Journal of Numerical Mathematics and Stochastics, **6** (1) : 62-72, 2014

http://www.jnmas.org/jnmas6-5.pdf
Euclidean Press, LLC
Online: ISSN 2151-2302

Mean-field Reflected Backward Doubly Stochastic DE With Continuous Coefficients*

N. CHAOUCHKHOUAN, B. LABED, and B. MANSOURI

Laboratory of Applied Mathematics, University of Biskra, POB 145, Algeria,
E-mail: mansouri.badreddine@gmail.com

Abstract. *We study the existence and uniqueness of the solutions to mean-field reflected backward doubly stochastic differential equation (MF-RBDSDE), when the driver f is Lipschitzian. We also study the existence in the case where the driver is of linear growth and continuous. In this case we establish a comparison theorem.*

Key words : Backward Doubly SDE, Mean-field, Continuous Coefficients, Comparison Theorem.

AMS Subject Classifications : 60H10, 60H05

1. Introduction

After the earlier work of Pardoux & Peng (1990), the theory of backward stochastic differential equations (BSDEs in short) has a significant headway thanks to the many application areas. Several authors contributed in weakening the Lipschitzian assumption required on the drift of the equation (see Lepaltier & San Martin (1996), Kobylanski (1997), Mao (1995), Bahlali (2000)).
A new kind of backward stochastic differential equations was introduced by Pardoux & Peng [5] (1994),

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s\, dW_s, \quad 0 \le t \le T$$

with two different directions of stochastic integrals, i.e., the equation involves both a standard (forward) stochastic integral dW_t and a backward stochastic integral dB_t. They have proved the existence and uniqueness of solutions for BDSDEs under uniformly Lipschitzian conditions. Shi et al. [6] (2005) provided a comparison theorem which is very important in studying viscosity solution of SPDEs with stochastic tools.
Bahlali et al. [2] (2009) proved the existence and uniqueness of the solution to the following

*This work is partially supported by the CNPRU project B01420140103

reflected backward doubly stochastic differential equations(RBDSDEs) with one continuous barrier and uniformly Lipschitzian coefficients:

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s \\ &\quad + K_T - K_t - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \end{aligned}$$

In a recent work of Buckdahn et al. [3] (2009), a notion of mean-field backward stochastic differential equation (MF-BSDEs in short) of the form

$$Y_t = \xi + \int_t^T E' f(s, \omega', \omega, Y_s(\omega), Y_s(\omega'), Z_s)ds - \int_t^T Z_s \, dW_s,$$

with $t \in [0,T]$, was introduced. The authors deepened the investigation of such mean-field BSDEs by studying them in a more general framework, with a general driver. They established the existence and uniqueness of the solution under uniformly Lipschitzian conditions. The theory of mean-field BSDE has been developed by several authors. Du et al. [4] (2001); established a comparison theorem and existence in the case linear growth and continuous condition. Shi et al. [6]; introduced and studied mean-field backward stochastic Volterra integral equations.

Mean-field Backward doubly stochastic differential equations

$$\begin{aligned} Y_t &= \xi + \int_t^T E' f(s, \omega', \omega, Y_s(\omega), Y_s(\omega'), Z_s)ds \\ &\quad + \int_t^T E' g(s, \omega', \omega, Y_s(\omega'), Z_s)dB_s - \int_t^T Z_s \, dW_s, \end{aligned}$$

with $t \in [0,T]$, are deduced by Ruimin Xu [7] (2012), who obtained the existence and uniqueness result of the solution with uniformly Lipschitzian coefficients and presented the connection between McKean-Vlasov SPDEs and mean-field BDSDEs.

In this paper, we study the case where the solution is forced to stay above a given stochastic process, called the obstacle. We obtain the real valued mean-field reflected backward doubly stochastic differential equation : with $t \in [0,T]$

$$\begin{aligned} Y_t &= \xi + \int_t^T E' f(s, \omega, \omega', Y_s, Y'_s, Z_s, Z'_s)ds \\ &\quad + \int_t^T E' g(s, \omega, \omega', Y_s, Y'_s, Z_s, Z'_s)dB_s + K_T - K_t - \int_t^T Z_s \, dW_s. \end{aligned} \tag{1}$$

We establish the existence and uniqueness of solutions for equation (1) under uniformly Lipschitz conditions on the coefficients. In the case where the coefficient f is only continuous, we establish the existence of maximal and minimal solutions.

In the case where the coefficient f is continuous with linear growth, we approximate f by a sequence of Lipschitz functions (f_n) and use a comparison theorem established here for MF-RBDSDEs.

The paper is organized as follows : In Sections 2, we give some notations, assumptions, and we define a solution of RBDSDE. In Section 3, we state our main results for existence and uniqueness in the case where the coefficients are Lipschtzian, and we present a comparison theorem. The case where the generator is continuous and linear growth is treated in section 4.

2. Notation, Assumptions and Definitions

Let $(\Omega,\mathcal{F},P)$ be a complete probability space, and $T>0$. Let $\{W_t, 0\le t\le T\}$ and $\{B_t, 0\le t\le T\}$ be two independent standard Brownian motions defined on $(\Omega,\mathcal{F},P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively. For $t\in[0,T]$, we put,

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B, \quad \text{and} \quad \mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_T^B,$$

where $\mathcal{F}_t^W = \sigma(W_s; 0\le s\le t)$ and $\mathcal{F}_{t,T}^B = \sigma(B_s - B_t; t\le s\le T)$, completed with P-null sets. It should be noted that $(\mathcal{F}_t)$ is not an increasing family of sub σ–fields, and hence it is not a filtration. However $(\mathcal{G}_t)$ is a filtration.

Let $M_T^2(0,T,\mathbb{R}^d)$ denote the set of d–dimensional, jointly measurable stochastic processes $\{\varphi_t; t\in[0,T]\}$, which satisfy :

(a) $E\int_0^T |\varphi_t|^2 dt < \infty$.

(b) φ_t is $\mathcal{F}_t$ –measurable, for any $t\in[0,T]$.

We denote by $S_T^2([0,T],\mathbb{R})$, the set of continuous stochastic processes φ_t, which satisfy :

(a') $E\left(\sup_{0\le t\le T}|\varphi_t|^2\right) < \infty$.

(b') For every $t\in[0,T]$, φ_t is $\mathcal{F}_t$ –measurable.

Let $(\overline{\Omega},\overline{\mathcal{F}},\overline{P}) = (\Omega\times\Omega, \mathcal{F}_t\otimes\mathcal{F}_t, P\otimes P)$ be the (non-completed) product of $(\Omega,\mathcal{F},P)$ with itself. We denote the filtration of this product space by $\overline{\mathcal{F}} = \{\overline{\mathcal{F}}_t = \mathcal{F}_t\otimes\mathcal{F}_t, 0\le t\le T\}$. A random variable $\xi\in L^0(\Omega,\mathcal{F},P;\mathbb{R}^n)$ originally defined on Ω is extended canonically to $\overline{\Omega}$: $\xi'(\omega',\omega) = \xi(\omega'), (\omega',\omega)\in\overline{\Omega} = \Omega\times\Omega$. For every $\theta\in L^1(\overline{\Omega},\overline{\mathcal{F}},\overline{P})$, the variable $\theta(.,\omega) : \Omega\to\mathbb{R}$ belongs to $L^1(\overline{\Omega},\overline{\mathcal{F}},\overline{P})$, $P(d\omega)-a.s,$. We denote its expectation by

$$E'[\theta(.,\omega)] = \int_\Omega \theta(\omega',\omega)P(d\omega').$$

Notice that $E'[\theta] = E'[\theta(.,\omega)]\in L^1(\Omega,\mathcal{F},P)$, and

$$\overline{E}[\theta] = \left(\int_{\overline{\Omega}} \theta d\overline{P} = \int_\Omega E'[\theta(.,\omega)]P(d\omega)\right) = E[E'[\theta]].$$

Then we consider the following assumptions,

H1) Let $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ and $g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ be two measurable functions and such that for every $(y,z,y',z')\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$, $f(.,y,z,y',z')$ and, $g(.,y,z,y',z')$ belongs in $M^2(0,T,\mathbb{R})$

H2) There exist constants $L>0$ and $0<\alpha<\frac{1}{2}$, such that for every $(t,\omega)\in\Omega\times[0,T]$ and $(y,z,y',z')\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$,

$$\begin{aligned}
&|f(t,y_1,z_1,y_1',z_1') - f(t,y_2,z_2,y_2',z_2')| \\
&\quad \le L(|y_1-y_2| + |y_1'-y_2'| + |z_1-z_2| + |z_2'-z_2'|) \\
&\qquad \times |g(t,y_1,z_1,y_1',z_1') - g(t,y_2,z_2,y_2',z_2')|^2 \\
&\quad \le L\left(|y_1-y_2|^2 + |y_1'-y_2'|^2\right) + \alpha\left(|z_1-z_2|^2 + |z_2'-z_2'|^2\right).
\end{aligned}$$

H3) Let ξ be a square integrable random variable which is $\mathcal{F}_T$ –mesurable.

H4) The obstacle $\{S_t, 0 \le t \le T\}$, is a continuous $\mathcal{F}_t$ –progressively measurable real-valued process satisfying $E\left(\sup_{0\le t\le T}(S_t)^2\right) < \infty$.
We assume also that $S_T \le \xi \ a.s.$

Definition 2.1. A solution of equation (1) is a $(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}_+)$ -valued $\mathcal{F}_t$ –progressively measurable process $(Y_t, Z_t, K_t)_{0\le t\le T}$ which satisfies equation (1) and
i) $(Y, Z, K_T) \in S^2 \times M^2 \times L^2(\Omega)$.
ii) $Y_t \ge S_t$.
iii) (K_t) is continuous and nondecreasing, $K_0 = 0$ and $\int_0^T (Y_t - S_t)dK_t = 0$.

3. Existence of a Solution to the RBDSDE With a Lipschitz Condition

Theorem 3.1. *Under conditions, H1), H2), H3) and H4), the MF-RBDSDE (1) has a unique solution.*

Proof. For any (y, z) we consider the following MF-RBDSDE, with $t \in [0, T]$

$$Y_t = \xi + \int_t^T E'f(s,\omega,\omega',Y_s,y'_s,Z_s,z'_s)ds + \int_t^T E'g(s,\omega,\omega',Y_s,y'_s,Z_s,z'_s)\, dB_s$$
$$+ K_T - K_t - \int_t^T Z_s\, dW_s.$$

According to Theorem 1 in Bahlali et al. [2], there exists a unique solution $(Y, Z) \in S^2 \times M^2$ i.e., if we define the process

$$K_t = Y_0 - Y_t - \int_0^t E'f(s,\omega,\omega',Y_s,y'_s,Z_s,z'_s)ds$$
$$- \int_0^t E'g(s,\omega,\omega',Y_s,y'_s,Z_s,z'_s)\, dB_s \ + \int_0^t Z_s\, dW_s,$$

then (Y, Z, K) satisfies Definition 2.1. Hence, if we define $\Theta(y, z) = (Y, Z)$, then Θ maps $S^2 \times M^2$ itself. We show now that Θ is contractive. To this end, take any $(y^i, z^i) \in S^2 \times M^2$ $(i = 1, 2)$, and let $\Theta(y^i, z^i) = (Y^i, Z^i)$. We denote $(\overline{Y}, \overline{Z}, \overline{K}) = (Y^1 - Y^2, Z^1 - Z^2, K^1 - K^2)$ and $(\overline{y}, \overline{z},) = (y^1 - y^2, z^1 - z^2)$. Therefore, Itô's formula applied to $|\overline{Y}|^2 e^{\beta t}$ where $\beta > 0$, and the inequality $2ab \le \left(\frac{1}{\delta}\right)a^2 + \delta b^2$, lead to

$$E|\overline{Y}_t|^2 e^{\beta t} + \left(\beta - 3L - \frac{8L^2}{1-2\alpha}\right) E\int_t^T |\overline{Y}_s|^2 e^{\beta S} ds + \frac{1}{2} E\int_t^T e^{\beta S}|\overline{Z}_s|^2 ds$$
$$\le + E\int_t^T e^{\beta S}\overline{Y}_s(dK_s^1 - dK_s^2)$$
$$+ E\int_t^T e^{\beta S}\left(\left(L + \frac{1-2\alpha}{2L}\right)|\overline{y}_s|^2 + \left(\frac{1+2\alpha}{4}\right)|\overline{z}_s|^2\right)ds$$

Choosing $\beta = 3L + \frac{8L^2}{1-2\alpha} + \frac{1}{2}\left(\frac{4}{1+2\alpha}\right)\left(L + \frac{1-2\alpha}{2L}\right)$ and setting $M = \left(\frac{4}{1+2\alpha}\right)\left(L + \frac{1-2\alpha}{2L}\right)$ yield

$$E|\overline{Y}_t|^2 e^{\beta t} + \frac{1}{2} ME\int_t^T |\overline{Y}_s|^2 e^{\beta S} ds + \frac{1}{2} E\int_t^T e^{\beta S} |\overline{Z}_s|^2 ds$$
$$\leq E\int_t^T e^{\beta S}\overline{Y}_s(dK_s^1 - dK_s^2) \; + \frac{1+2\alpha}{4} E\int_t^T e^{\beta S}\left(M|\overline{y}_s|^2 + |\overline{z}_s|^2\right)ds.$$

As

$$E\int_t^T e^{\beta S}\overline{Y}_s(dK_s^1 - dK_s^2) < 0,$$

and

$$E\int_t^T e^{\beta S}\left(M|\overline{Y}_s|^2 + |\overline{Z}_s|^2\right)ds \leq \frac{1+2\alpha}{2} E\int_t^T e^{\beta S}\left(M|\overline{y}_s|^2 + |\overline{z}_s|^2\right)ds,$$

consequently the mapping Θ is a strict contraction on $S^2 \times M^2$ equipped with the norm

$$\|(Y,Z)\|_\beta = \left(E\int_t^T e^{\beta S}\left(M|\overline{Y}_s|^2 + |\overline{Z}_s|^2\right)ds\right)^{\frac{1}{2}}.$$

Moreover, it has a unique fixed point, which is the unique solution of the MF-RBDSDE with data (ξ, f, g, S). ■

4. RBDSDEs With a Continuous Coefficient

In this section we prove the existence of a solution to the MF-RBDSDE where the coefficient is only continuous.

Towards this end, we consider the following assumption.

H5) i) for $a.e\ (t,w)$, the mapping $(y,y',z,z') \mapsto f(t,y,y',z,z')$ is continuous. ii) There exist constants $L > 0$ and $\alpha \in (0,\frac{1}{2})$, such that for every $(t,\omega) \in \Omega \times [0,T]$ and $(y,z,y',z') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{cases} |f(t,y,y',z,z')| \leq L(1 + |y| + |y'| + |z| + |z'|) \\ |g(t,y_1,y_1',z_1,z_1') - g(t,y_2,y_2',z_2,z_2')|^2 \leq L\left(|y_1' - y_2'|^2 + |y_1 - y_2|^2\right) \\ \qquad + \alpha\left(|z_1' - z_2'|^2 + |z_1 - z_2|^2\right) \end{cases}$$

Theorem 4.1. *Under assumption H1), H3), H4) and H5), the MF-RBDSDE (1) has an adapted solution (Y,Z,K) which is a minimal one, in the sense that, if (Y^*,Z^*) is any other solution we have $Y \leq Y^*,\ P-a.s.$*

Before giving a proof to this theorem, we invoke first the following classical lemma, which can be proved by adapting the proof given in Alibert and Bahlali [1].

Lemma 4.1. *Let $f : [0,T] \times \overline{\Omega} \times R \times R \times R^d \longmapsto R$ be a measurable function such that:*
(a) For almost every $(t,\overline{\omega}) \in [0,T] \times \overline{\Omega}$, $x \longmapsto f(t,\overline{\omega},x)$ is continuous,
(b) There exists a constant $K > 0$ such that for every $(t,y',y,z) \in [0,T] \times R \times R \times R^d$

$|f(t,y',y,z)|\leq K(1+|y'|+|y|+|z|)\ a.s.$

(c) For almost every $y,z,f(t,y',y,z)$ *is increasing in* y'.
Then, the sequence of functions

$$f_n(t,y',y,z) = \inf_{(u,v,w)\in\mathbb{Q}^{2+d}} \{f(t,u,v,w)+n(y'-u)^+ + n|y-v|+n|z-w|\}$$

is well defined for each $n\geq K$ *and satisfies:*
(1) for every $(t,y',y,z)\in[0,T]\times R^{2+d}$, $|f_n(t,y',y,z)|\leq K(1+|y'|+|y|+|z|)$,
(2) for every $(t,y',y,z)\in[0,T]\times R^{2+d}$, $n\to f_n(t,x)$ *is increasing,*
(3) for every $(t,y',y,z)\in[0,T]\times R^{2+d}$, $y'\to f_n(t,y',y,z)$ *is increasing,*
(4) for every $n\geq K$, $(t,y'^1,y^1,z^1)\in[0,T]\times R^{2d}$,$(t,y'^2,y^2,z^2)\in[0,T]\times R^{2d}$
$|f_n(t,y'^1,y^1,z^1)-f_n(t,y'^2,y^2,z^2)|\leq n(|y'^1-y'^2|+|y^1-y^2|+|z'^1-z'^2|)$,
(5) If $(y'_n,y_n,z_n)\to(y',y,z)$, *as* $n\to\infty$ *then for every* $t\in[0,T]$ $f_n(t,y'_n,y_n,z_n)\to f(t,y',y,z)$ *as* $n\to\infty$.
Since ξ *satisfies H3), we get from theorem 3.1, that for every* $n\in N^*$, *there exists a unique solution* $\{(Y^n_t,Z^n_t,K^n_t),\ 0\leq t\leq T\}$ *for the following MF-RBDSDE*

$$\begin{cases} Y^n_t = \xi+\int_t^T f_n(s,(Y^n_s)',Y^n_s,Z^n_s)ds+K^n_T-K^n_t+\int_t^T g(s,(Y^n_s)',Y^n_s,Z^n_s)dB_s \\ \qquad -\int_t^T Z^n_s dW_s, \quad 0\leq t\leq T, \\ \qquad Y^n_t\geq S_t\ ,\ \int_0^T (Y^n_s-S_s)dK^n_s = 0. \end{cases} \tag{2}$$

Since, $|f^1(t,u,v,w)-f^1(t,u',v',w')|\leq L(|u-u'|+|v-v'|+|w-w'|)$, we consider the function defined by

$f^1(t,u,v,w) := L(1+|u|+|v|+|w|)$,

then a similar argument as before shows that there exists a unique solution $((U_s,V_s,K_s),0\leq s\leq T)$ to the following MF-RBDSDE:

$$\begin{cases} U_t = \xi+\int_t^T f^1(s,U'_s,U_s,V_s)ds+K_T-K_t+\int_t^T g(s,U'_s,U_s,V_s)dB_s-\int_t^T V_s dW_s \\ \qquad U_t\geq S_t, \\ \qquad \int_0^T (U_s-S_s)dK_s = 0. \end{cases} \tag{3}$$

We would also need the following comparison theorem.

Theorem 4.2. *(Comparison theorem) Let* (ξ^1,f^1,g,S^1) *and* (ξ^2,f^2,g,S^2) *be two MF-RBDSDEs. Each one satisfying all the previous assumptions H1), H2), H3) and H4). Assume moreover that :*
i) $\xi^1\leq\xi^2$ *a.s.*
ii) $f^1(t,y',y,z',z)\leq f^2(t,y',y,z',z)\ dP\times dt\ a.e.\ \forall(y',y,z',z)\in R\times R^d$.
iii) $S^1_t\leq S^2_t,\ 0\leq t\leq T\ a.s.$
Let (Y^1,Z^1,K^1) *be a solution of MF-RBDSDE* (ξ^1,f^1,g,S^1) *and* (Y^2,Z^2,K^2) *be a solution of MF-RBDSDE* (ξ^2,f^2,g,S^2). *We suppose also :*
a) One of the two generators is independent of z'.

b) One of the two generators is nondecreasing in y'.
then

$Y_t^1 \leq Y_t^2, \ 0 \leq t \leq T \quad a.s.$

Pr*oof*. Suppose that (a) is satisfied by f^1 and (b) by f^2. Applying Itô's formula to $|(Y_t^1 - Y_t^2)^+|^2$, and passing to expectation, we have

$$E|(Y_t^1 - Y_t^2)^+|^2 + E\int_t^T 1_{\{Y_s^1 > Y_s^2\}}|Z_s^1 - Z_s^2|^2 ds$$
$$= 2\,E\int_t^T (Y_s^1 - Y_s^2)^+ E'\Big(f^1\big(s,(Y_s^1)', Y_s^1, Z_s^1\big) - f^2\big(s,(Y_s^2)', Y_s^2, (Z_s^2)', Z_s^2\big)\Big)ds$$
$$+ 2\,E\int_t^T (Y_s^1 - Y_s^2)^+ dK_s^1 - dK_s^2$$
$$+ E\int_t^T \Big|E'\Big(g\big(s,(Y_s^1)', Y_s^1, (Z_s^1)', Z_s^1\big) - g\big(s,(Y_s^2)', Y_s^2, (Z_s^2)', Z_s^2\big)\Big)\Big|^2 1_{\{Y_s^1 > Y_s^2\}} ds.$$

Since on the set $\{Y_s^1 > Y_s^2\}$, we have $Y_t^1 > S_t^2 \geq S_t^1$, then

$$\int_t^T (Y_s^1 - Y_s^2)^+ (dK_s^1 - dK_s^2) = -\int_t^T (Y_s^1 - Y_s^2)^+ dK_s^2 \leq 0.$$

Since f^1 and f^2 are Lipschitzian, we have on the set $\{Y_s > Y_s'\}$,

$$E|(Y_t^1 - Y_t^2)^+|^2 + E\int_t^T 1_{\{Y_s^1 > Y_s^2\}}|Z_s^1 - Z_s^2|^2 ds$$
$$\leq E\int_t^T \left\{\left(6L + \tfrac{L^2}{1-2\alpha}\right)|(Y_t^1 - Y_t^2)^+|^2 + |Z_s^1 - Z_s^2|^2\right\} ds\,,$$

then

$$E|(Y_t^1 - Y_t^2)^+|^2 \leq E\int_t^T \left(6L + \tfrac{L^2}{1-2\alpha}\right)|(Y_t^1 - Y_t^2)^+|^2 ds.$$

The required result follows by using Gronwall's lemma. ■

Lemma 4.2. *i) $a.s.$ for all t, $Y_t^0 \leq Y_t^n \leq Y_t^{n+1} \leq U_t$. ii) There exists $Z \in M^2$, such that Z^n converges to Z.*

Proof. Assertion i) follows from the comparison theorem. We therefore need to prove ii) only. Itô's formula yields

$$E|Y_0^n|^2 + E\int_0^T |Z_s^n|^2 ds = E|\xi|^2 + 2E\int_0^T Y_s^n E'\big(f_n(s,(Y_s^n)', Y_s^n, Z_s^n)\big)ds + 2E\int_0^T S_s dK_s^n$$
$$+ E\int_0^T E'\Big(\big|g(s,(Y_s^n)', Y_s^n, Z_s^n)\big|^2\Big)ds.$$

From assumption H5) , and the inequality $2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2$ for $\varepsilon > 0$, we get :

$$E\int_0^T |Z_s^n|^2 ds \le E|\xi|^2 + \frac{LT}{\varepsilon} + E\int_0^T |g(s,0,0,0)|^2 ds + (3L\varepsilon + \frac{L}{\varepsilon} + 4L)E\int_0^T |Y_s^n|^2 ds$$
$$+\left(\frac{L}{\varepsilon}+\alpha\right)E\int_0^T \frac{1}{\varepsilon}|Z_s^n|^2 ds + 2E\int_0^T S_s dK_s^n.$$

On the other hand, we have from (2)

$$K_T^n = Y_0^n - \xi - \int_0^T E' f_n(s,(Y_s^n)', Y_s^n, Z_s^n)ds - \int_0^T E' g(s,(Y_s^n)', Y_s^n, Z_s^n)dB_s$$
$$+ \int_0^T Z_s^n\, dW_s. \tag{4}$$

Then

$$E(K_T^n)^2 \le C\left(1 + E\int_0^T \|Z_s^n\|^2 ds\right).$$

We also have

$$2\,E\int_0^T S_s dK_s^n \le \frac{1}{\beta}E\left(\sup_t |S_t|^2\right) + \beta E(K_T^n)^2$$
$$\le \frac{1}{\beta}E\left(\sup_t |S_t|^2\right) + \beta C\left(1 + E\int_0^T \|Z_s^n\|^2 ds\right),$$

which leads to

$$E\int_0^T |Z_s^n|^2 ds \le E|\xi|^2 + \frac{LT}{\varepsilon} + \beta C + E\int_0^T |g(s,0,0,0)|^2 ds + (3L\varepsilon + \frac{L}{\varepsilon} + 4L)E\int_0^T |Y_s^n|^2 ds$$
$$+\left(\frac{L}{\varepsilon}+\alpha+\beta C\right)E\int_0^T \frac{1}{\varepsilon}|Z_s^n|^2 ds + \frac{1}{\beta}E\left(\sup_t |S_t|^2\right).$$

Choosing ε, β such that $(\frac{L}{\varepsilon} + \alpha + \beta C) < 1$, we obtain

$$E\int_0^T \|Z_s^n\|^2 ds \le C.$$

For $n,\ p \ge K$, Itô's formula gives,

$$E(Y_0^n - Y_0^p)^2 + E\int_0^T \|Z_s^n - Z_s^p\|^2 ds$$
$$= 2\,E\int_0^T (Y_s^n - Y_s^p)E'(f_n(s,Y_s^n,(Y_s^n)',Z_s^n) - f_p(s,Y_s^p,(Y_s^p)',Z_s^p))ds$$
$$+ 2\,E\int_0^T (Y_s^n - Y_s^p)dK_s^n + 2\,E\int_0^T (Y_s^p - Y_s^n)dK_s^p$$
$$+ E\int_0^T \left\| E'\left(g(s,Y_s^n,,(Y_s^n)',Z_s^n) - g(s,Y_s^p,(Y_s^p)',Z_s^p)\right)\right\|^2 ds.$$

But

$$E\int_0^T (Y_s^n - Y_s^p)dK_s^n = E\int_0^T (S_s - Y_s^p)dK_s^n \le 0.$$

Similarly, we have $E\int_0^T (Y_s^p - Y_s^n)dK_s^p \le 0$. Therefore,

$$E\int_0^T\|Z_s^n-Z_s^p\|^2ds\leq 2E\int_0^T(Y_s^n-Y_s^p)E'(f_n(s,Y_s^n,(Y_s^n)',Z_s^n)-f_p(s,Y_s^p,(Y_s^p)',Z_s^p))ds$$
$$+E\int_0^T\left\|E'\Big(g(s,Y_s^n,(Y_s^n)',Z_s^n)-g(s,Y_s^p,(Y_s^p)',Z_s^p)\Big)\right\|^2ds.$$

By Hôlder's inequality and the fact that g is Lipschitzian, we get

$$E\int_0^T\|Z_s^n-Z_s^p\|^2ds$$
$$\leq\left(E\int_0^T(Y_s^n-Y_s^p)^2ds\right)^{\frac{1}{2}}\left(E\int_0^T E'(f_n(s,Y_s^n,(Y_s^n)',Z_s^n)-f_p(s,Y_s^p,(Y_s^p)',Z_s^p))^2ds\right)^{\frac{1}{2}}$$
$$+LE\int_0^T\Big(|Y_s^n-Y_s^p|^2+|(Y_s^n)'-(Y_s^p)'|^2\Big)ds+\alpha E\int_0^T|Z_s^n-Z_s^p|^2ds$$

Since $\sup_n E\int_0^T|f_n(s,Y_s^n,(Y_s^n)',Z_s^n)|^2\leq C$, we obtain,

$$E\int_0^T\|Z_s^n-Z_s^p\|^2ds\leq C\left(E\int_0^T(Y_s^n-Y_s^p)^2ds\right)^{\frac{1}{2}}.$$

Hence

$$E\int_0^T\|Z_s^n-Z_s^p\|^2ds\longrightarrow 0,\ \text{as}\ n,\ p\to\infty.$$

Thus $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $\mathrm{M}^2(\mathbb{R}^d)$. ■

4.1. **Proof of Theorem 4.1**.

Let $Y_t=\sup_n Y_t^n$, and we have $(Y^n,Z^n)\to(Y,Z)$ in $S^2(\mathbb{R}^d)\times M^2(\mathbb{R}^d)$. Then, along a subsequence which we will still denote as (Y^n,Z^n), we have

$$(Y^n,Z^n)\to(Y,Z),\quad dt\otimes d\mathrm{P}\ a.e.$$

Then, by using Lemma 4.1, we get $f_n(t,Y_t^n,(Y_s^n)',Z_t^n)\to f(t,Y_t,(Y_t)',Z_t)\quad dPdt\ a.e.$ On the other hand, since $Z^n\longrightarrow Z$ in $\mathrm{M}^2(\mathbb{R}^d)$, then there exists an $\Lambda\in M^2(\mathbb{R})$ and a subsequence, which we continue to denotes as denote Z^n, such that $\forall n,|Z^n|\leq\Lambda$, $Z^n\longrightarrow Z,\ dt\otimes dP\ a.e.$

Moreover, from H5), and Lemma 4.2, we have

$$|f_n(t,Y_t^n,(Y_t^n)',Z_t^n)|\leq\kappa(1+\sup_n|Y_t^n|+\sup_n|(Y_t^n)'|+\Lambda_t)\in\mathrm{L}^2([0,T],\ dt),\quad P-a.s.$$

It follows from the dominated convergence theorem that,

$$E\int_0^T\left|E'\Big(f_n(s,Y_s^n,(Y_t^n)',Z_s^n)-f(s,Y_s,(Y_s)',Z_s)\Big)\right|^2ds\longrightarrow 0,\quad n\to\infty.$$

Subsequently,

$$E\int_0^T \|E'(g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s))\|^2 ds$$
$$\leq CE\int_0^T E'\Big(|Y_s^n-Y_s|^2+|(Y_s^n)'-(Y_s)'|^2\Big)ds$$
$$+\alpha\, E\int_0^T \|Z_s^n-Z_s\|^2 ds \longrightarrow 0, \quad \text{as } n\to\infty.$$

It is not difficult to show that (Y,Z) is a solution to our MF-RBDSDE. Indeed, let

$$\overline{Y}_t = \xi+\int_t^T E'f\Big(s,Y_s,(Y_s)',Z_s\Big)ds+K_T-K_t$$
$$+\int_t^T E'g\Big(s,Y_s,(Y_s)',Z_s\Big)dB_s-\int_t^T \overline{Z}_s dW_s,$$

where $\overline{Z}\in M^2$, $\overline{Y}\in S^2$, $K_T\in L^2$, $\overline{Y}_t\geq S_t$, (K_t) is continuous and nondecreasing, $K_0=0$ and $\int_0^T(\overline{Y}_t-S_t)dK_t=0$, and (Y^*,Z^*,K^*) be a solution of (1). Then, by theorem 4.2, we have for every $n\in\mathbb{N}^*$, $Y^n\leq Y^*$. Therefore, $\overline{Y}$ is a minimal solution of (1).

Remark 4.1. Using the same arguments and the following approximating sequence

$$Eh_n(t,x,y,z)=\sup_{(u,v,w)\in\mathbb{Q}^p}\{h(u,v,w)-n|x-u|-n|y-v|-n|z-w|\},$$

one can prove that the MF-RBDSDE (1) has a maximal solution.

Acknowledgements
The authors would like to thank the referee for some comments which have led to an improvement of the paper.

References

[1] J. J. Alibert, and K. Bahlali, Genericity in deterministic and stochastic differential equations, *Séminaire de Probabilités XXXV, Lecture Notes in Mathematics* **1755**, (2001), 220-240.

[2] K. Bahlali, M. Hassani, B. Mansouri, and N. Mrhardy, One barrier reflected backward doubly stochastic differential equations with continuous generator, *Comptes Rendus Mathematique* **347**(19), (2009), 1201-1206.

[3] R. Buckdan, B. Djehiche, J. Li, and S. Peng, Mean-field backward stochastic differential equations : a limit approach, *Annals of Probability* **37**, (2009), 1524-1565.

[4] H. Du, J. Li, and Q. M. Wei, Mean-field backward stochastic differential equations with continuous coefficients, In *Proceedings of the 30th Chinese Control Conference*, July 22-24, Yantai, China, 2012.

[5] E. Pardoux, and S. Peng, Backward doubly stochastic differential equations and systems of quasilinear SPDEs, *Probability Theory and Related Fields* **98**, (1994), 209-227.

[6] Y. Shi, Y. Gu, and K. Liu, Comparison theorems of backward doubly stochastic differential equations and applications, *Stochastic Analysis and Applications* **23**, (2005), 97-110.

[7] R. Xu, Mean-field backward doubly stochastic differential equation and related SPDEs, *Boundary Value Problems* **2012**:**114**, (2012).

[8] L. Zhi, and J. Luo, Mean-field reflected backward stochastic differential equations, *Statistics and Probability Letters* **82**, (2012), 1961-1968.

Article history: Submitted October, 11, 2014; Accepted February, 05, 2015.