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Existence Results for Time-Dependent Stochastic Neutral Functional Integrodifferential Equations Driven by a Fractional Brownian Motion*

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Abstract. *This article presents some results on existence and uniqueness of mild solutions to neutral stochastic functional evolution integrodifferential equations driven by a fractional Brownian motion. The existence of mild solutions for the equations are discussed by means of theory of resolvent operators. Under some sufficient conditions, results are obtained by using a Banach contraction principle.*

Key words : Resolvent Operators, Evolution Operators, Existence, Uniqueness, C_0-semigroup, Wiener Process, Mild Solutions, Fractional Brownian Motion.

AMS Subject Classifications : 60H15, 93E15, 35R12

1. Introduction

In this paper, we study the existence of mild solutions for a class of abstract stochastic partial neutral functional integro-differential equations modeled in the form

$$\left.\begin{aligned} d[u(t)+G(t,u(t-r(t)))] &= A(t)[u(t)+G(t,u(t-r(t)))]dt \\ &\quad + \left[\int_0^t B(t,s)[u(s)+G(s,u(s-r(s)))]ds + F(t,u(t-\delta(t)))\right]dt \\ &\quad + \sigma(t)dB^H(t) \qquad for \quad t\in[0,T], \\ u_0(.) &= \varphi, \qquad -\tau\le t\le 0, \end{aligned}\right\} \tag{1}$$

where $A(t)$ is a linear operator which generates a linear evolution system $\{R(t,s),\ t\ge 0\}$ on a

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Hilbert space X, $B(t)$ is a closed linear operator on X with domain $D(B) \supset D(A)$ which is independent of t, B^H is a fractional Brownian motion on a real and separable Hilbert space Y. The functions r,δ defined from$[0,+\infty)$ into $[0,\tau](\tau > 0)$ are measurable, and $G,F : [0,+\infty) \times X \longrightarrow X$, σ $[0,+\infty) \longrightarrow \mathcal{L}_2^0(Y,X)$ are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see Section 2).

Neutral integro-differential equations arise in many areas of applied mathematics. For instance, the system of rigid heat condition with finite wave speeds, studied in [6], can be modeled in the form of integrodifferential equations of neutral type with delay, and for this reason these equations (with an initial condition or a nonlocal condition) have received much attention in the last few decades. One of the important techniques to discuss these topics is the semigroup approach; see , for example [13, 8, 9]. In the paper [5], Caraballo and Diop investigated the existence of solutions for the following stochastic functional differential equation:

$$\left.\begin{aligned} d[u(t) + G(t,u(t-r(t)))] &= A[u(t) + G(t,u(t-r(t)))]dt \\ &\quad + \Big[\int_0^t B(t,s)[u(s) + G(s,u(s-r(s)))]ds + F(t,u(t-\delta(t)))\Big]dt \\ &\quad + \sigma(t)dB^H(t) \qquad for\ \ t \in [0,T], \\ u_0(.) &= \varphi, \qquad -\tau \le t \le 0, \end{aligned}\right\} \tag{2}$$

which differs from (1) only in its $A(t)$, by using a semigroup approach and classical fixed point arguments.

In this work, the linear part in our equation is an operator independent of time t and generates a strongly continuous semigroup, so that the semigroup approach can be employed. Our purpose in the present paper is to establish some results concerning existence and uniqueness of the solutions for the non-autonomous stochastic integrodifferential equations (1). A motivating example for this type of equations is the following non-autonomous boundary problem:

$$\left.\begin{aligned} \frac{\partial}{\partial t}[x(t,\xi) + g(t,x(t-\rho_1,\xi))] &= \Big[\frac{\partial^2}{\partial \xi^2} + \gamma(t,\xi)\Big][x(t,\xi) + g(t,x(t-\rho_1,\xi))] \\ &\quad + \int_0^t b(t-s)\frac{\partial^2}{\partial \xi^2}[x(s,\xi) + g(s,x(s-\rho_1,\xi))]ds \\ &\quad + f(t,x(t-\rho_2,\xi)) + \sigma(t)\frac{dB^H}{dt}(t), \\ x(t,0) + g(t,x(t-r(t),0)) &= 0 \quad for \quad t \ge 0, \\ x(t,\pi) + g(t,x(t-r(t),\pi)) &= 0 \quad for \quad t \ge 0, \\ x(\theta,\xi) = x_0(\theta,\xi),\varphi(s,.) &\in L^2[0,T], \quad -\rho \le \theta \le 0, \quad 0 \le \xi \le \pi. \end{aligned}\right\} \tag{3}$$

Such problems arise in the study of stochastic systems in the presence of hereditary influences on the state variable. For example, stochastic integrodifferential systems which cover a large area of system dynamics including reactor dynamics [4, 14, 16], heat transfer by conduction and radiation [17, 15], mathematical modeling of system hysteresis [11, 14], models of transmission of infection of diseases [3]. Therefore, it is meaningful to deal with (1) to acquire some results applicable to problem (3).

As we know, non-autonomous evolution equations are much more complicated, to be dealt with, than autonomous ones. Our approach here is to assume that $\{A(t) \ : t \geq 0\}$ is a family of linear operators on X with dense domain such that it generates a linear evolution system. The results in this paper are natural continuation and generalization of the results reported by Caraballo and Diop [5].

Let us now describe the remaining contents of the paper. In Section 2, we introduce some notations, concepts of resolvent operators, basic results about fractional Brownian motion and Wiener integral over Hilbert space. In Section 3, we prove the existence and uniqueness of mild solutions for the system (1). An example to illustrate our previous abstract results is analyzed in Section 4.

2. Wiener Process and Deterministic Integrodifferential Equations

2.1. **Wiener process**

In this section we introduce the fractional Brownian motion as well as the Wiener integral with respect to it. We also need to establish some important results which will be needed throughout the paper. So, first, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 2.1. Given $H \in (0,1)$, a continuous centered Gaussian process $\beta^H(t),\ t \in \mathbb{R}$, with covariance function

$$R_H(s,t) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t,s \in \mathbb{R},$$

is called a two-sided one-dimensional fractional Brownian motion (fBm), and H is the Hurst parameter.

Now we aim at introducing the Wiener integral with respect to the one-dimensional fBm β^H. Let $T > 0$ and denote by Λ the linear space of $\mathbb{R}$-valued step function on $[0,T]$, that is $\phi \in \Lambda$ if

$$\phi(t) = \sum_{i=1}^{n-1} x_i \ 1_{[t_i,t_{i+1})}(t),$$

where $t \in [0,T]$, $x_i \in \mathbb{R}$ and $0 = t_1 < t_2 < \cdots < t_n = T$. For $\phi \in \Lambda$ we define its Wiener integral with respect to β^H as

$$\int_0^T \phi(s)\, d\beta^H(s) = \sum_{i=1}^{n-1} x_i(\beta^H(t_{i+1}) - \beta^H(t_i)).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of Λ with respect to the scalar product $\langle 1_{[0;t]}, 1_{[0;s]}\rangle_{\mathcal{H}} = R_H(t,s)$. Then the mapping

$$\phi(t) = \sum_{i=1}^{n-1} x_i \ 1_{[t_i,t_{i+1})}(t) \to \int_0^T \phi(s)\, d\beta^H(s)$$

is an isometry between Λ and the linear space span$\{\beta^H, \quad t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{\text{span}}^{L^2(\Omega)}\{\beta^H, \quad t \in [0,T]\}$ (see [21]). The image of an element $\varphi \in \mathcal{H}$ by this isometry is called the Wiener integral of φ with respect to β^H. Our next goal is to give an explicit expression for this integral. To this end, consider the Kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du ,$$

where $c_H = \sqrt{\frac{H(2H-1)}{\hat{B}(2-2H, H-\frac{1}{2})}}$, with $\hat{B}$ denoting the Beta function and $t \leq s$. It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \Big(\frac{t}{s}\Big)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}.$$

Consider the linear operator $K_H^* : \Lambda \to L^2([0,T])$ given by

$$(K_H^*\varphi)(s) = \int_s^t \varphi(t) \frac{\partial K}{\partial t}(t,s)dt.$$

Then

$$K_H^* 1_{[0;t]} \ (s) = K_H(t,s) 1_{[0;t]}(s),$$

and K_H^* is an isometry between Λ and $L^2([0,T])$ that can be extended to Λ (see [1]).
Considering $W = \{W(t), \quad t \in [0,T]\}$ defined by

$$W(t) = \beta^H((K_H^*)^{-1} 1_{[0;t]}),$$

it turns out that W is a Wiener process and β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s) \, dW(s).$$

In addition, for any $\varphi \in \Lambda$,

$$\int_0^T \phi(s) \, d\beta^H(s) = \int_0^T (K_H^*\varphi)(t) dW(t),$$

if and only if $K_H^*\varphi \in L^2([0,T])$.
Also denoting $L^2_{\mathcal{H}}([0,T]) = \{\varphi \in \Lambda, \ K_H^*\varphi \in L^2([0,T])\}$, since $H > \frac{1}{2}$, allows for

$$L^{\frac{1}{H}}([0,T]) \subset L^2_{\mathcal{H}}([0,T]), \tag{4}$$

see [18]. Moreover, the following useful result holds.

Lemma 2.1. [19] *For* $\varphi \in L^{\frac{1}{H}}([0,T])$,

$$H(2H-1) \int_0^T \int_0^T |\varphi(r)||\varphi(u))||r-u|^{2H-2} drdu \leq c_H \|\varphi\|^2_{L^{\frac{1}{H}}([0,T])}.$$

Next we are interested in considering a fBm with values in a Hilbert space and giving the definition of the corresponding stochastic integral. Let $(X, \|.\|_X, (.,.)_X)$ and $(Y, \|.\|_Y, (.,.)_Y)$ be separable Hilbert spaces. Let $\mathcal{L}(X,Y)$ denote the space of all bounded linear operator from X to Y. Let $Q \in \mathcal{L}(X,Y)$ be a non-negative self-adjoint operator. Denote by $\mathcal{L}_2^0(Y,X)$ the space of $\vartheta \in L(Y,X)$ such that $\vartheta Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$|\vartheta|^2_{\mathcal{L}_2^0(Y,X)} = \left|\vartheta Q^{\frac{1}{2}}\right|_{HS} = tr(\vartheta Q \vartheta^*).$$

Then ϑ is called a Q-Hilbert-Schmidt operator from Y to X.
Let $\{\beta_n^H(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega,\mathcal{F},\mathbb{P})$. When one considers the following series

$$\sum_{n=1}^{\infty}\beta_n^H(t)e_n, \qquad t\geq 0,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in X, this series does not necessarily converge in the space Y. Thus we consider a Y-valued stochastic process

$$B_Q^H(t)=\sum_{n=1}^{\infty}\beta_n^H(t)Q^{\frac{1}{2}}e_n, \qquad t\geq 0.$$

If Q is a non-negative self-adjoint trace class operator, this series converges in the space Y, that is, it holds that $B_Q^H(t)\in L^2(\Omega,Y)$. Then, we say that the above $B_Q^H(t)$ is a Y-valued Q-cylindrical fractional Brownian motion with covariance operator Q. For example, if $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n=\sigma_ne_n$, assuming that Q is a nuclear operator in Y (that is, $\sum_{n=1}^{\infty}\sigma_n<\infty$), then the stochastic process

$$B_Q^H(t)=\sum_{n=1}^{\infty}\beta_n^H(t)Q^{\frac{1}{2}}e_n=\sum_{n=1}^{\infty}\sqrt{\sigma_n}\,\beta_n^H(t)e_n, \qquad t\geq 0,$$

is well-defined as a Y-valued Q-cylindrical fractional Brownian motion.
Then let $\varphi : [0,T]\to L_Q^0(Y,X)$ such that

$$\sum_{n=1}^{\infty}\left\|K_H^*(\varphi Q^{\frac{1}{2}}e_n)\right\|_{\mathcal{L}^2([0,T];X)}<\infty. \tag{5}$$

Definition 2.2. Given $H\in(0,1)$, and let $\varphi : [0,T]\to L_H^0(Y,X)$ satisfy (5). Then, its stochastic integral with respect to the fBm B_Q^H is defined, for $t\geq 0$, as follows

$$\int_0^t\varphi(s)dB_Q^H(s) := \sum_{n=1}^{\infty}\int_0^t\varphi(s)Q^{\frac{1}{2}}e_n\beta_n^H(s)=\sum_{n=1}^{\infty}\int_0^t(K_H^*(\varphi Q^{\frac{1}{2}}e_n))(s)dW(s). \tag{6}$$

Notice that if

$$\sum_{n=1}^{\infty}\|\varphi Q^{\frac{1}{2}}e_n\|_{L^{\frac{1}{H}}([0,T];X)}<\infty,$$

then in particular (5) holds, which follows immediately from (4).

Now we end this subsection by stating the following result which is crucial for proving our main result. It can be proved by similar arguments to those used in Lemma 2 in Caraballo et al. [6].

Lemma 2.2. *If $\psi : [0,T] \to L_2^0(X,Y)$ satisfies $\int_0^T \|\psi\|^2_{\mathcal{L}_2^0} ds < \infty$, then the above sum in (6) is well defined as a X-valued random variable and*

$$\mathbb{E}\left\| \int_0^t \psi(s) dB^H(s) \right\|^2 \leq 2H\, t^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

Proof. See [2]. ■

2.2. **The stochastic convolution integral**

Here we present some properties of the stochastic convolution integral of the form

$$\tilde{R}(t) \quad = \quad \int_0^t R(t,s)\sigma(s)\, dB^H(s), \quad t \in [0,T],$$

where $\sigma(s) \in \mathcal{L}_2^0(X,Y)$ and $\left\{R(t,s),\ t \geq 0\right\}$ is an evolution system of operators.
The following result on the stochastic convolution integral $\tilde{R}$ should always hold.

Lemma 2.3. *Suppose that $\sigma : [0,T] \to L_2^0(X,Y)$ satisfies $\sup_{t\in[0,T]} \|\sigma(t)\|^2_{\mathcal{L}_2^0(X,Y)} < \infty$, and suppose that $\left\{R(t,s),\ t \geq 0\right\}$ is an evolution system of operators satisfying $\|R(t,s)\| \leq Me^{-\beta(t-s)}$, for some constants $\beta > 0$ and $M \geq 1$, for all $t > s$. Then*

$$\mathbb{E}\left\| \int_0^t R(t,s)\sigma(s)\, dB^H(s) \right\|^2 \leq CM^2 t^{2H} (\sup_{t\in[0,T]} \|\sigma(t)\|^2_{\mathcal{L}_2^0(X,Y)}).$$

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal basis of Y and $\{\beta_n^H\}_{n\in\mathbb{N}}$ is a sequence of independent, real-valued standard fractional Brownian motion each with the same Hurst parameter $H \in (\frac{1}{2}, 1)$. Thus, using the fractional Itô isometry one can write

$$\begin{aligned}
&\mathbb{E}\left\| \int_0^t R(t,s)\sigma(s) dB^H(s) \right\|^2 \\
&= \sum_{n=1}^{\infty} \mathbb{E}\left\| \int_0^t R(t,s)\sigma(s) e_n d\beta_n^H(s) \right\|^2 \\
&= \sum_{n=1}^{\infty} \int_0^t \int_0^t \langle\, R(t,s)\sigma(s)e_n, R(t,r)\sigma(r)e_n \rangle H(2H-1)|s-r|^{2H-2} ds dr \\
&\leq H(2H-1) \int_0^t \left\{ \|R(t,s)\sigma(s)\| \int_0^t \|R(t,r)\sigma(r)\| |s-r|^{2H-2} dr \right\} ds \\
&\leq H(2H-1)M^2 \int_0^t \left\{ e^{-\beta(t-s)} \|\sigma(s)\|_{\mathcal{L}_2^0(X,Y)} \int_0^t e^{-\beta(t-r)} |s-r|^{2H-2} \|\sigma(r)\|_{\mathcal{L}_2^0(X,Y)} dr \right\} ds.
\end{aligned}$$

Since σ is bounded, one can then conclude that

$$\begin{aligned}
\mathbb{E}\left\| \int_0^t R(t,s)\sigma(s) dB^H(s) \right\|^2 &\leq H(2H-1)M^2 (\sup_{t\in[0,T]} \|\sigma(t)\|^2_{\mathcal{L}_2^0(X,Y)})^2 \times \\
&\times \int_0^t \left\{ e^{-\beta(t-s)} \int_0^t e^{-\beta(t-r)} |s-r|^{2H-2} dr \right\} ds.
\end{aligned}$$

Performing the change of variables $v = t - s$ for the first integral, and $u = t - r$ for the second one, we obtain

$$\mathbb{E}\left\| \int_0^t R(t,s)\sigma(s)dB^H(s) \right\|^2 \leq H(2H-1)M^2 \Big(\sup_{t\in[0,T]} \|\sigma(t)\|^2_{\mathcal{L}_2^0(X,Y)} \Big)^2 \times$$
$$\times \int_0^t \left\{ e^{-\beta v} \int_0^t e^{-\beta u}|u-v|^{2H-2}du \right\} dv.$$

From [29] it follows that

$$\mathbb{E}\left\| \int_0^t R(t,s)\sigma(s)dB^H(s) \right\|^2 \leq CM^2 t^{2H} \Big(\sup_{t\in[0,T]} \|\sigma(t)\|^2_{\mathcal{L}_2^0(X,Y)} \Big)^2,$$

and the proof is then complete. ■

2.3. **Partial integro-differential equations in Banach spaces**

Let us recall some fundamental results needed to establish our results. The resolvent operators play an important role to study the existence of solutions and to give a variation of constants formula for nonlinear systems. We need to know when the linear system (7) has a resolvent operator. For more details on resolvent operators, we refer the reader to [12]. The following assumptions are:
- (i) $A(t)$ generates a strongly continuous semigroup of evolution operators.
- (ii) Suppose Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_Y := |Ay| + |y| \quad \text{for} \quad y \in Y.$$

$A(t)$ and $B(t,s)$ are in the set of bounded linear operators from Y to X, $\mathcal{L}(Y,X)$ for $0 \leq t \leq T$ and $0 \leq s \leq T$ respectively. $A(t)$ and $B(t,s)$ are continuous on $0 \leq t \leq T$ and $0 \leq s \leq t \leq T$, respectively, into $\mathcal{L}(Y,X)$.
To obtain the results, we consider the following abstract integrodifferential Cauchy problem

$$\left.\begin{aligned} dv(t) &= \left[A(t)v(t) + \int_0^t B(t,s)v(s)ds \right] dt, \quad \text{for} \quad 0 \leq s \leq t \leq T, \\ v(0) &= v_0 \in X. \end{aligned}\right\} \tag{7}$$

Definition 2.3. [12] A resolvent operator for Eq(7) is a bounded linear operator valued function $R(t,s) \in \mathcal{L}(X)$ for $0 \leq s \leq t \leq T$, satisfying the following properties:
- (i) $R(t,t) = I$ and $|R(t,s)| \leq Ne^{\beta(t-s)}$, $t,s \in [0,T]$ for some constants N and β.
- (ii) $R(t,s)$ is strongly continuous in s and t.
- (iii) For $y \in Y$, $R(t,s)y$ is continuously differentiable in s and t, and for $0 \leq s \leq t \leq T$,

$$\frac{\partial}{\partial t}R(t,s)y = A(t)R(t,s)y + \int_s^t B(t-r)R(r,s)ydr,$$

$$\frac{\partial}{\partial s}R(t,s)y = -R(t,s)A(s)y - \int_s^t R(t,r)B(r-s)ydr,$$

with $\frac{\partial}{\partial t}R(t,s)y$ and $\frac{\partial}{\partial s}R(t,s)$ are strongly continuous on $0 \leq s \leq t \leq T$. Here $R(t,s)$ can be extracted from the evolution operator of the generator $A(t)$.

For the family of linear operators $\{A(t) : 0 \leq t \leq T\}$, the following assumptions need to be imposed:

- (**H1**) The domain $D(A)$ of $\{A(t) : 0 \leq t \leq T\}$ is dense in X and independent of t; $A(t)$ is a closed linear operator.
- (**H2**) For each $t \in [0,T]$, the resolvent operator $\hat{R}(\lambda, A(t))$ exists for all λ with $Re\lambda \leq 0$ and there exists $K > 0$ such that $\|\hat{R}(\lambda, A(t))\| \leq \frac{K}{(|\lambda|+1)}$.
- (**H3**) There exists $0 < \delta \leq 1$ and $K > 0$ such that $\|(A(t) - A(s))A^{-1}(r)\| \leq K|t-s|^{\delta}$ for all $t,s,r \in [0,T]$.
- (**H4**) For each $t \in [0,T]$ and some $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, the resolvent $\hat{R}(\lambda, A(t))$, is a compact operator.

Under these assumptions, the family $\{A(t) : 0 \leq t \leq T\}$ generates a unique linear evolution system, also called linear evolution operator.

Definition 2.4. [20] A two parameter family of bounded linear operators
$U(t,s),\ 0 \leq s \leq t \leq T$, on X is called an evolution system if the following two conditions holds

- (i) $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$, for $0 \leq s \leq r \leq t \leq T$.
- (ii) $(t,s) \longrightarrow U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

Lemma 2.4. [20] *Assume that $(H1)-(H3)$ hold. Then, there exist a unique evolution system $U(t,s),\ 0 \leq s \leq t \leq T$ and a constant $K > 0$ such that*

- *(i) $U(t,s) \leq K$ for $0 \leq s \leq t \leq T$,*
- (ii) *for $0 \leq s \leq t \leq T$, $U(t,s) : X \to Y$ and $t \to U(t,s)$ is strongly differentiable in X. The derivative $\frac{\partial}{\partial t}U(t,s)$ belongs to $L(X)$ and it is strongly continuous on $0 \leq s \leq t \leq T$. Moreover, for all $0 \leq s \leq t \leq T$, it holds*

$$\frac{\partial}{\partial t}U(t,s) + A(t)U(t,s) = 0,$$

$$\left\| \frac{\partial}{\partial t}U(t,s) \right\| = \|A(t)U(t,s)\| \leq \frac{K}{t-s},$$

$$\|A(t)U(t,s)A(s^{-1})\| \leq K,$$

- (iii) *for each $y \in Y$ and $t \in [0,T]$, $U(t,s)y$ is differentiable with respect to s on $0 \leq s \leq t \leq T$ and $\frac{\partial}{\partial t}U(t,s)y = U(t,s)A(s)y$.*

Lemma 2.5. [10] *Let $\{A(t), t \in [0,T]\}$ be a family of linear operators satisfying (H1)-(H4). If $\{U(t,s),\ 0 \leq s \leq t \leq T\}$ is the linear evolution system generated by $\{A(t), t \in [0,T]\}$, then $\{U(t,s), 0 \leq s \leq t \leq T\}$ is a compact operator whenever $t - s > 0$.*

3. Existence of Mild Solutions for Eq (1)

In this section, we establish the existence and uniqueness of mild solutions of Eq (1) using a contraction mapping principle. For this reason we introduce the following technical assumptions.

- (**H5**) There exists a resolvent operator $R(t,s)$ which is compact and continuous in the uniform operator topology for $t > s$.

• (**H6**) The function $F : [0,+\infty) \times X \to X$ satisfies the following conditions: there exist positive constants C_1, C_2 such that, for all $t \in [0,T]$ and $x,y \in X$

$$\|F(t,x) - F(t,y)\| \leq C_1\|x - y\|,$$

$$\|F(t,x)\|^2 \leq C_2(1 + \|x\|^2).$$

• (**H7**) The function $G : [0,+\infty) \times X \to X$ satisfies the following conditions: there exist positive constants $C_3, C_4, 0 < C_3 < 1$ such that, for all $t \in [0,T]$ and $x,y \in X$

$$\|G(t,x) - G(t,y)\| \leq C_3\|x - y\|,$$

$$\|G(t,x)\|^2 \leq C_4(1 + \|x\|^2).$$

• (**H8**) The function G is continuous in the mean square sense. For all $x \in u \in \mathcal{C}([0,T], L^2(\Omega,X))$, it holds that $\lim_{t\to s} \mathbb{E}\|G(t,x(t)) - G(s,x(s))\|^2 = 0$.

• (**H9**) The function $\sigma : [0,+\infty) \to \mathcal{L}_2^0(Y,X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \quad \forall T > 0.$$

Moreover, we assume that $\varphi \in \mathcal{C}([-\tau,0], L^2(\Omega,X))$. Next, we introduce the concept of mild solution for Eq (1).

Definition 3.1. An X-valued process $\{u(t),\ t \in [-\tau,T]\}$, is called a mild solution of Eq (1) if $u \in \mathcal{C}([-\tau,T], L^2(\Omega,X)), u(t) = \varphi(t)$ for $t \in [-\tau,0]$, and, for $t \in [0,T]$, satisfies

$$\begin{aligned} u(t) + G(t,u(t - r(s))) &= R(t,0)[\varphi(0) - G(0,\varphi(-r(0)))] + \int_0^t R(t,s)F(s,u(s - \delta(s)))ds \\ &+ \int_0^t R(t,s)\sigma(s)dB^H(s) \quad \mathbb{P} - a.s. \end{aligned}$$

To prove our main results we first recall the next lemma, which is Lemma 1 of [7] by Caraballo et al.

Lemma 3.1. [7] *For $x,y \in X$ and $0 < c < 1$,*

$$\|x\|_X^2 \leq \frac{1}{1-c}\|x - y\|_X^2 + \frac{1}{c}\|y\|_X^2.$$

Theorem 3.1. *Under the assumptions $(H1) - (H9)$, for every $\varphi \in C([-\tau,0], L^2(\Omega,X))$ there exists a unique mild solution u to Eq (1).*

Proof. Assume that $T > 0$ is a fixed time and let $C_T := \mathcal{C}([-\tau,T], L^2(\Omega,X))$ be the Banach space of all continuous functions from $[-\tau,T]$ into $L^2(\Omega,X)$ equipped with the supremum norm $\|\zeta\|_{C_T} = \sup_{z\in[-\tau,T]} (\mathbb{E}\|\zeta(z)\|^2)^{1/2}$, and let us consider the set

$$S_T(\varphi) := \{u \in \mathcal{C}([-\tau,T],\ L^2(\Omega,X)) \ : u(s) = \varphi(s),\ for \quad s \in [-\tau,0]\}.$$

$S_T(\varphi)$ is a closed subset of C_T provided with the norm $\|.\|_{C_T}$. Define the operator Γ on $S_T(\varphi)$ by $\Gamma(u)(t) = \varphi(t)$ for $t \in [-\tau, 0]$, and for $t \in [0, T]$

$$\begin{aligned}\Gamma(u)(t) &= R(t,0)[\varphi(0) - G(0,\varphi(-r(0)))] - G(t,u(t-r(t))) \\ &+ \int_0^t R(t,s)F(s,u(s-\delta(s)))ds + \int_0^t R(t,s)\sigma(s)dB^H(s).\end{aligned} \tag{8}$$

It is clear that, proving the existence of mild solutions to Eq (1) is equivalent to finding a fixed point for the operator Γ.

Next we will show, by using the Banach fixed point theorem that Γ has a fixed point. We split the proof into two steps.

Step 1: For arbitrary $u \in S_T(\varphi)$, let us prove that $t \to \Gamma(u)(t)$ is continuous on the interval $[0, T]$ in the $L^2(\Omega, X)$-sense. Let $0 < t < T$ and $|h|$ be sufficiently small. Then, for any fixed $u \in S_T(\varphi)$, we have

$$\begin{aligned}\|(\Gamma(u)(t+h) - (\Gamma(u)(t)\| &\leq \|(R(t+h,0) - R(t,0))[\varphi(0) - G(0,\varphi(-r(0)))]\| \\ &+ \|G(t+h,u(t+h-r(t+h))) - G(t,u(t-r(t)))\| \\ &+ \left\|\int_0^{t+h} R(t+h,s)F(s,u(s-\delta(s)))ds - \int_0^t R(t,s)F(s,u(s-\delta(s))ds\right\| \\ &+ \left\|\int_0^{t+h} R(t+h,s)\sigma(s)dB^H(s) - \int_0^t R(t,s)\sigma(s)dB^H(s)\right\| \\ &= \sum_{1\leq i\leq 4} I_i(h).\end{aligned}$$

Using the continuity of $R(t,s)$, we obtain

$$\lim_{h\to 0} (R(t+h,0) - R(t,0))(\varphi(0) - G(0,\varphi(-r(0)))) = 0.$$

From $(H4)$, we have

$$\begin{aligned}&\|(R(t+h,0) - R(t,0))(\varphi(0) - G(0,\varphi(-r(0))))\| \\ &\qquad \leq [Ne^{\beta(t+h)} + Ne^{\beta t}]\|\varphi(0) - G(0,\varphi(-r(0)))\|_{L^2(\Omega)}.\end{aligned}$$

Then, by the Lebesgue Majorant Theorem, we conclude that

$$\lim_{h\to 0} \mathbb{E}|I_1(h)|^2 = 0.$$

Moreover, assumption (H8) ensures that

$$\lim_{h\to 0} \mathbb{E}|I_2(h)|^2 = 0.$$

For the third term $I_3(h)$, we suppose $h > 0$ (similar estimates hold for $h < 0$), then we have

$$\begin{aligned}I_3(h) &\leq \left\|\int_0^t (R(t+h,s) - R(t,s))F(s,u(s-\delta(s)))ds\right\| + \left\|\int_t^{t+h} R(t,s)F(s,u(s-\delta(s)))ds\right\| \\ &\leq I_{31}(h) + I_{32}(h),\end{aligned}$$

and due to Hölder's inequality,

$$\mathbb{E}|I_{31}(h)|^2 \leq t\,\mathbb{E}\int_0^t \|(R(t+h,s) - R(t,s))F(s,u(s-\delta(s)))\|^2 ds.$$

Again exploiting the continuity of $R(t,s)$, we have for each $s \in [0, t]$,

$$\lim_{h\to 0} (R(t+h,s)-R(t,s))F(s,u(s-\delta(s))) = 0,$$

and

$$\|(R(t+h,s)-R(t,s))F(s,u(s-\delta(s)))ds\| \leq \tilde{N}\|F(s,u(s-\delta(s)))ds\| \in L^2([0,t]\times\Omega),$$

where $\tilde{N} = [2N^2e^{-2\beta(t+h)} + 2N^2e^{-2\beta t}]$. Then, by the Lebesgue Majorant Theorem once more, we conclude that

$$\lim_{h\to 0} \mathbb{E}|I_{31}(h)|^2 = 0.$$

Next, by Hölder's inequality, it follows that

$$\mathbb{E}|I_{32}(h)|^2 \leq C_2hN^2(1-e^{2\beta h})\int_0^T(\mathbb{E}\|u(s-\delta(s))\|^2+1)ds,$$

and then

$$\lim_{h\to 0} \mathbb{E}|I_{32}(h)|^2 = 0.$$

Now, for the term $I_4(h)$, we have

$$\begin{aligned} I_4(h) &\leq \left\|\int_0^t(R(t+h,s)-R(t,s))\sigma(s)dB^H(s)\right\| + \left\|\int_t^{t+h}R(t+h,s)\sigma(s)dB^H(s)\right\| \\ &\leq I_{41}(h)+I_{42}(h). \end{aligned}$$

By lemma 2.2,

$$\mathbb{E}|I_{41}(h)|^2 \leq 2Ht^{2H-1}\int_0^t\|(R(t+h,s)-R(t,s))\sigma(s)\|^2_{\mathcal{L}_2^0}\, d\sigma.$$

Since $\lim_{h\to 0}\|(R(t+h,s)-R(t,s))\sigma(s)\|^2_{\mathcal{L}_2^0} = 0$ and

$$\|(R(t+h,s)-R(t,s))\sigma(s)\|_{\mathcal{L}_2^0} \leq [Ne^{-\beta(t+h)}+Ne^{-2\beta t}]\|\sigma(s)\|^2_{\mathcal{L}_2^0} \in L^1([0,T],ds),$$

the Lebesgue Majorant Theorem then implies that

$$\lim_{h\to 0} \mathbb{E}|I_{41}(h)|^2 = 0.$$

Again by lemma 2.2, we obtain that

$$\mathbb{E}|I_{42}(h)|^2 \leq 2HN^2(1-e^{2\beta h})t^{2H-1}\int_t^{t+h}\|\sigma(s)\|_{\mathcal{L}_2^0}ds \quad \to 0 \quad when \quad h\to 0.$$

The above arguments show that $\lim_{h\to 0}\mathbb{E}\|\Gamma(u)(t+h)-\Gamma(u)(t))\|^2 = 0$. Hence, we conclude that the function $t\to\Gamma(u)(t)$ is continuous on $[0,T]$ in the L^2-sense.

Step 2: Now we show that Γ is a contracting mapping in $S_{T_1}(\varphi)$ for some small enough $T_1 < T$ For every $u,\ v\in S_T(\varphi)$ and $t\in[0,T]$, by using lemma 3.1 we have

$$\begin{aligned} \|\Gamma(u)(t)-\Gamma(v)(t)\|^2 &\leq \frac{1}{C_3}\|G(t,u(t-r(t)))-G(t,v(t-r(t)))\|^2 \\ &+ \frac{1}{1-C_3}\left\|\int_0^t R(t,s)(F(s,u(s-\delta(s)))-F(s,v(s-\delta(s))))ds\right\|^2. \end{aligned}$$

Owing to the Lipschitz properties of F and G combined with Hölder's inequality, there holds

$$\mathbb{E}\|\Gamma(u)(t)-\Gamma(v)(t)\|^2 \leq C_3\mathbb{E}\|u(t-r(t)) - v(t-r(t))\|^2 + \frac{1}{1-C_3}N^2C_1^2\left(\frac{1-e^{2\beta t}}{2\beta}\right)\int_0^t \mathbb{E}\|u(s-\delta(s)) - v(s-\delta(s))\|^2 ds.$$

Hence

$$\sup_{s\in[-\tau,t]} \mathbb{E}\|\Gamma(u)(t)-\Gamma(v)(t)\|^2 \leq \alpha(t) \sup_{s\in[-\tau,t]} \mathbb{E}\|u(s)-v(s)\|^2,$$

where

$$\alpha(t) = C_3 + \frac{1}{1-C_3}N^2C_1^2\left(\frac{1-e^{2\beta t}}{2\beta}\right)t.$$

By condition (iii) in (H4) we have $\alpha(0) = C_3 < 1$. Then there exists $0 < T_1 \leq T$ such that $0 < \alpha(T_1) < 1$ and Γ is a contraction mapping on $S_{T_1}(\varphi)$ and therefore has a unique fixed point, which is a mild solution of Eq. (1) on $[-\tau, T_1]$. This procedure can be repeated a finite number of times in order to extend the solution to the entire interval $[-\tau, T]$. This completes the proof. ■

4. Application

As we mentioned in the Introduction, neutral stochastic differential equations arise in many real world problems such as physics, population dynamics, ecology, biological systems, biotechnology, optimal control, theory of elasticity, electrical networks, etc. Now, to illustrate our results, we consider the stochastic partial functional integrodifferential equation with finite delays ($\infty > \rho > \rho_i \geq 0$, $i = 1,2$):

$$\left.\begin{aligned}
\frac{\partial}{\partial t}[x(t,\xi)+g(t,x(t-\rho_1,\xi))] &= \left[\frac{\partial^2}{\partial\xi^2}+\gamma(t,\xi)\right][x(t,\xi)+g(t,x(t-\rho_1,\xi))]\\
&\quad+\int_0^t b(t-s)\frac{\partial^2}{\partial\xi^2}[x(s,\xi)+g(s,x(s-\rho_2,\xi))]ds\\
&\quad+f(t,x(t-\delta(t),\xi)) + \sigma(t)\frac{dB^H}{dt}(t)\\
x(t,0)+g(t,x(t-\rho_1,0)) &= 0 \quad for \quad t\geq 0,\\
x(t,\pi)+g(t,x(t-\rho_1,\pi)) &= 0 \quad for \quad t\geq 0,\\
x(\theta,\xi) = x_0(\theta,\xi),\varphi(s,.) &\in L^2[0,T], \quad -\rho\leq\theta\leq 0, \quad 0\leq\xi\leq\pi,
\end{aligned}\right\} \tag{9}$$

where B^H denotes a fractional Brownian motion, $g,f : \mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$, $r,\delta : [0,+\infty)\to[0,\tau]$, and $b:\mathbb{R}^+\to\mathbb{R}$ are continuous functions. Let $X = L^2([0,\pi])$ and $e_n := \sqrt{\frac{2}{\pi}}\sin(nx)$, $(n = 1,2,3,\cdots)$. Then $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis in X. Let $A = \frac{\partial^2}{\partial z^2}$, whose domain is $Y := D(A) = H^2([0,\pi]) \cap H_0^1([0,\pi])$. Then, it is well known that $Az = \sum_{n=1}^\infty n^2\langle z,e_n\rangle e_n$ for any $z\in X$, and that A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{R(t)\}_{t\geq 0}$ on X, which is given by $R(t)\phi = \sum_{n=1}^\infty e^{-n^2t}\langle\phi,e_n\rangle e_n$, $\phi\in D(A)$.In addition, it follows that $R(t)$ is compact for every $t\geq 0$ and $\|R(t)\|\leq e^{-t}$ for every $t\geq 0$.In order to define the operator $Q : Y\to Y$, we choose a sequence $\{\sigma_n\}_{n\geq 1}\subset\mathbb{R}^+$ and set $Qe_n = \sigma_n e_n$, and assume that $tr(Q) = \sum_{n=1}^\infty\sqrt{\sigma_n} < \infty$. Define the process $B^H(s)$ by

$$B^H = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\, \gamma_n(t) e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent. Now we define an operator $A(t) : D(A) \subset X \rightarrow X$ by

$$A(t)l(\xi) = Al(\xi) + \gamma(t,\xi)l(\xi).$$

Let $b(\bullet)$ be continuous and $\gamma(t,\xi) \leq -\alpha(\alpha < 0)$, for every $t \in \mathbb{R}$. Then, the system

$$\left.\begin{array}{l} x'(t) = A(t)x(t) \quad t > s \\ x(s) = x \in X \end{array}\right\} \tag{10}$$

has an associated evolution family, given by

$$U(t,s)x(\xi) = \left[R(t-s) e^{\int_s^t \gamma(\eta,\,\xi) d\eta} x \right](s).$$

From the above expression, it follows that $U(t,s)$ is a compact operator for every $t,s \in [0,T]$ with $t > s$ and $\|U(t,s)\| \leq e^{(1+\alpha)(t-s)}$.
Suppose then that the following conditions hold:

- (A1) For $t \geq 0,\ f(t,0) = g(t,0) = 0.$
- (A2) There exist positive constants $l_1,\ l_2,\ l_3,\ l_4,$ with $0 < \pi l_3^2 < 1$, such that

$$|f(t,\zeta_1) - f(t,\zeta_2)| \leq l_1|\zeta_1 - \zeta_2|;$$

$$|g(t,\zeta_1) - g(t,\zeta_2)| \leq l_2 K|\zeta_1 - \zeta_2|.$$

- (A3)

$$|f(t,\zeta)| \leq l_3(1 + |\zeta|^2);$$

$$|g(t,\zeta)| \leq l_4(1 + |\zeta|^2),$$

for $t \geq 0 \ \in \zeta, \zeta_1, \zeta_2 \in \mathbb{R}$.

- (A4) The function $\sigma \ : \ [0,+\infty[\rightarrow \mathcal{L}_2^0(L^2([0,\pi]), L^2([0,\pi]))$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \quad \forall T > 0.$$

Let $C = C([-\rho,0],\ X)$ and define the operators $F, G : \mathbb{R}^+ \times C \rightarrow X$ for $\xi \in [0,\pi]$ by

$$G(t,\phi)(\xi) = g(t,\phi(-\rho_1)(\xi));$$

$$F(t,\phi)(\xi) = f(t,\phi(-\rho_2)(\xi)).$$

If we put

$$u(t)(\xi) \quad = x(t,\xi) \quad \textit{for} \quad t \geq 0 \quad \textit{and} \quad \xi \in [0,\pi],$$

$$\varphi(\theta)(\xi) \quad = x_0(\theta,\xi) \quad \textit{for} \quad \theta \in [-r,0] \quad \textit{and} \quad \xi \in [0,\pi],$$

then Eq (10) takes the following abstract form

$$
\begin{aligned}
d[u(t)+G(t,u(t-\rho_1))] &= A(t)[u(t)+G(t,u(t-\rho_1))]dt \\
&\quad + \left[\int_0^t B(t-s)[u(s)+G(s,u(s-\rho_1))]ds + F(t,u(t-\rho_2))\right]dt \\
&\quad + \sigma(t)dB^H(t) \quad for \quad t \geq 0, \\
u(t) &= \varphi, \quad t \in [-\rho, 0].
\end{aligned}
$$

By assumption $(A1)$ we have

$$
\begin{aligned}
\|F(t,\phi_1)-F(t,\phi_2)\|_X &= \left(\int_0^\pi |F(t,\phi_1)(\xi)-F(t,\phi_2)(\xi)|^2 d\xi\right)^{1/2} \\
&= \left(\int_0^\pi |f(t,\phi_1(-\rho_2)(\xi)) - f(t,\phi_2(-\rho_2)(\xi))|^2 d\xi\right)^{1/2} \\
&\leq l_1\left(\int_0^\pi |\phi_1(-\rho_2)(\xi)-\phi_2(-\rho_2)(\xi)|^2 d\xi\right)^{1/2} \\
&\leq l_1\|\phi_1-\phi_2\|_X,
\end{aligned}
$$

demonstrating that $F(t,x)$ satisfies a Lipschitz condition. Further, by assuming $(A2)$ it follows that

$$
\begin{aligned}
\|F(t,\phi_1)\|_X^2 &= \left(\int_0^\pi |F(t,\phi_1)(\xi)|^2 d\xi\right)^{1/2} \\
&= \left(\int_0^\pi |f(t,\phi_1(-\rho_2)(\xi))|^2 d\xi\right)^{1/2} \\
&\leq l_3^2\left(\int_0^\pi [1+|\phi_1(-\rho_2)(\xi)|^2] d\xi\right)^{1/2} \\
&\leq l_3^2\pi\left[1+\|\phi_1\|_X^2\right].
\end{aligned}
$$

The remaining conditions can be verified similarly. Thus, all the assumptions of theorem 3.1 are fulfilled. Therefore, the existence of a mild solutions for (9) is verified.

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