of neutral SPDEs with infinite delays. On the other hand, recently SPDEs driven by jump process have received a great deal of attention. For example, Ren and Sakthivel [12] have established the existence and uniqueness of mild solution for a class of second-order neutral stochastic evolution equations with infinite delay and Poisson jumps by means of the successive approximation, Nan Ding [4] have established the exponential stability in mean square of mild solution for neutral stochastic partial functional differential equations with impulses, Cui, Yan, and Sun [2] studied the stability of neutral partial differential equations with delays and Poisson jumps, motivated by the previous problems, our current consideration is on neutral SPDEs with delays infinity and Poisson jumps.

In this paper, our goal is to study and extend the existence and uniqueness of the mild solution, satisfying Caratheodory conditions, of (1) with infinite delays and Poisson jumps by means of fixed-point theory. Even in the special case ($a = 0$), the result obtained here appears to be new. The rest of this paper is organized as follows. In section 2, we introduce some preliminaries. In section 3, we prove the existence and uniqueness of the mild solution. Finally, in the fourth section, we give an example to illustrate the theory.

2. Preliminaries

Throughout this paper, we work in the frameworks used by [10]. Let $\{\Omega, \mathfrak{F}_t, P\}$ be a complete probability space equipped with some filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathfrak{F}_o$ contains all P-null sets.

Suppose $P(t), t \geq 0$ is a σ-finite stationary $\mathfrak{F}_t$-adapted Poisson point process taking values in a measurable space $(U, \mathcal{B}(\mathcal{U}))$. The random measure N_p defined by $N_p((0,t] \times \Lambda) \doteq \sum_{s\in(0,t]} 1_\Lambda(p(s))$ for $\Lambda \in \mathcal{B}(\mathcal{U})$ is called the Poisson random measure induced by $p(\cdot)$, thus, we can define the measure $\widetilde{N}$ by $\widetilde{N}(dt,dy) = N_p(dt,dy) - \nu(dy)dt$, where ν is the characteristic measure of N_p, which is called the compensated Poisson random measure.

Let H, K be two real separable Hilbert spaces and we denote by $< \cdot, \cdot >_H$, $< \cdot, \cdot >_K$ their inner products and by $\|\cdot\|_H, \|\cdot\|_K$ their vector norms, respectively, We denote by $\mathcal{L}(K,H)$ the set of all linear bounded operators from K into H, equipped with the usual operator norm $\|\cdot\|$. In this paper, we always use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the space potentially involved when no confusion possibly arises. Let $\{W(t), t \geq 0\}$ denote a K-valued $\{\mathfrak{F}_t\}_{t\geq 0}$-Wiener process defined on $\{\Omega, \mathfrak{F}_t, P\}$ with covariance operator Q, i.e.,

$$E < W(t),x >_K \ < W(s),y >_K \ = (t \wedge s) < Qx,y >_K \quad for \quad all \quad x,y \in K,$$

where Q is a positive, self-adjoint, trace class operator on K. In particular, we shall call such $W(t), t \geq 0$, a K-valued Q-Wiener process with respect to $\{\mathfrak{F}_t\}_{t \geq 0}$.

In order to define stochastic integrals with respect to the Q-Wiener process $W(t)$, we introduce the subspace $K_0 = Q^{1/2}(K)$ of K which, endowed the inner product

$$< u,v >_{K_0} \ = Q^{-1/2}u, Q^{-1/2}v >_K ,$$

is a Hilbert space. Let $\mathcal{L}_2^0 = \mathcal{L}_2(K_0,H)$ denote the space of all Hilbert-Schmidt operators from K_0 into H. It turns out to be a separable Hilbert space, equipped with the norm

$$\|\Psi\|_{\mathcal{L}_2^0}^2 = tr((\Psi Q^{1/2})(\Psi Q^{1/2})^*) \quad for \quad any \quad \Psi \in \mathcal{L}_2^0.$$

Clearly, for any bounded operators $\Psi \in \mathcal{L}(K,H)$, this norm reduces to $\|\Psi\|_{\mathcal{L}_2^0} = tr(\Psi Q \Psi^*)$. For arbitrarily given $T \geq 0$, let $J(t,w), t \in [0,T]$, be an $\mathfrak{F}_t$-adapted, $\mathcal{L}_2^0$-valued process, and we define the following norm for arbitrary $t \in [0,T]$:

$$|J|_t = \{E\int_0^t tr(J(s,w)Q^{1/2})(J(s,w)Q^{1/2})^*)ds\}^{1/2}.$$

In particular, we denote all $\mathcal{L}_2^0$-valued predictable processes J satisfying $|J|_T < \infty$ by $\mathcal{U}^2([0,T];\mathcal{L}_2^0)$. The stochastic integral

$$\int_0^t j(s,w)dW(s) = L^2 - \lim_{n \to \infty} \sum_{i=1}^n \int_0^t \sqrt{\lambda_i}\, J(s,w)e_i dB_s^i, \quad t \in [0,T],$$

where $W(t) = \sum_{i=1}^\infty \sqrt{\lambda_i}\, B_t^i e_i$. Here $(\lambda_i \geq 0, i \in N)$ are the eigenvalues of Q and $(e_i, i \in N)$ are the corresponding eigenvectors, $(B_t^i, i \in N)$ are independent standard real-valued Brownian motions. The reader is referred to [3] for a systematic theory concerning stochastic integrals of this kind.

Let $\tau(t),\ \delta(t) \in C(\mathbb{R}_+,\mathbb{R}_+)$ satisfy $t - \tau(t) \to \infty, t - \delta(t) \to \infty \quad as \quad t \to \infty$, and

$$m(0) = \max\{\inf(s - \tau(s), s \geq 0), \inf(s - \delta(s), s \geq 0), \inf(s - \alpha(s), s \geq 0)\}.$$

We use $D_{\mathfrak{F}_0}^b([m(0),0];H)$ to denote the family of all almost surely bounded, $\mathfrak{F}_0$-measurable, continuous random variables from $[m(0),0]$ to H. Denote the norm $\|\varphi\|_D$ by

$$\|\varphi\|_D = \sup_{m(0) \leq \theta \leq 0} E\|\varphi(\theta)\|_H.$$

A semigroup $\{S(t), t \geq 0\}$ is said to be exponentially stable if there exist positive constants M and a such that $\|S(t)\| \leq Me^{-at}, t \geq 0$. If $M = 1$ the semigroup is said to be a contraction. If $\{S(t), t \geq 0\}$ is an analytic semigroup, see Pazy [11] with infinitesimal generator A such that $0 \in \rho(A)$ (the resolvent set of A) then it is possible to define the fractional $(-A)^\alpha$, for $0 \leq \alpha \leq 1$ as a closed linear operator on its domain $D((-A)^\alpha) = H_\alpha$ Furthermore, the subspace $D((-A)^\alpha)$ is dense in H and

$$\|x\|_\alpha = \|(-A)^\alpha x\|_H \quad x \in D((-A)^\alpha).$$

For convenience of the reader, we will state the following lemmas that will be used in the sequel.

Lemma [11] **2.1**. *Let A be the infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$, If $0 \in \rho(A)$ then,*

(i) *For every $t \geq 0$, $\alpha \geq 0$,*

$$S(t) : H \to H_\alpha. \tag{2}$$

(ii) *For every* $x \in H_\alpha$ *one has*

$$S(t)(-A)^\alpha x = (-A)^\alpha S(t)x. \tag{3}$$

(iii) *For every* $t \geq 0$ *the operator*

$$\|(-A)^\alpha S(t)\|_H \leq \mu_\alpha t^{-\alpha} e^{-at}, \quad a \geq 0. \tag{4}$$

(iv) *Let* $0 < \alpha \leq 1$ *and* $x \in H_\alpha$ *Then,*

$$\|S(t)x - x\|_H \leq \gamma_\alpha t^\alpha \|(-A)^\alpha x\|_H. \tag{5}$$

Lemma [8] **2.2**. *Let-A be the infinitesimal generator of an analytic semigroup of bounded linear operators* $\{S(t), t \geq 0\}$ *in K. Then, Then, for any stochastic process* $F : [0,\infty) \to H$ *which is strongly measurable with*$\int_0^T E\|(-A)^\alpha F(t)\|_H^p \, dt < \infty$, $p \geq 2, 0 < T \leq \infty$, *for* $0 < t \leq T$ *the following inequality holds:*

$$E\|\int_0^t (-A)S(t-s)F(s)\, ds\|_H^p \leq k(p,a,\alpha)\int_0^t E\|(-A)^\alpha F(s)\|_H^p \, ds, \tag{6}$$

provided that $1/p < \alpha < 1$, where

$$k(p,a,\alpha) = M_{1-\alpha}^p \frac{(p-1)^{p\alpha-1}[\Gamma((p\alpha - 1/(p-1)))]^{p-1}}{(pa)^{p\alpha-1}}, \tag{7}$$

and $\Gamma(\bullet)$ *is the Gamma function.*

3. Existence and Uniqueness

In this section, using Caratheodory conditions, we establish the existence and uniqueness of a mild solution to (1). For precision we let $-A : D(A) \subseteq H \to H$ be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ defined on H. Let the functions $f(t,u), a(t,u), h(t,u,y)$ and $g(t,u)$ be defined as follows: $f : \mathbb{R}_+ \times H \to H$ $\quad a : \mathbb{R}_+ \times H_\alpha \to \mathcal{L}(K,H)$ $\quad g : \mathbb{R}_+ \times H \to \mathcal{L}(K,H)$ $\quad h : \mathbb{R}_+ \times H \times U \to H$ are Borel measurable. We also Let the following assumptions hold a.s.: (H1) $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in H and that the semigroup is a contraction.

(H2) There exists a function $H(t,r) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $H(t,r)$ is locally integrable in $t \geq 0$ for any fixed $r \geq 0$, and is continuous monotone nondecreasing and concave in r for any fixed $t \in [0,T]$. Moreover, for any fixed $t \in [0,T]$ and $\xi \in H$, the following inequality is satisfied.

$$\|f(t,\xi)\|_H^2 + \|g(t,\xi)\|_{L_0^2} + \int_z \|h(t,\xi,z)\|_H^2 \nu(dz) \leq H(t, \|\xi\|_D^2), \quad t \in [0,T],$$

and for any $K > 0$, the differential equation

$$\frac{du}{dt} = KH(t,u), \quad t \in [0,T],$$

has a global solution for any initial value u_0.

(H3) There exists a function $G(t,r) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $G(t,r)$ is locally integrable in $t \leq 0$ for any fixed $r \leq 0$, and is continuous monotone nondecreasing and concave in r for any fixed $t \in [0,T], G(t,0) = 0$ for any fixed $t \in [0,T]$, Moreover, for any fixed $t \in [0,T]$ and $\xi, \eta \in H$, the following inequality

$$\|f(t,\xi) - f(t,\eta)\|_H^2 + \|g(t,\xi) - g(t,\eta)\|_{\mathfrak{L}_0^2}^2 + \int_z \|h(t,\xi,z) - h(t,\eta,z)\|_H^2 \nu(dz) \leq G(t, \|\xi - \eta\|_D^2), \quad t \in [0,T].$$

is satisfied for any constant $\bar{K} > 0$, if a nonnegative function $z(t)$ satisfies

$$z(t) \leq \bar{K} \int_0^t G(s, z(s))ds, \quad t \in [0,T],$$

with $z(t) = 0$ holding for any $t \in [0,T]$.

(H4) The mapping $a(t,x)$ satisfied that exists a number $\alpha \in [0,1]$ and a positive K_0 such that, for any $\xi, \eta \in H$ and $t \geq 0$, $a(t,x) \in \mathcal{D}((-A^\alpha)$ and

$$\|(-A)^\alpha a(t,\xi) - (-A)^\alpha a(t,\eta)\|_H \leq K_0 \|\xi - \eta\|_D.$$

Moreover, we assume that $a(t,0) = 0$.

Definition 3.1. A stochastic process $\{X(t), t \in [0,T]\}, 0 \leq T < \infty$, is called a mild solution of (1) if :
(i) $X(t)$ is adapted to $\mathfrak{F}_t, t \geq 0$;
(ii) $X(t) \in H$ has cadlag paths on $t \in [0,T]$ almost surely, and for arbitrary $0 \leq t \leq T$,

$$\begin{aligned} X(t) &= S(t)[\xi(0) + a(0,\xi)] - a(t,X_t) - \int_0^t AS(t-s)a(s,X_s)ds \\ &+ \int_0^t S(t-s)f(s,X(s-\tau(s)))ds + \int_0^t S(t-s)g(s,X(s-\delta(s)))dW(s) \\ &+ \int_0^t \int_z S(t-s)h(t,X(s-\alpha(s)),y)\widetilde{N}(ds,dy), \end{aligned}$$

and

$$X_0 = \xi \in D_{\mathfrak{F}_0}^b([m(0),0],H).$$

Theorem 3.1. *Suppose that the assumptions* $(H1)$-$(H4)$ *are satisfied. Then, there exists a unique mild solution to (1).*

Proof. Denote by $\mathcal{S}$ the space of all $\mathfrak{F}_0$-adapted processes $\phi(t,w) : [m(0),\infty) \times \Omega \to \mathbb{R}$, which is a.s. continuous in t for fixed $w \in \Omega$. Moreover, $\phi(s,w) = \xi(s)$ for $s \in [m(0),0]$ and $E\|\phi(t,w)\|_H^p \to 0$ as $t \to \infty$. It is then routine to check that $\mathcal{S}$ is a Banach space when it is equipped with a norm defined by

$$\|\phi\|_{\mathcal{S}} = \sup_{t\geq 0} E\|\phi(t)\|_H^p \quad for \quad each \quad \phi \in \mathcal{S}.$$

Next, define an operator $\Theta : \mathcal{S} \to \mathcal{S}$ by $\Theta(x)(t) = \varphi(t)$ for $t \in [m(0), 0]$ and for $t \geq 0$,

$$\begin{aligned}\Theta(x)(t) &= S(t)[\xi(0) + a(0,\xi)] - a(t, X(t)) - \int_0^t AS(t-s)a(X(s))ds \\ &+ \int_0^t S(t-s)f(s, X(s-\tau(s)))ds + \int_0^t S(t-s)g(s, X(s-\delta(s)))dW(s) \\ &+ \int_0^t \int_z S(t-s)h(t, X(s-\alpha(s)), y)\widetilde{N}(ds, dy), \\ &:= \sum_{i=1}^7 I_i(t). \end{aligned} \tag{8}$$

We first verify the mean square continuity of Θ on $[0, \infty)$. Let $x \in \mathcal{S}$, $t_1 \geq 0$, and $|r|$ be sufficiently small, then

$$E\|\Theta(x)(t_1 + r) - \Theta((x)(t_1)\|_H^2 \leq 7 \sum_{i=1}^7 E\|I_i(t_1 + r) - Ii(t_1)\|_H^2.$$

By virtue of closed ness of $(-A)^\alpha$ and the fact that $S(t)$ commutes with $(-A)^\alpha$ on H_α, we have by lemma 2.1 and the assumption $(H4)$ that

$$\begin{aligned} E\|I_1(t_1 + r) - I_1(t_1)\|_H^2 &= \|S(t_1 + r) - S(t_1)\xi(0)\|_H^2 \\ &= E\|(S(r) - I)S(t_1)\xi(0)\|_H^2 \leq \gamma_\alpha^2 \mu_\alpha^2 t_1^{-2\alpha} h^{2\alpha} e^{-2at_1} E\|\xi\|_D^2, \\ E\|I_2(t_1 + r) - I_2(t_1)\|_H^2 &= E\|(S(r) - I)S(t_1)(-A)^\alpha(-A)^\alpha a(0, \xi)\|_H^2 \\ &\leq \gamma_\alpha^2 \mu_\alpha^2 t_1^{-2\alpha} h^{2\alpha} e^{-2at_1} 2C^2 \|(-A)^{-\alpha}\|_H^2 (1 + E\|\xi\|_D^2) \\ E\|I_3(t_1 + r) - I_3(t_1)\|_H^2 &\leq \|(-A)^{-\alpha}\|_H^2 E\|(-A)^{-\alpha} a(t_1 + r, X_{t_1+r}) - (-A)^\alpha a(t_1, X_{t_1})\|_H^2. \end{aligned}$$

Next, using lemmas 2.1 and 2.2 and assumption (H4), we obtain

$$\begin{aligned} E\|I_4(t_1 + r) - I_4(t_1)\|_H^2 &= E \left\| \int_0^{t_1} (-A)S(t_1 - s)(S(r) - I)\, a(s, X_s)\, ds \right\|_H^2 \\ &\leq \frac{M_{1-\alpha}^2 \Gamma(2\alpha - 1)}{(2a)^{2\alpha-1}} \left\{ \int_0^{t_1} E\|(-A)^\alpha (S(r) - I)a(s, X_s)\|_H^2\, ds \right. \\ &\left. + \int_{t_1}^{t_1+r} E\|(-A)^\alpha S(r)a(s, X_s)\|_H^2\, ds \right\} \\ &\leq \frac{M_{1-\alpha}^2 \Gamma(2\alpha - 1)}{(2a)^{2\alpha-1}} \left\{ \gamma_\beta^2 h^{2\beta} \int_0^{t_1} E\|(-A)^{\alpha+\beta} a(s, X_s)\|_H^2\, ds \right. \\ &\left. + e^{-2ah} \int_{t_1}^{t_1+r} E\|(-A)^\alpha a(s, X_s)\|_H^2\, ds \right\} \end{aligned}$$

$$\leq 4C^2 2\frac{M_{1-\alpha}^2\,\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}[\gamma_\beta^2 h^2\beta t + he^{-2ah}](1 + E\|X_{t_1}\|_D^2),$$

Next,

$$\begin{aligned}E\|I_5(t_1+r) - I_5(t_1)\|_H^2 &= E\|\int_0^{t_1}(S(r)-I)S(t_1-s)\,f(s,X(s-\tau(s))\,ds \\ &+ \int_0^{t_1+r} S(t_1+r-s)\,f(s,X(s-\tau(s))\,ds\|_H^2 \\ &\leq 2c\int_0^{t_1}(t-s)^{-2\alpha}e^{-2a(t-s)}E\|f(s,X(s-\tau(s)))\|_H^2\,ds \\ &+ 2c\int_{t_1}^{t_1+r}(t-s)^{-2\alpha}e^{-2a(t-s)}E\|f(s,X(s-\tau(s)))\|_H^2\,ds.\end{aligned}$$

Then, using (H2), we may write

$$\begin{aligned}E\|I_5(t_1+r) - I_5(t_1)\|_H^2 \leq 2c&\int_0^{t_1}(t-s)^{-2\alpha}e^{-2a(t-s)}H(s,E\|X(s-\tau(s)))\|_D^2\,ds) \\ &+ 2c\int_{t_1}^{t_1+r} e^{-2a(t_1+r-s)}H(s,E\|X(s-\tau(s))\|_D^2)\,ds.\end{aligned}$$

Hence, using similar arguments as in Ahmed [1], Theorem 6.3.2, one can find constants K_1 and $K_2 > 0$ depending on the parameters μ,α,γ,k,h such that

$$E\|I_5(t_1+r) - I_5(t_1)\|_H^2 \leq 2k\gamma_\alpha\mu_\alpha[K_1h^{2\alpha} + K_2h](1 + H(t_1,E\|X(t_1-\tau(t_1))\|_D^2)).$$

Further assumption of (H2) indicates that there is a solution u_t given by

$$u_t = mE\|\varphi\|_D^2 + l\int_0^t H(r,u_r)dr,$$

with $m = l = 2k\gamma_\alpha\mu_\alpha[K_1h^{2\alpha} + K_2h]$, which satisfies

$$E\|I_5(t_1+r) - I_5(t_1)\|_H^2 \leq u_t < \infty.$$

Moreover,

$$\begin{aligned}E\|I_6(t_1+r) - I_6(t_1)\|_H^2 &\leq 2E\|\int_0^{t_1}(S(t_1+r-s) - S(t_1-s))g(s,X(s-\delta(s)))\,dW(s)\|_H^2 \\ &+ 2E\|\int_{t_1}^{t_1+r} S(t_1+r-s)g(s,X(s-\delta(s)))\,dW(s)\|_H^2\end{aligned}$$

$$\begin{aligned}E\|I_6(t_1+r) - I_6(t_1)\|_H^2 &\leq 2c_p\int_0^{t_1} E\|S(t_1+r-s) - S(t_1-s))g(s,X(s-\delta(s)))\|_H^2\,ds \\ &+ 2c_p\int_{t_1}^{t_1+r} E\|S(t_1+r-s)g(s,X(s-\delta(s)))\|_H^2\,ds\end{aligned}$$

$$\begin{aligned}E\|I_6(t_1+r) - I_6(t_1)\|_H^2 &\leq 2c_p\int_0^{t_1}(t-s)^{-2\alpha}e^{-2a(t-s)}E\|g(s,X(s-\tau(s)))\|_H^2 ds \\ &+ 2c_p\int_{t_1}^{t_1+r} e^{-2a(t_1+r-s)}E\|g(s,X(s-\tau(s))\|_H^2 ds.\end{aligned}$$

$$\begin{aligned}E\|I_6(t_1+r) - I_6(t_1)\|_H^2 &\leq 2c_p\int_0^{t_1}(t-s)^{-2\alpha}e^{-2a(t-s)}H(s,E\|X(s-\tau(s)))\|_D^2)ds \\ &+ 2c_p\int_{t_1}^{t_1+r} e^{-2a(t_1+r-s)}H(s,E\|X(s-\tau(s))\|_D^2)ds.\end{aligned}$$

wherein we could have used the work of Da Prato and Zabczyk [3], [7](Theorem 6.10, page 160), or Lemma 2.4 [11]. Arguing as before, we find constants K_3 and $K_4 > 0$ such that

$$E\|I_6(t_1+r)-I_6(t_1)\|_H^2 \le 2k\gamma_\alpha\mu_\alpha[K_3h^{2\alpha}+K_4h](1+H(s,E\|X(t_1-\tau(t_1)\|_D^2)).$$

Assumption of (H2) implies that there is a solution u_t given by

$$u_t = mE\|\varphi\|_D^2 + l\int_0^t H(r,u_r)dr,$$

with $m = l = 2k\gamma_\alpha\mu_\alpha[K_3h^{2\alpha}+K_4h]$, which satisfies

$$E\|I_6(t_1+r)-I_6(t_1)\|_H^2 \le u_t < \infty.$$

Similarly,

$$\begin{aligned}
E\|I_7(t_1+r)-I_7(t_1)\|_H^2 &\le 2E\|\int_0^{t_1}\int_z (S(t_1+r-s)-S(t_1-s))h(t,X(t-\alpha(s)),y)\widetilde{N}(ds,dy)\|_H^2 \\
&\quad + 2E\|\int_{t_1}^{t_1+r}\int_z S(t_1+r-s)h(t,X(t-\alpha(s)),y)\widetilde{N}(ds,dy)\|_H^2 \\
&\le 2E\|\int_0^{t_1} e^{-2\zeta(t-s)}\int_z \|h(t,X(t-\alpha(s)),y)\nu(dy)ds\|_H^2 \\
&\quad + 2E\int_{t_1}^{t_1+r} e^{-2\zeta(t-s)}\int_z \|h(t,X(t-\alpha(s)),y)\nu(dy)ds\|_H^2 \\
&\le 2c_p\int_0^{t_1}(t-s)^{-2\alpha}e^{-2a(t-s)}H(s,E\|X(s-\alpha(s)))\|_D^2 ds \\
&\quad + 2c_p\int_{t_1}^{t_1+r} e^{-2a(t_1+r-s)}H(s,E\|X(s-\alpha(s))\|_D^2)ds,
\end{aligned}$$

and this leads to

$$E\|I_7(t_1+r)-I_7(t_1)\|_H^2 \le u_t < \infty,$$

as $h \to 0$. Thus Θ is indeed continuous in 2-nd moment on $(0,\infty]$.

Next, we show that $\Theta(\mathcal{S}) \subset \mathcal{S}$. It follows from (8) that

$$\begin{aligned}
E\|\Theta(x)(t)\|_H^2 \le 7\{&E\|S(t)\xi(0)\|_H^2 + E\|S(t)a(0,\xi)\|_H^2 + E\|a(t,X_t)\|_H^2 \\
&+ E\|\int_0^t (-A)S(t-s)a(s,X_s)ds\|_H^2 \\
&+ E\|\int_0^t S(t-s)f(s,X(s-\tau(s)))\|_H^2\, ds \\
&+ E\|\int_0^t S(t-s)g(s,X(s-\delta(s)))\|_H^2\, dWs \\
&+ E\|\int_0^t S(t-s)\int_z h(s,X(s-\alpha(s)),y)\widetilde{N}(ds,dy)\|_H^2\Big\} \\
:= \sum_{i=1}^{7} J_i \quad . && (9)
\end{aligned}$$

We now estimate each term of (9), starting with

$$J_1 \le 7e^{2ar}e^{-2at}E\|\xi\|_D^2.$$

By lemma 2.1 and assumption (H4), we have

$$J_2 \le 7\|(-A)^{-\alpha}\|^2 e^{-2at}C_5^2(1+E\|\xi\|_D^2,$$

$$J_3 \le 7\|(-A)^{-\alpha}\|^2 C_5^2(1+E\|X(t_1)\|_D^2.$$

Next, by observing assumption (H4) and employing lemma 2.2, we have

$$J_4 \le 7\frac{M_{1-\alpha}^2\,\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}\int_0^t E\|(-A)^\alpha a(s,X_s)\|_H^2\,ds$$

$$\times\, 14TC_5^2\frac{M_{1-\alpha}^2\,\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}(1+E\|X_{t_1}\|_D^2),$$

and by assumption (H2) and lemma 2.1, we get

$$J_5 \le 7E\sup_{-r\le\theta\le 0}\int_0^{t+\theta} e^{-2a(t+\theta-s)}H(s,E\|X(s-\tau(s)\|)_H^2)\,ds$$

$$\le 7TC_3^2(1+H(t_1,E\|X(t_1)\|_D^2)).$$

Lastly, by [3], Theorem 6.10, and assumption (H2), we arrive at

$$J_6 \le 7k\int_0^t E\|b(s,X(s-\delta(s))\|_H^2\,ds$$

$$\times\, 7kTC_3^2(1+H(t_1),E\|X(t_1)\|_D^2),$$

and

$$J_7 \le 7kE\int_0^t e^{-2\gamma(t-s)}\int_z \|h(s,X(s-\alpha(s)),y)\widetilde{N}\nu(dy)\,ds$$

$$\times\, 7k(1+H(t_1,E\|X(t_1)\|_D^2)).$$

Consequently, $E\|(\Theta(X)(T)\|_D^2 < \infty$, implying that Θ maps $\mathcal{S}$ into itself. Thirdly, we will show that Θ is contractive. For $x,y\in\mathcal{S}$, proceeding as we did previously, we can obtain

$$\sup_{s\in[0,T]} E\|\Theta(x)(t)-\Theta(y)(t)\|_H^2 \le 5E\sup_{s\in[0,T]}\|a(t,X(t))-a(t,Y(t))\|_H^2$$

$$\times 5\,E\sup_{s\in[0,T]}\|\int_0^t(-A)S(t-s)[a(t,X(t))-a(t,Y(t))]ds\|_H^2$$

$$\times 5\,E\sup_{s\in[0,T]}\|\int_0^t S(t-s)(f(s,X(s-\tau(s)))-f(s,Y(s-\tau(s))))ds\|_H^2$$

$$\times 5\,E\sup_{s\in[0,T]}\|\int_0^t S(t-s)(g(s,X(s-\delta(s)))-g(s,Y(s,\delta(s))))dW(s)\|_H^2$$

$$\times 5\,kE\sup_{s\in[0,T]}\|\int_0^t S(t-s)\int_z h(s,X(s-\alpha(s)),y)-h(s,Y(s-\alpha(s)),y)\nu(dy)ds\|_H^2$$

$$\le 5C_4^2\|-A^{-\alpha}\|^2\sup_{s\in[0,T]}E\|X(t)-Y(t)\|_D^2$$

$$\times 5\frac{M_{1-\alpha}^2\,\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}TC_4^2\sup_{s\in[0,T]}E\|X(t)-Y(t)\|_D^2$$

$$\times 5T\int_0^t H(r, E(\sup_{s\in[0,T]} \|X(t)-Y(t)\|_D^2)dr$$

$$\times 5C_1\int_0^t H(r, E(\sup_{s\in[0,T]} \|X(t)-Y(t)\|_D^2)dr$$

$$\times k\int_0^t H(r, E(\sup_{s\in[0,T]} \|X(t)-Y(t)\|_D^2)dr.$$

Now choosing $T > 0$ sufficiently small, we can find a positive number$K(T) \in [0,1]$ such that

$$\|\Theta(X)-\Theta(Y)\|_{\mathcal{S}} \leq K(T)[\|X(t)-Y(t)\|_D + H(t, X(t)-Y(t))].$$

For any $X, Y \in \mathcal{S}$. Hence, by the Banach fixed point theorem and assumption (H3), Θ has a unique fixed point $X \in \mathcal{S}$ and this fixed point is the unique mild solution of (1) on $[0,T]$. ■

4. Illustrative Example

Consider the neutral stochastic partial functional differential equation with finite delays $\tau_1(t), \tau_2(t), \tau_3(t), \tau_4 \to \infty, t \to \infty$:

$$d[u(t,x) + \frac{l_3(t)}{\|(-A)^{3/4}\|}\int_{-\tau_3(t)}^0 u(t+w,x)dw] = \left[\frac{\partial^2}{\partial x^2}u(t,x) + l_1(t)\int_{-\tau_1(t)}^0 u(t+w,x)dw\right]dt$$

$$+l_2(t)u(t-\tau_2(t),x)d\beta(t) + \int_z \sqrt{\alpha_4}\, zX(t-\tau_4(t))\widetilde{N}(dz,dt), \quad t > 0,$$

$$l_i : \mathbb{R}^+ \to \mathbb{R}^+, \quad i = 1,2,3; \quad u(t,0) = u(t,\pi) = 0 \quad t > 0,$$

$$u(s,x) = \xi,$$

where $\beta(t)$ is a standard one-dimensional Wiener process, $l_i(t)$, $i = 1,2,3$, are continuous functions and $E\|\xi\|^2 < \infty$. Let $H = L^2[0,\pi]$, $K = \mathbb{R}^1$, and define $-A : H \to H$ by $-A = \partial^2/\partial x^2$ with domain $D(-A) = z \in H$. And assume z and $\partial z/\partial x$ to be absolutely continuous, $\partial^2 z/\partial x^2 \in H$, $z(0) = z(\pi) = 0$. Then

$$-Az = \sum_{n=1}^{\infty} n^2(z,z_n)z_n, \quad z \in D(-A), \tag{10}$$

where $z_n(x) = \sqrt{2/\pi}\, sin\; nx, n = 1,2,3\ldots$ is the orthonormal set of eigenvectors of $-A$. It is well known that $-A$ is the infinitesimal of an analytic semigroup $S(t), t \geq 0$ in H and is given by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}(z,z_n)z_n, \quad z \in H, \tag{11}$$

that satisfies $\|\; S(t)\; \| \leq e^{-\pi^2 t}$, $t \geq 0$, and hence is a contraction semigroup.

Define now

$$a(t,w_t) = \frac{l_3(t)}{\|(-A)^{3/4}\|}\int_{-\tau_3(t)}^{0} u(t+w,x)dw,$$
$$f(t,w_t) = l_1(t)\int_{-\tau_3(t)}^{0} u(t+w,x)dw,$$
$$g(t,w_t) = l_2 u(t-\tau_2(t),x),$$
$$h(t,X(t-\tau_4),y) = \alpha_4\, yX(t-\tau_4).$$
$$\begin{aligned}\| a(t,w_t) \|_{3/4} &= \frac{l_3(t)}{\|(-A)^{3/4}\|} \ | \ (-A)^{3/4}\int_{-\tau_3(t)}^{0} u(t+w,x)dw \ | \\ &\leq l_3(T)\tau_3(t) \ \| \ u \ \| \quad a.s. \end{aligned} \tag{12}$$

Let $u(t,v) = \theta(t)\bar{u}(v), t \in [0,T]$ where $\theta(t) \geq 0$ is locally integrable and $\bar{u}(v)$ is a concave nondecreasing function from $\mathbb{R}^+$ *to* $\mathbb{R}^+$ such that $\bar{u}(0) = 0, \bar{u}(v) > 0$ for $u > 0$ and $\int_{0^+}\frac{dv}{\bar{u}(v)} = \infty$. The comparison theorem of ordinary differential equation shows that assumption (H_2). Now let us give some concrete examples of the function $\bar{u}$. Let $K > 0$ and let $\delta \in (0,1)$ be sufficiently small. Define then

$$\bar{u}_1(v) = Kv, \quad v \geq 0,$$

with

$$\bar{u}_2(v) = \begin{cases} v\ ln(v^{-1}), & 0 \leq v \leq \delta, \\ \delta\ ln(\delta^{-1}) + \bar{u}_2^{'}(\delta -)(v-\delta), & v > \delta \end{cases} \tag{13}$$

$$\bar{u}_3(v) = \begin{cases} vln(v^{-1})ln\ ln(v^{-1}), & 0 \leq \delta \\ \delta ln(\delta^{-1})ln\ ln(\delta^{-1}) + \bar{u}_3^{'}(\delta -)(v-\delta), & v > \delta \end{cases} \tag{14}$$

where $\bar{u}^{'}$ denotes the derivative of function $\bar{u}$. All $\bar{u}$ are concave nondecreasing functions satisfying $\int_0 \frac{dv}{\bar{u}(v)} = +\infty$. Hence, there exists a unique mild solution by theorem 3.2.

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