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Optimality Conditions by Means of the Generalized HJB Equation

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Abstract. *This paper studies necessary as well as sufficient conditions of optimality for a general class of controlled diffusions with Jumps. The state of the system is described by a nonlinear stochastic differential equation, driven by a Poisson random measure and an independent Brownian motion. A discussion on verification results is carried out, by using recent results on the relationship between the adjoint processes and the value function, in terms of viscosity solutions and the associated super-differentials. In a second step, we prove the nonsmooth version of the necessity part of the verification theorem in terms of sub-differentials, rather than derivatives of the value function.*

Key words : Stochastic Systems With Jumps, Optimal Controls, Maximum Principle, Dynamic Programming, Generalized HJB Equation, Viscosity Solutions.

AMS Subject Classifications : 93E20, 60H30

1. Introduction

In this paper we study stochastic control models which are driven by a stochastic differential equation (SDE) with jumps, i.e. the dynamics of the controlled system is described by

$$\begin{cases} dx(t) = b(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dB(t) + \int_E \beta(t,x(t-),u(t),e)\widetilde{N}(dt,de), \\ x(s) = y. \end{cases} \tag{1}$$

The objective is to minimize the expected cost functional, that is earned during a finite time horizon,

$$J(u) = \mathbb{E}\left[\int_s^T f(t,x(t),u(t))dt + g(x(T))\right], \tag{2}$$

over the set of the admissible controls u. We shall give a more precise formulation of the problem and of the hypothesis on the data in the next sections. Here let us briefly survey the stochastic background for the model. As for the Brownian motion version, a major approach to studying this kind of stochastic control problems, is the Bellman dynamic programming principle. The associated Hamilton-Jacobi-Bellman (HJB) equation is a nonlinear second order parabolic integrodifferential equation. Pham studied in [16] a mixed optimal stopping and stochastic control of jump diffusion processes by using the viscosity solutions approach. Some verification theorems of various types of problems for systems governed by this kind of SDEs are discussed in øksendal and Sulem [15]. The stochastic maximum principle is another powerful tool for solving stochastic control problems. Some results that cover the controlled jump diffusion processes are discussed in [2], [11], and [18]. Necessary and sufficient conditions of optimality for partial information control problems are given in [2]. In [11], the sufficient maximum principle and its link with the dynamic programming principle are discussed. The second order stochastic maximum principle for optimal controls of nonlinear dynamics, with jump and convex state constraints, was developed via a spike variation method, by Tang and Li [18]. These underlying conditions are described in terms of two adjoint processes, which are linear backward SDEs. Such equations have important applications in hedging problems, see e.g. [10]. Existence and uniqueness of solutions to BSDE with jumps and nonlinear coefficients have been addressed by Tang and Li [18], and by Barles et al. [4]. The link with integral-partial differential equations is studied in [4]. For a discrete time approximation of decoupled FBSDE with jumps, the reader is referred to Bouchard and Elie [6].

Under certain differentiability conditions the relationship between the maximum principle and dynamic programming is essentially the relationship between the solution of the adjoint equation, with the spatial gradient of the value function evaluated along the optimal trajectory, see e.g. [19], in the classical case. For diffusions with jumps, the relationship between the maximum principle and dynamic programming, was investigated by Framstad et al. in [11]. The smoothness conditions do not hold in general and are difficult to verify a priori. This has led to the development of the notion of viscosity solution of HJB equations, see [19]. However, there is a vast literature dealing with the study of the viscosity solutions of integral-partial differential equations in different contexts. For more details the reader is referred to [1], [5].

The study of the characterization of optimal control by verification theorems is one of the fundamental tasks of stochastic control theory. When the value function is smooth, it is based on the fact that the value function is the maximum solution of the HJB equation and the maximum condition (31). But in many cases the value function is not differentiable. So the verification theorem does not apply. Without assuming any differentiability conditions, the problem will be overcome by the viscosity solutions theory to the HJB equation. See [12], [13], [14], [19], [21], and [20] for more detail.

The organization of the paper is as follows, in the second section, we formulate the problem and give the notations used throughout the paper. In sections 3 and 4, we give the stochastic maximum principle (SMP) as well as the dynamic programming principle (DPP) for systems governed by this kind of SDE. Finally, without assuming the smoothness of the value function, we derive the necessary and sufficient conditions for optimality, by using some relationships between the value function and the adjoint processes. These relationships are presented via the "super- and sub-differential" which is related to the viscosity solution.

2. Problem Formulation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$ be a probability space such that $\mathcal{F}_0$ contains the $\mathbb{P}$ –null sets, $\mathcal{F}_T = \mathcal{F}$ and $(\mathcal{F}_t)_{t\leq T}$, that satisfies the usual assumptions. We assume that, $(\mathcal{F}_t)_{t\leq T}$ is generated by a d –dimensional standard Brownian motion B and an independent Poisson measure N on $[0,T] \times E$, where $E = \mathbb{R}^m \backslash \{0\}$ for some $m \geq 1$. We denote by $(\mathcal{F}_t^B)_{t\leq T}$ (resp. $(\mathcal{F}_t^N)_{t\leq T}$) the $\mathbb{P}$ –augmentation of the natural filtration of B (resp. N). Obviously, we have

$$\mathcal{F}_t = \sigma\left[\int\int_{A\times(0,s]} N(dr,de); s \leq t, A \in \mathcal{B}(E)\right] \vee \sigma[B_s; s \leq t] \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of ν –null sets, and $\sigma_1 \vee \sigma_2$ denotes the σ –field generated by $\sigma_1 \cup \sigma_2$. We assume the compensator of N has the form $\mu(dt,de) = \nu(de)dt$ for some positive and σ –finite Lévy measure ν on E, endowed with its Borel measure $\mathcal{B}(E)$. Also we suppose that $\int_E 1 \wedge |e|^2 \nu(de) < \infty$ and define the measure $\mathbb{P} \otimes \mu$ on $(\Omega \times [0,T] \times E, \mathcal{F} \times \mathcal{B}([0,T]) \times \mathcal{B}(E))$ by

$$\mathbb{P} \otimes \mu(G) = \mathbb{E}\left[\int\int_{[0,T]\times E} 1_G(\omega,t,e)\mu(dt,de)\right], \text{ for } G \in \mathcal{F} \times \mathcal{B}([0,T]) \times \mathcal{B}(E).$$

This is called the measure generated by μ. Then write $\widetilde{N}(dt,de) = N(dt,de) - \nu(de)dt$ for the compensated jump martingale random measure of N.

Notation. Any element $x \in \mathbb{R}^n$ will be identified to a column vector with an n-th component, and the norm $|x| = |x_1| + \ldots + |x_n|$. The scalar product of any two vectors x and y on $\mathbb{R}^n$ is denoted by $x.y$. We denote by M^{T} the transpose of any vector or matrix M. For a function h, we denote by h_x (resp. h_{xx}) the gradient or Jacobian (resp. the Hessian) of h with respect to the variable x. C always represents a generic constant, which can be different from line to line.
Additionally given $s < t$,

- $\mathcal{L}_\nu^2(E;\mathbb{R}^n)$ or $\mathcal{L}_\nu^2$ is the set of square integrable functions $\mathrm{l}(.) : E \to \mathbb{R}^n$ such that
$$\|\mathrm{l}(e)\|^2_{\mathcal{L}_\nu^2(E;\mathbb{R}^n)} := \int_E |\mathrm{l}(e)|^2 \nu(de) < \infty.$$
- $L^2([s,t];\mathbb{R}^n)$ the set of $\mathbb{R}^n$ –valued adapted càdlàg processes $\mathrm{P}(.)$ such that
$$\|\mathrm{P}(.)\|_{L^2([s,t];\mathbb{R}^n)} := \mathbb{E}\left[\sup_{r\in[s,t]} |\mathrm{P}(r)|^2\right]^{\frac{1}{2}} < \infty.$$
- $\mathcal{M}^2([s,t];\mathbb{R}^n)$ is the set of progressively measurable $\mathbb{R}^n$ –valued processes $\mathrm{Q}(.)$ such that
$$\|\mathrm{Q}(.)\|_{\mathcal{M}^2([s,t];\mathbb{R}^n)} := \mathbb{E}\left[\int_s^t |\mathrm{Q}(r)|^2 dr\right]^{\frac{1}{2}} < \infty.$$
- $\mathcal{L}^2([s,t];\mathbb{R}^n)$ is the set of $\mathcal{B}([0,T] \times \Omega) \otimes \mathcal{B}(E)$ measurable maps $\mathrm{R} : [0,T] \times \Omega \times E \to \mathbb{R}^n$ such that
$$\|\mathrm{R}(.,.)\|_{\mathcal{L}^2([s,t];\mathbb{R}^n)} := \mathbb{E}\left[\int_s^t \int_E |\mathrm{R}(r,e)|^2 \nu(de) dr\right]^{\frac{1}{2}} < \infty.$$

Definition 2.1. Let T be a strictly positive real number and U is a nonempty subset of $\mathbb{R}^n$. An admissible control is defined as a function $u : [0,T] \times \Omega \to U$ which is Borel measurable and $\mathcal{F}_t$ –predictable, such that, the SDE (3) has a unique solution, and $u \in \mathcal{U}$.

Let us consider the following stochastic control problem.
For $u(.) \in \mathcal{U}$, we assume that the state $x(.)$, of a controlled jump diffusion in $\mathbb{R}^n$, is described, for $t \in [0,T]$, by the following stochastic differential equation

$$\begin{cases} dx(t) = & b(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dB_t + \int_E \beta(t,x(t-),u(t),e)\widetilde{N}(dt,de), \\ x(s) = & y, \end{cases} \tag{3}$$

where $(s,y) \in [0,T) \times \mathbb{R}^n$be given, representing the initial time and initial state respectively, of the system. As before $\widetilde{N}(dt,de) = \left(\widetilde{N}_1(dt,de),\ldots,\widetilde{N}_l(dt,de)\right)^{\mathrm{T}}$,
and $\widetilde{N}_j(dt,de) = N_j(dt,de) - \nu_j(de)dt,\ 1 \leq j \leq n.$

Suppose next that the cost functional has the form

$$J(u) = \mathbb{E}\left[\int_s^T f(t,x(t),u(t))dt + g(x(T))\right], \quad \text{for } u(.) \in \mathcal{U}, \tag{4}$$

where $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$. Here $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}, \beta : [0,T] \times \mathbb{R}^n \times U \times E \to \mathbb{R}^n$,
$f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, are measurable functions.

The objective of the optimality problem, is to minimize the functional $J(u(.))$ over all $u(.) \in \mathcal{U}$, i.e. we seek $u^*(.)$ such that $J(u^*(.)) = \inf_{u(.) \in \mathcal{U}} J(u(.))$. Any admissible control $u^*(.)$ that achieves the minimum is called an optimal control, and it implies an associated optimal state evolution $x^*(.)$ from (3). Here $(x^*(.),u^*(.))$ is an optimal solution. Finally, we introduce the value function associated to the control problem by

$$\begin{cases} V(t,x) = & \inf\limits_{u(.) \in \mathcal{U}} J(u(.)),\ \forall (t,x) \in [0,T) \times \mathbb{R}^n, \\ V(T,x) = & g(x). \end{cases} \tag{5}$$

3. The Stochastic Maximum Principle

It is well known that the maximum principle for a stochastic optimal control problem by which a necessary or a sufficient condition of optimality can be realized, involves the adjoint processes which solves the corresponding adjoint equation. In fact, the adjoint equation is in general a linear backward stochastic differential equation (BSDE in short) for which a terminal condition on the state has been specified. In this section, we give second-order necessary optimality condition for problem $(3) - (4)$. Note that, general statements on the maximum principle were made in [18].

At this point, we need the following basic assumptions
(H1) For each $(t,x,u,e) \in [0,T] \times \mathbb{R}^n \times U \times E$, The maps b, σ, and β are twice continuously

differentiable in x and all derivatives are bounded. There exists a constant $M > 0$ such that, for $h = b, \sigma$

$$|h(t,x,u) - h(t,x',u)| + \|\beta(t,x,u,e) - \beta(t,x',u,e)\|_{\mathcal{L}^2_\nu} \le M|x - x'|, \tag{6}$$

$$|h_x(t,x,u) - h_x(t,x',u)| + \|\beta_x(t,x,u,e) - \beta_x(t,x',u,e)\|_{\mathcal{L}^2_\nu} \le M|x - x'|, \tag{7}$$

$$|h(t,x,u)| + |\beta(t,x,u,e)| \le M(1 + |x|).\ \forall t \in [0,T],\ x,x' \in \mathbb{R}^n. \tag{8}$$

(**H2**) For each $(t,x,u) \in [0,T] \times \mathbb{R}^n \times U$, The maps f and g are twice continuously differentiable in x and all derivatives are bounded. There exists a constant $M > 0$ such that

$$|f(t,x,u) - f(t,x',u)| + |g(x) - g(x')| \le M|x - x'|, \tag{9}$$

$$|f_x(t,x,u) - f_x(t,x',u)| + |g_x(x) - g_x(x')| \le M|x - x'|, \tag{10}$$

$$|f(t,x,u)| + |g(x)| \le M(1 + |x|). \tag{11}$$

Remark 3.1. Under the above hypotheses, the SDE (3) has a unique strong solution, and by standard arguments, see e.g. [16], it is easy to show that for any $p > 0$,

$$\mathbb{E}\left[\sup_{0\le t\le T} |x(t)|^p\right] < \infty,$$

and the cost functional (4) is well defined from $\mathcal{U}$ into $\mathbb{R}$.

The maximum principle involves an admissible pair $(u^*(.), x^*(.))$ and a pair of adjoint variables $(\psi(.), \phi(.), \gamma(.,.))$ and $(\Psi(.), \Phi(.), \Gamma(.,.))$ associated with $(u^*(.), x^*(.))$. Here $\psi(.)$ is the unique solution to the first-order adjoint equation

$$\begin{cases} d\psi(t) = & -(b_x(t,x^*(t),u^*(t))^{\mathrm{T}}\psi(t) + \sigma_x(t,x^*(t),u^*(t))^{\mathrm{T}}\phi(t) \\ & +\int_E \beta_x(t,x^*(t),u^*(t),e)^{\mathrm{T}}\gamma(t,e)\nu(de) \\ & + f_x(t,x^*(t),u^*(t)))dt + \phi(t)dB_t + \int_E \gamma(t,e)\widetilde{N}(dt,de), \\ \psi(T) = & -g_x(x^*(T)), \end{cases} \tag{12}$$

and $\Psi(.)$ is the unique solution to the second-order adjoint equation

$$\begin{aligned} d\Psi(t) = & -\Big(b_x(t,x^*(t),u^*(t))^{\mathrm{T}}\Psi(t) + \Psi(t).b_x(t,x^*(t),u^*(t)) \\ & +\sigma_x(t,x^*(t),u^*(t))^{\mathrm{T}}\Psi(t)\sigma_x(t,x^*(t),u^*(t)) \\ & +\sigma_x(t,x^*(t),u^*(t))^{\mathrm{T}}\Phi(t) + \Phi(t).\sigma_x(t,x^*(t),u^*(t)) \\ & +\int_E \beta_x(t,x^*(t),u^*(t),e)^{\mathrm{T}}(\Gamma(t,e) + \Psi(t))\beta_x(t,x^*(t),u^*(t),e)\nu(de) \\ & +\int_E (\Gamma(t,e).\beta_x(t,x^*(t),u^*(t),e) + \beta_x(t,x^*(t),u^*(t),e)^{\mathrm{T}}\Gamma(t,e))\nu(de)\Big)dt \\ & +H_{xx}(t,x^*(t),u^*(t),\psi(t),\phi(t),\gamma(t,e))dt + \Phi(t)dB_t + \int_E \Gamma(t,e)\widetilde{N}(dt,de), \end{aligned} \tag{13}$$

$\Psi(T) = -g_{xx}(x^*(T))$.

Note that under assumptions $(\mathbf{H1}) - (\mathbf{H2})$, the linear BSDEs (12) and (13) admit unique $\mathcal{F}_t$-adapted solutions $(\psi,\phi,\gamma) \in \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n$ and $(\Psi,\Phi,\Gamma) \in \mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times n})^d \times \mathbb{R}^{n\times n}$, where $\Phi = \left(\Phi_j, \text{ for } j = 1,..,d\right)$ and $\Phi_j \in \mathbb{R}^{n\times n}$, with ψ and Ψ being càdlàg processes. Moreover, since the coefficients $b_x, \sigma_x, \beta_x, f_x, g_x, b_{xx}, \sigma_{xx}, \beta_{xx}, f_{xx}$ and g_{xx} are bounded, we deduce from standard arguments that, there exists a constant C, independent of (x,u), such that the solutions of (12) and (13) have the following estimates

$$\|\psi(.)\|^2_{L^2([s,T];\mathbb{R}^n)} + \|\phi(.)\|^2_{\mathcal{M}^2([s,T];\mathbb{R}^{n\times d})} + \|\gamma(.,.)\|^2_{\mathcal{L}^2_\nu([s,T];\mathbb{R}^n)} \leq C,$$

$$\|\Psi(.)\|^2_{L^2([s,T];\mathbb{R}^{n\times n})} + \|\Phi_j(.)\|^2_{\mathcal{M}^2([s,T];\mathbb{R}^{n\times n})} + \|\Gamma(.,.)\|^2_{\mathcal{L}^2_\nu([s,T];\mathbb{R}^{n\times n})} \leq C, \text{ for } j = 1,..,d.$$

Define the usual Hamiltonian for $(t,x,u,p,q,\mathrm{X}) \in [s,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n$, by

$$H(t,x,u,p,q,\mathrm{X}) = -f(t,x,u) + pb(t,x,u) + q\sigma(t,x,u) + \int_E \mathrm{X}(e)\beta(t,x,u,e)\nu(de). \tag{14}$$

Furthermore, we define the $\mathcal{H}$ –function corresponding to a given admissible pair (x,u) as follows

$$\begin{aligned}\mathcal{H}(t,x,u) &= H(t,x,u,\psi(t),\phi(t),\gamma(t,e)) - \sigma(t,x,u)^{\mathrm{T}}\Psi(t)\sigma(t,x^*(.),u^*(.)) \\ &\quad + \frac{1}{2}\sigma(t,x,u)^{\mathrm{T}}\Psi(t)\sigma(t,x,u) - \int_E \Big(\beta(t,x,u,e)^{\mathrm{T}}(\Psi(t) + \gamma(t,e))\beta(t,x^*(.),u^*(.),e) \\ &\quad - \frac{1}{2}\beta(t,x,u,e)^{\mathrm{T}}(\Psi(t) + \gamma(t,e))\beta(t,x,u,e)\Big)\nu(de),\end{aligned} \tag{15}$$

for $(t,x,u) \in [s,T] \times \mathbb{R}^n \times U$, where the processes $\psi(t), \phi(t), \gamma(t,e)$, and $\Psi(t)$ are determined by the adjoint equations (12) and (13) corresponding to $(x^*(.),u^*(.))$. An immediate consequence of theorem $2.1.$ in [18] is the theorem that follows.

Theorem 3.1. *Let* $(\mathbf{H1}) - (\mathbf{H2})$ *hold and let* $(x^*(.),u^*(.))$ *be an optimal pair of problem* $(3),(4)$. *Then there are pairs of processes*

$$(\psi(.),\phi(.),\gamma(.,.)) \in L^2([s,T];\mathbb{R}^n) \times \mathcal{M}^2([s,T];\mathbb{R}^{n\times d}) \times \mathcal{L}^2([s,T];\mathbb{R}^n),$$

$$(\Psi(.),\Phi(.),\Gamma(.,.)) \in L^2([s,T];\mathbb{R}^{n\times n}) \times (\mathcal{M}^2([s,T];\mathbb{R}^{n\times n}))^d \times \mathcal{L}^2([s,T];\mathbb{R}^{n\times n}),$$

where $(\psi(.),\phi(.),\gamma(.,.))$ *and* $(\Psi(.),\Phi(.),\Gamma(.,.))$ *are the solutions to* (12) *and* (13) *respectively, corresponding to* $(x^*(.),u^*(.))$, *such that the following maximum condition holds*

$$\mathcal{H}(t,x^*(t),u^*(t)) = \max_{u\in\mathcal{U}} \mathcal{H}(t,x^*(t),u),\ a.e.\ t,\ \mathbb{P}-a.s.$$

4. Relation to Dynamic Programming

In a jump-diffusion setting, the connection between the SMP and DPP was reported in Framstad et al. [11]. Moreover,Theorem 2.1 of [11] says that if $V(.,.) \in \mathcal{C}^{1,3}([0,T] \times \mathbb{R}^n)$, then the relationship between the value function (5) of the control problem $(3),(4)$ and the adjoint variable $(\psi(.),\phi(.),\gamma(.,.))$ along an optimal solution $(x^*(.),u^*(.))$ is given by

$$\begin{cases} \psi(.) = & -V_x(t,x^*(t)), \\ \phi(.) = & -V_{xx}(t,x^*(t))\sigma(t,x^*(t),u^*(t)), \\ \gamma(.,.) = & -V_x(t,x^*(t-)+r(t,x^*(t-),u^*(t)),e)+V_x(t,x^*(t)). \end{cases}$$

Unfortunately, the HJB equation (20) does not necessarily admit smooth solutions. Without assuming any differentiability conditions the connection between the DPP and SMP is proved in [13] by using the viscosity solution theory of general nonlinear PDE. In [13] all the derivatives involved are replaced by the so-called superdifferentials and subdifferentials of the value function. In this section, we recall some related concepts and results in the theory of viscosity solution to the generalized Hamilton-Jacobi-Bellman equations to impose the following conditions

(**H3**) b,σ,r are uniformly continuous in (t,x,u), and there exists a constant $M>0$ such that, for $h=b,\sigma$

$$|h(t,x,u)-h(t,x',u')|+\|\beta(t,x,u,e)-\beta(t,x',u',e)\|_{\mathcal{L}^2_v} \leq M(|x-x'|+|u-u'|), \tag{16}$$

$$|h(t,x,u)|+\|\beta(t,x,u,e)\|_{\mathcal{L}^2_v} \leq M(1+|x|),\ \forall t\in[0,T],\ x,x'\in\mathbb{R}^n, u,u'\in U. \tag{17}$$

(**H4**) f,g are uniformly continuous in (t,x,u). There exists a constant $M>0$ and an increasing continuous function $\vartheta:[0,\infty)\times[0,\infty)\to[0,\infty)$ which satisfies $\vartheta(r,0)=0,\forall r\geq 0,$ such that

$$|f(t,x,u)-f(t,x',u)|+|g(x)-g(x')| \leq \vartheta(|x|\vee|x'|,|x-x'|) \tag{18}$$

$$|f(t,0,u)|,|g(0)|\leq M,\ \forall t\in[0,T],x,x'\in\mathbb{R}^n,u\in U. \tag{19}$$

The standard approach adopted in the literature to determine an optimal control is to solve the HJB equation

$$-V_t(t,x)+\sup_{u\in U} G(t,x,u,-V(t,x),-V_x(t,x),-V_{xx}(t,x))=0, \tag{20}$$

where the generalized Hamiltonian function, associated with a function $\varphi,$ is defined by

$$\begin{aligned} G(t,x,u,\varphi(t,x),\varphi_x(t,x),\varphi_{xx}(t,x)) &:= \varphi_x(t,x).b(t,x,u)+\tfrac{1}{2}tr(\varphi_{xx}(t,x)\sigma(t,x,u)\sigma(t,x,u)^{\mathrm{T}}) \\ &-f(t,x,u)-\int_E\{\varphi(t,x+\beta(t,x,u,e))-\varphi(t,x)-\varphi_x(t,x).\beta(t,x,u,e)\}v(de). \end{aligned} \tag{21}$$

The discussion we shall give in this section follows the same lines as of that one given in [13]. Hence it would be possible to replace the smoothness assumption by a weaker one. To this end we start with the definition of viscosity solutions of the HJB equation (20).

Definition 4.1. (i) A function $W\in\mathcal{C}([0,T]\times\mathbb{R}^n)$ is called a viscosity subsolution of (20) if $W(T,x)\leq g(x),\forall x\in\mathbb{R}^n,$ and for any test function $\varphi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n),$ whenever $W-\varphi$ attains a global maximum at $(t,x)\in[0,T)\times\mathbb{R}^n$, then

$$-\varphi_t(t,x)+\sup_{u\in U} G(t,x,u,-\varphi(t,x),-\varphi_x(t,x),-\varphi_{xx}(t,x))\leq 0,$$

(ii) A function $W\in\mathcal{C}([0,T]\times\mathbb{R}^n)$ is called a viscosity supersolution of (4.5) if $v(T,x)\geq g(x),\forall x\in\mathbb{R}^n,$ and for any test function $\chi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n),$ whenever $W-\chi$ attains a global minimum at $(t,x)\in[0,T]\times\mathbb{R}^n$, then

$$-\chi_t(t,x)+\sup_{u\in U} G(t,x,u,-\chi(t,x),-\chi_x(t,x),-\chi_{xx}(t,x))\geq 0,$$

(iii) If $W \in \mathcal{C}([0,T] \times \mathbb{R}^n)$ is both a viscosity subsolution and a viscosity supersolution of (4.5), then it is called a viscosity solution of (20).

In proving the uniqueness result for a viscosity solution of second order equations, it is convenient to give an intrinsic characterization of viscosity solutions. Here we need to recall the notion of parabolic semijets as introduced in [9].

Given $W \in \mathcal{C}([0,T] \times \mathbb{R}^n)$ and $(t',x') \in [0,T) \times \mathbb{R}^n$, the right parabolic superjet of W at (t',x') is the set triple

$$\mathcal{D}^{1,2,+}_{t+,x} W(t',x') := \{(q,p,Q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} | W(t,x) \leq W(t',x') + q(t-t') + p.(x-x') + \tfrac{1}{2}(x-x')^{\mathrm{T}} P(x-x') + o\left(|t-t'| + |x-x'|^2\right), \text{ as } t \downarrow t', x \to x'\Big\}. \tag{22}$$

Similarly, we consider the right parabolic subjet of W at (s,y) by the set triple

$$\mathcal{D}^{1,2,-}_{t+,x} W(t',x') := \{(q,p,Q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} | W(t,x) \geq W(t',x') + q(t-t') + p.(x-x') + \tfrac{1}{2}(x-x')^{\mathrm{T}} P(x-x') + o\left(|t-t'| + |x-x'|^2\right), \text{ as } t \downarrow t', x \to x'\Big\}. \tag{23}$$

According to standard estimates of the theory of SDEs, the assumptions $(\mathbf{H3}) - (\mathbf{H4})$ on the coefficient functions guarantee the fulfillment of the following results.

Proposition 4.1. *Let $V \in C([0,T] \times \mathbb{R}^n)$ be a value function. then there exist increasing continuous functions $\vartheta_1 : [0,\infty) \to [0,\infty)$ and $\vartheta_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ which, satisfies $\vartheta_2(r,0) = 0$, $\forall r \geq 0$, such that*

$$V(s,y) \leq \vartheta_1(|y|), \ \forall (s,y) \in [0,T) \times \mathbb{R}^n, \tag{24}$$

$$|V(s,y) - V(s',y')| \leq \vartheta_2(|y| \vee |y'|, |y-y'| + |s-s'|), \ \forall (s,y),(s',y') \in [0,T) \times \mathbb{R}^n. \tag{25}$$

The next result is the existence and uniqueness of a viscosity solution to the generalized HJB equation (20).

Theorem 4.1. *Suppose that* $(\mathbf{H3}) - (\mathbf{H4})$ *hold, then the following equivalent results also hold.*
(i) The value function $V \in C([0,T] \times \mathbb{R}^n)$ defined by (5) *is the unique viscosity solution of the HJB equation* (20) *in the class of functions satisfying* (24) *and* (25).
(ii) The value function $V \in C([0,T] \times \mathbb{R}^n)$ is the unique function that satisfies (24), (25) *and the following. For all $(t,x) \in [0,T) \times \mathbb{R}^n$,*

$$-q + \sup_{u \in U} G(t,x,u,-\varphi(t,x),-p,-P) \leq 0, \forall (q,p,P) \in \mathcal{D}^{1,2,+}_{t+,x} V(t,x), \tag{26}$$

$$-q + \sup_{u \in U} G(t,x,u,-\chi(t,x),-p,-P) \geq 0, \forall (q,p,P) \in \mathcal{D}^{1,2,-}_{t+,x} V(t,x), \tag{27}$$

with the condition $V(T,x) = g(x)$. Here φ and $\chi \in C^{1,2}([0,T] \times \mathbb{R}^n)$, such that $\varphi(s,y) > V(s,y)$ and $\chi(s,y) < V(s,y)$, $\forall (s,y) \neq (t,x) \in [t,T] \times \mathbb{R}^n$.

Proof. This theorem can be sorted out from [13]. ■

Now we can present a technical lemma needed later in this study; see Zhou et al. [20] for its proof.

Lemma 4.1. *Let* $v \in C([0,T]\times\mathbb{R}^n)$ *be a given function that satisfies*

$$|v(t,x)-v(t',x')| \le c\Big(|t-t'|^{\frac{1}{2}}+|x-x'|\Big).$$

For any $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$, *if* $(p,q,Q)\in D^{+}_{t+,x}v(t_0,x_0)$ *(resp.,* $(p,q,Q)\in D^{-}_{t+,x}v(t_0,x_0)$*), then there exists a function* $\varphi:[t_0,T]\times\mathbb{R}^n\to\mathbb{R}$ *satisfying*

1) $\varphi\in C([t_0,T]\times\mathbb{R}^n)\cap C^{1,2}((t_0,T]\times\mathbb{R}^n)$,

2) $\varphi(t_0,x_0)=v(t_0,x_0)$ *and* $\varphi(t,x)>v(t,x)$ *(resp.,* $\varphi(t,x)<v(t,x)$*) for any* $(t,x)\neq(t_0,x_0)$,

3) $\lim\limits_{t\to t_0+,x\to x_0}\varphi_t(t,x)=p$, *where* $|x-x_0|\le N|t-t'|^{\frac{1}{2}}$ *for any fixed* $N>0$, $\varphi_x(t_0,x_0)=q$ *and* $\varphi_{xx}(t_0,x_0)=Q$.

4) $|\varphi_t(t,x)|\le C_2\Big(1+|x-x_0|\big/|t-t_0|^{\frac{1}{2}}\Big)$, $\forall(t,x)\in(t_0,T]\times\mathbb{R}^n$.

5) $|\varphi_x(t,x)|+|\varphi_{xx}(t,x)|\le C(1+|x|+|x|^2+|x|^3)$, $\forall(t,x)\in[t_0,T]\times\mathbb{R}^n$.

Associated with an optimal pair $(x^*(.),u^*(.))$, with a corresponding adjoint processes satisfying (14) and (15), we define a $\mathcal{G}$ –function, where $\mathcal{G}:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}$ such that

$$\mathcal{G}(t,x,u):=\mathcal{H}(t,x,u)+\tfrac{1}{2}tr\Big(\int_E\Gamma(t,e)\beta(t,x^*(t),u^*(t),e)\beta(t,x^*(t),u^*(t),e)^{\mathrm{T}}\nu(de)\Big), \tag{28}$$

where the $\mathcal{H}$ –function is defined in the section 3. The relationship between the SMP and DPP is established through the following theorem.

Theorem 4.2. *Suppose that* $(\mathbf{H3})-(\mathbf{H4})$ *hold and let* $(s,y)\in[0,T)\times\mathbb{R}^n$ *be fixed. Let* (u^*,x^*) *be an optimal pair for the present stochastic optimal control problem. Let* $(\psi(.),\phi(.),\gamma(.,.))$ *and* $(\Psi(.),\Phi(.),\Gamma(.,.))$ *be first-order and second-order adjoint processes, respectively. Then*

$$[\mathcal{G}(t,x^*(t),u^*(t)),\infty)\times\{-\psi(t)\}\times[-\Psi(t),\infty)\subseteq\mathcal{D}^{1,2,+}_{t+,x}V(t,x^*(t)),\ a.e.\,t\in[s,T],\mathbb{P}-a.s. \tag{29}$$

$$\mathcal{D}^{1,2,-}_{t+,x}V(t,x^*(t))\subseteq[-\infty,\mathcal{G}(t,x^*(t),u^*(t)))\times\{-\psi(t)\}\times[-\infty,\Psi(t)),\ a.e.\,t\in[s,T],\mathbb{P}-a.s. \tag{30}$$

Proof. This result is just Theorem 3.2. in [13]. ■

5. The Main Results

The classical verification theorem is of significant importance in the DPP. It says that if an admissible control satisfies the maximum condition, then the control is indeed optimal for the stochastic control problem. However, this theorem is based on the assumption that the value function is smooth. But this smoothness assumption should not necessarily always hold, as illustrated in example 5.1.

Without assuming any differentiability of the value function, it turns out that the viscosity solution theory, provides an excellent framework to deal with the problem.

Theorem 5.1. *Suppose that* $(\mathbf{H3})-(\mathbf{H4})$ *hold and let* $W\in C^{1,2}([0,T]\times\mathbb{R}^n)$ *be a solution of* *the HJB equation* (20). *Then* $W(s,y) \leq J(u(.)), \quad \forall u(.) \in U, \quad (s,y) \in [0,T) \times \mathbb{R}^n$. *Furthermore, if there exists an admissible control* $u^*(.)$ *such that*

$$\begin{aligned} W_t(t,x^*(t)) &= \max_{u \in U} G(t,x^*(t),u,-W(t,x^*(t)),-W_x(t,x^*(t)),-W_{xx}(t,x^*(t))), \\ &= G(t,x^*(t),u^*(t),-W(t,x^*(t)),-W_x(t,x^*(t)),-W_{xx}(t,x^*(t))). \end{aligned} \tag{31}$$

Then
$W(s,y) \geq J(u^*(.))$.

More specifically, (31) gives both the necessary and sufficient condition for a given admissible pair $(x^*(.),u^*(.))$ to be optimal.

Example 5.1. We now give an example where the verification theorem 5.1. does not hold. Consider the SDE

$$\begin{cases} dx(t) &= x(t)u(t)dt + x(t)dB(t) + x(t-)u(t)\widetilde{N}(dt), \ t \in [s,T], \\ x(s) &= y. \end{cases} \tag{32}$$

Here N is a Poisson process with the intensity λdt and $\widetilde{N}(dt) := N(dt) - \lambda dt$, $(\lambda > 0)$, as the compensated martingale measure, with $u \in [0,1]$, and the cost function $J(u(.)) = \mathbb{E}[-x(T)]$. The HJB equation is

$$\begin{cases} -W_t(t,x) - \lambda W(t,x) - \frac{1}{2}x^2 W_{xx}(t,x) + \sup_{0 \leq u \leq 1} \{(1-\lambda)W_x(t,x)xu + \lambda W(t,x(1+u))\} \\ \qquad = 0, \ t \in [s,T], \\ W(T,x) = -x. \end{cases}$$

Its unique viscosity solution is

$$W(t,x) = \begin{cases} -x & \text{if } x \leq 0, \\ -xe^{T-t} & \text{if } x > 0. \end{cases}$$

For $x \neq 0$, W is differentiable and satisfies the HJB equation. If we consider an admissible pair $(x^*(.),u^*(.)) \equiv (0,0)$, then condition (31) can not tell if the pair is optimal, since $W_x(t,x^*(t))$ does not exist on the entire trajectory $x^*(.)$.

If the value function is not differentiable, equation (20) should be interpreted in a weaker sense, then the result (31) does not apply. Adapting the notion of viscosity solutions, we can derive the nonsmooth version of theorem 5.1. by using the fact that, the value function is the unique viscosity solution to the generalized HJB equation (20), and to theorem 4.2.

First, let us present a nonsmooth version of the necessity part of theorem 5.1. For this we need the following abbreviations: For $l_1 = b,\sigma,f,b_x,\sigma_x,f_x$, and $l_2 = \beta,\beta_x$, we define $l_1(t,x^*(t),u^*(t)) = l_1^*(t)$, and $l_2(t,x^*(t),u^*(t),e) = l_2^*(t,e)$.

Proposition 5.1. *Suppose that* $(\mathbf{H1})-(\mathbf{H4})$ *hold and let* $(s,y)\in[0,T)\times\mathbb{R}^n$ *be fixed. Let* $(x^*(.),u^*(.))$ *be the optimal pair of the stochastic optimal control* (3), (4). *Let* $(\psi(.),\phi(.),\gamma(.,.))$ *and* $(\Psi(.),\Phi(.),\Gamma(.,.))$ *be first-order and second-order adjoint processes, respectively, then* $a.e.\ t\in[s,T],\ P-a.s.$

$$0\leq tr(\sigma^*(t)^T\{\phi(t)-\Psi(t)\sigma^*(t)\})-\tfrac{1}{2}tr\left(\Psi(t)\int_E\beta^*(t,e)\beta^*(t,e)^T\nu(de)\right)$$
$$+\int_E(\varphi(t,x^*(t)+\beta^*(t,e))-\varphi(t,x^*(t))+\{\psi(t)+\gamma(t,e)\}.\beta^*(t,e))\nu(de),$$

where
$\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$, *such that* $\varphi(t',x')>V(t',x'),\ \forall(t',x')\neq(t,x)\in[s,T]\times\mathbb{R}^n$.

Proof. By (29) and by the fact that V is the viscosity solution of the HJB equation (20) it follows, on one hand, that

$$\mathcal{G}(t,x^*(t),u^*(t))\geq\sup_{u\in U}G(t,x^*(t),u^*(t),-\varphi(t,x^*(t)),\psi(t),\Psi(t)),\ a.e.\ t\in[s,T],\ \mathbb{P}-a.s.$$

On the other hand, according to (28) we observe that

$$\mathcal{G}(t,x^*(t),u^*(t))=G(t,x^*(t),u^*(t),-\varphi(t,x^*(t)),\psi(t),\Psi(t))$$
$$+tr(\sigma^*(t)^T\{\phi(t)-\Psi(t)\sigma^*(t)\})-\tfrac{1}{2}tr\left(\Psi(t)\int_E\beta^*(t,e)\beta^*(t,e)^T\nu(de)\right)$$
$$+\int_E(\varphi(t,x^*(t)+\beta^*(t,e))-\varphi(t,x^*(t))+\{\psi(t)+\gamma(t,e)\}.\beta^*(t,e))\nu(de).$$

Then, the required result directly follows. ■

The next theorem establishes a sufficient condition for optimality in terms of a system of PDEs. This result is therefore similar to the verification theorem in [13]. The main difference consists in the fact that in this case we have to work under relationship (29).

Theorem 5.2. *Suppose that* $(\mathbf{H1})-(\mathbf{H4})$ *hold. Let* $W\in C([0,T]\times\mathbb{R}^n)$ *be the unique viscosity solution of the HJB equation* (20). *Then*

$$W(s,y)\leq J(u(.)),\ \forall u(.)\in U. \tag{33}$$

Furthermore, let $u^*(.)$ *be an admissible control and* $x^*(.)$ *the corresponding solution of the state SDE* (3). *Let* $(\psi(.),\phi(.),\gamma(.,.))$ *and* $(\Psi(.),\Phi(.),\Gamma(.,.))$ *be first-order and second-order adjoint processes, respectively. If for* $a.e.\ t\in[s,T]$

$$G(t,x^*(t),u^*(t))=G(t,x^*(t),u^*(t),-\varphi(t,x^*(t)),\psi(t),\Psi(t)),\ P-a.s., \tag{34}$$

where $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$, *such that* $\varphi(t',x')>W(t',x'),\ \forall(t',x')\neq(t,x)\in[s,T]\times\mathbb{R}^n$. *Then* $(x^*(.),u^*(.))$ *is an optimal pair for* (s,y).

Proof. The conclusion (33) is clear, because W coincides with the value function due to the uniqueness of the viscosity solution to the HJB equation. Fix $t\in[s,T]$ so as the conditions of theorem 5.2 are satisfied at t for all $\omega\in\Omega_0$, and $\mathbb{P}(\Omega_0)=1$. Then fix $\omega\in\Omega_0$ so as the regular conditional probability $\mathbb{P}(.|\mathcal{F}_t^s)(\omega)$, given $\mathcal{F}_t^s$, is well defined on $(\Omega_0,\mathcal{F})$. The space is now equipped with a new filtration $\{\mathcal{F}_r^t\}_{r\in[t,T]}$ and the control process is adapted to this new filtration. Since $(\mathcal{G}(t,x^*(t),u^*(t)),-\psi(t),-\Psi(t))\in\mathcal{D}_{t+,x}^{1,2,+}V(t,x^*(t))$, there exists a φ which satisfies the conditions of lemma 4.1. Therefore for $h>0$ we obtain by Ito's formula

$$
\begin{aligned}
&W(t+h,x^*(t+h)) - W(t,x^*(t)) \le \varphi(t+h,x^*(t+h)) - \varphi(t,x^*(t)) \\
&\quad = \int_t^{t+h} (\varphi_t(r,x^*(r)) + \varphi_x(r,x^*(r))b_x^*(r) + \tfrac{1}{2}\sigma_x^*(r)^T\varphi_{xx}(r,x^*(r))\sigma_x^*(r))dr \\
&\quad + \int_t^{t+h} \varphi_x(r,x^*(r))\sigma_x^*(r)dB_r + \int_t^{t+h}\int_E [\varphi(r,x^*(r-)+\beta^*(r,e)) - \varphi(r,x^*(r-))]\widetilde{N}(dr,de) \\
&\quad + \int_t^{t+h}\int_E [\varphi(r,x^*(r)+\beta^*(r,e)) - \varphi(r,x^*(r)) - \varphi_x(r,x^*(r))\beta_x^*(r,e)]\nu(de)dr. \qquad (35)
\end{aligned}
$$

Now we calculate, for any fixed $N > 0,$

$$\frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}\{\varphi_t(r,x^*(r)) - \mathcal{G}(t,x^*(t),u^*(t))\}dr|\mathcal{F}_t^s\Big] = I_1(N,h) + I_2(N,h),$$

where the processes $I_1(N,h),\ I_2(N,h)$ are respectively given by

$$I_1(N,h) = \frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}\{\varphi_t(r,x^*(r)) - \mathcal{G}(t,x^*(t),u^*(t))\}\mathbf{1}_{|x^*(r)-x^*(t)|>N|r-t|^{\frac{1}{2}}}dr|\mathcal{F}_t^s\Big],$$

$$I_2(N,h) = \frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}\{\varphi_t(r,x^*(r)) - \mathcal{G}(t,x^*(t),u^*(t))\}\mathbf{1}_{|x^*(r)-x^*(t)|\le N|r-t|^{\frac{1}{2}}}dr|\mathcal{F}_t^s\Big].$$

For fixed $N > 0,$ it can be easily deduced from lemma 4.1. that

$$\sup_{t<r\le t+h}\{\varphi_t(r,x^*(r)) - \mathcal{G}(t,x^*(t),u^*(t))\} \to 0, \text{ as } h \to 0+, \mathbb{P}-a.s.,$$

on $\Big\{|x^*(r)-x^*(t)| \le N|r-t|^{\frac{1}{2}}\Big\}$. Thus we conclude by the dominated convergence theorem that $I_2(N,h) \to 0$ as $h \to 0+$ for each fixed N. On the other hand, from standard estimates in the theory of SDEs, there are constants $C,\ C(\alpha) > 0,$ independent of t, such that $\mathbb{E}\Big[|x^*(r)-x^*(t)|^2|\mathcal{F}_t^s\Big] \le C|r-t|,\ \forall r \ge t,$ and $\sup_{s\le r\le T}\mathbb{E}\Big[|x^*(r)|^2|\mathcal{F}_t^s\Big] \le C(\alpha),\ \forall\alpha \ge 1.$ Thus, it follows from the Schwartz and Tschebyshev inequalities that

$$
\begin{aligned}
I_1(N,h) &\le \frac{1}{h}\int_t^{t+h}\mathbb{E}\Big[|\varphi_t(r,x^*(r)) - \mathcal{G}(t,x^*(t),u^*(t))|^2|\mathcal{F}_t^s\Big]^{\frac{1}{2}} \\
&\qquad \mathbb{P}\Big[|x^*(r)-x^*(t)| > N|r-t|^{\frac{1}{2}}|\mathcal{F}_t^s\Big]^{\frac{1}{2}}dr, \\
&\le \tfrac{C}{N} \to 0 \text{ uniformly in } h > 0 \text{ as } N \to \infty.
\end{aligned}
$$

This proves that

$$\frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}\varphi_t(r,x^*(r))dr|\mathcal{F}_t^s\Big] \to \mathbb{E}[\mathcal{G}(t,x^*(t),u^*(t))|\mathcal{F}_t^s], \text{ as } h \to 0+.$$

Moreover, due to lemma 4.1, and from the dominated convergence theorem we get by sending h to $0+$

$$\lim_{h\to 0+}\frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}\{\varphi_x(r,x^*(r))b_x^*(r)+\tfrac{1}{2}\sigma_x^*(r)^T\varphi_{xx}(r,x^*(r))\sigma_x^*(r)$$
$$+\int_E[\varphi(r,x^*(r)+\beta^*(r,e))-\varphi(r,x^*(r))-\varphi_x(r,x^*(r))\beta_x^*(r,e)]\nu(de)\Big\}dr|\mathcal{F}_t^s\Big]$$
$$=\mathbb{E}[-\psi(t)b_x^*(t)-\tfrac{1}{2}\sigma_x^*(t)^T\Psi(t)\sigma_x^*(t)$$
$$+\int_E[\varphi(t,x^*(t)+\beta^*(t,e))-\varphi(t,x^*(t))+\psi(t)\beta_x^*(t,e)]\nu(de)|\mathcal{F}_t^s\Big],$$

and

$$\lim_{h\to 0+}\frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}f_x^*(r)dr|\mathcal{F}_t^s\Big]=\mathbb{E}[f_x^*(t)|\mathcal{F}_t^s].$$

The last results, together with (35), are the tools we need to get the following inequality

$$\limsup_{h\to 0}\frac{1}{h}\mathbb{E}[W(t+h,x^*(t+h))-W(t,x^*(t))|\mathcal{F}_t^s]$$
$$\le\mathbb{E}[\mathcal{G}(t,x^*(t),u^*(t))-\psi(t)b_x^*(t)-\tfrac{1}{2}\sigma_x^*(t)^T\Psi(t)\sigma_x^*(t)$$
$$+\int_E[\varphi(t,x^*(t)+\gamma^*(t,e))-\varphi(t,x^*(t))+\psi(t)\gamma_x^*(t,e)]\nu(de)|\mathcal{F}_t^s\Big]. \tag{36}$$

Integrate, while interchanging expectation and integration, to obtain, by taking the expectation conditioned to $\mathcal{F}_t^s$,

$$\mathbb{E}[W(T-\varepsilon,x^*(T-\varepsilon))]-W(s,y)$$
$$=\lim_{h\to 0}\frac{1}{h}\mathbb{E}\Big[\int_{T-\varepsilon}^{T-\varepsilon+h}W(t,x^*(t))dt-\int_s^{s+h}W(t,x^*(t))dt\Big],$$
$$=\lim_{h\to 0}\frac{1}{h}\mathbb{E}\Big[\int_{s+h}^{T-\varepsilon+h}W(t,x^*(t))dt-\int_{s+h}^{T-\varepsilon}W(t,x^*(t))dt-\int_s^{s+h}W(t,x^*(t))dt\Big],$$
$$=\lim_{h\to 0}\frac{1}{h}\mathbb{E}\Big[\int_{s+h}^{T-\varepsilon+h}W(t,x^*(t))dt-\int_s^{T-\varepsilon}W(t,x^*(t))dt\Big],$$
$$=\lim_{h\to 0}\int_s^{T-\varepsilon}\mathbb{E}\Big[\frac{1}{h}\mathbb{E}[W(t+h,x^*(t+h))-W(t,x^*(t))|\mathcal{F}_t^s]\Big]dt.$$

From Fatou's lemma and (36), it follows that

$$\mathbb{E}[W(T-\varepsilon,x^*(T-\varepsilon))]-W(s,y)$$
$$\le\int_s^{T-\varepsilon}\mathbb{E}\Big[\limsup_{h\to 0}\tfrac{1}{h}\mathbb{E}[W(t+h,x^*(t+h))-W(t,x^*(t))|\mathcal{F}_t^s]\Big]dt,$$
$$\le\int_s^{T-\varepsilon}\mathbb{E}[\mathcal{G}(t,x^*(t),u^*(t))-\psi(t)b_x^*(t)-\tfrac{1}{2}\sigma_x^*(t)^T\Psi(t)\sigma_x^*(t)$$
$$+\int_E[\varphi(t,x^*(t)+\beta^*(t,e))-\varphi(t,x^*(t))+\psi(t)\beta_x^*(t,e)]\nu(de)\Big]dt.$$

Since W is continuous and $W(T,x_T^*)=g(x_T^*)$, it follows, from the condition (34) and the arbitrariness of ε, that

$$\begin{aligned}
&\mathbb{E}[g(x^*(T))] - W(s,y) \\
&\quad \leq \int_s^T \mathbb{E}[G(t,x^*(t),u^*(t),-\varphi(t,x^*(t)),\psi(t),\Psi(t)) - \psi(t)b_x^*(t) - \tfrac{1}{2}\sigma_x^*(t)^T\Psi(t)\sigma_x^*(t) \\
&\quad + \int_E [\varphi(t,x^*(t)+\beta^*(t,e)) - \varphi(t,x^*(t)) + \psi(t)\beta_x^*(t,e)]\nu(de)\Big]dt, \\
&\quad = \mathbb{E}\left[-\int_s^T f(t,x^*(t),u^*(t))dt\right].
\end{aligned}$$

This means that $J(u^*(.)) \leq W(s,y)$. ■

Now, let us revisit example 5.1. For $x \neq 0$ we have to check whether (26) and (27) hold. To see this, note that (27) is trivially fulfilled by the fact that $\mathcal{D}_{t+,x}^{1,2,-}W(t,0) = (-\infty,0] \times \varnothing \times \varnothing$, $t \in [0,T]$. On the other hand, it is worth noting that $\mathcal{D}_{t+,x}^{1,2,+}W(t,0) = [0,\infty) \times [-e^{T-t},-1] \times [0,\infty)$, $t \in [0,T]$. For $x = 0$ the HJB equation is $-W_t(t,0) = 0$, so (26) holds, too. Let $(s,y) = (0,0)$, to claim that $u^*(t) \equiv 0$ is an optimal control, and from the SDE (31), we get $x^*(t) \equiv 0$, the first-order and second-order adjoint processes are then $(\psi(t),\phi(t),\gamma(t,e)) = (1,0,0)$ and $(\Psi(t),\Phi(t),\Gamma(t,e)) = (0,0,0)$. So (29) is satisfied. Since $\mathcal{G}(t,x^*(t),u^*(t)) = \mathcal{G}(t,0,0) = 0$, $t \in [0,T]$, we have $(\mathcal{G}(t,x^*(t),u^*(t)),-\psi(t),-\Psi(t)) \equiv (0,-1,0) \in \mathcal{D}_{t+,x}^{1,2,+}W(t,0)$. For the optimality of $u^*(.)$ we have to check on the maximum principle. In fact, we have $\forall u \in [0,1]$, $\mathcal{H}(t,u,x) = 0$, so the maximum principle is trivially satisfied.

Proposition 5.2. *Condition* (34) *in theorem* 5.2 *is equivalent to the following*

$$\mathbb{E}\int_s^T \mathcal{G}(t,x^*(t),u^*(t))dt \leq \mathbb{E}\int_s^T G(t,x^*(t),u^*(t),-\varphi(t,x^*(t)),\psi(t),\Psi(t))dt, \tag{37}$$

where φ is the same test function as in theorem 4.1.

Proof. It is clear that (34) implies (37). Suppose now that (37) holds. Since V is the viscosity solution of the HJB equation (20), we have from theorem 4.1,

$$\sup_{u\in U} G(t,x^*(t),u,-\varphi(t,x^*(t)),\psi(t),\Psi(t)) \leq \mathcal{G}(t,x^*(t),u^*(t)).$$

The above inequality along with (37) yields (34). ■

Remark 5.1. The condition (34) implies that

$$G(t,x^*(t),u^*(t),-\varphi(t,x^*(t)),\psi(t),\Psi(t)) = \max_{u\in U} G(t,x^*(t),u,-\varphi(t,x^*(t)),\psi(t),\Psi(t)). \tag{38}$$

This is easily seen by recalling the fact that V is the viscosity solution of (20), and (29). Then

$$-\mathcal{G}(t,x^*(t),u^*(t)) + \sup_{u\in U} G(t,x^*(t),u,-\varphi(t,x^*(t)),\psi(t),\Psi(t)) \leq 0, \tag{39}$$

which yields (38) from (34).

To conclude this section, let us present a nonsmooth version of the necessity part of theorem 5.2 expressed in terms of subdifferentials.

Theorem 5.3. *Suppose that* (**H1**) – (**H4**) *hold then let* $(s,y) \in [0,T) \times \mathbb{R}^n$ *be fixed, and* V *be the value function. Let* $(x^*(.),u^*(.))$ *be the optimal pair of the present stochastic optimal control. If* $(\mathcal{G}(t,x^*(t),u^*(t)),-\psi(t),-\Psi(t)) \in D^{1,2,-}_{t+,x}V(t,x^*(t))$ *a.e.* $t \in [s,T]$, $P-a.s.$, *then*

$$\mathbb{E}[\mathcal{G}(t,x^*(t),u^*(t))] \leq \mathbb{E}[G(t,x^*(t),u^*(t),-\varphi(t,x^*(t)),\psi(t),\Psi(t))],\ a.e.\ t \in [s,T],$$

must hold.

Proof. Fix a $t \in [s,T)$ so as $(\mathcal{G}(t,x^*(t),u^*(t)),-\psi(t),-\Psi(t)) \in \mathcal{D}^{1,2,-}_{t+,x}V(t,x^*(t))$, $\mathbb{P}-a.s.$ Then there exists a φ which satisfies the conditions of Lemma 4.1. Therefore for $h > 0$ we obtain by Ito's formula

$$\begin{aligned}\mathbb{E}[V(t+h,x^*(t+h)) - V(t,x^*(t))] &\geq \mathbb{E}[\varphi(t+h,x^*(t+h)) - \varphi(t,x^*(t))]\\ &= \mathbb{E}\Big[\int_t^{t+h}(\varphi_t(r,x^*(r)) + \varphi_x(r,x^*(r))b^*_x(r) + \tfrac{1}{2}\sigma^*_x(r)^T\varphi_{xx}(r,x^*(r))\sigma^*_x(r)\\ &\quad + \int_E[\varphi(r,x^*(r)+\beta^*(r,e)) - \varphi(r,x^*(r)) - \varphi_x(r,x^*(r))\beta^*_x(r,e)]\nu(de)\Big)dr\Big].\end{aligned} \tag{40}$$

However, since $(x^*(.),u^*(.))$ is optimal, we have the following Principle of Optimality

$$V(r,x^*(r)) = \mathbb{E}\Big[\int_r^T f(\theta,x^*(\theta),u^*(\theta))d\theta + g(x^*(T))|\mathcal{F}_r\Big],\ \mathbb{P}-a.s.,\ \forall r \in [s,T],$$

which implies that

$$\mathbb{E}[V(t+h,x^*(t+h)) - V(t,x^*(t))] = -\mathbb{E}\Big[\int_t^{t+h} f(r,x^*(r),u^*(r))dr\Big]. \tag{41}$$

By the same argument as in the proof of theorem 5.2, we conclude by the dominated convergence theorem that

$$\begin{aligned}&\lim_{h\to 0+}\frac{1}{h}\mathbb{E}\Big[\int_t^{t+h}\{\varphi_t(r,x^*(r)) + \varphi_x(r,x^*(r))b^*_x(r) + \tfrac{1}{2}\sigma^*_x(r)^T\varphi_{xx}(r,x^*(r))\sigma^*_x(r)\\ &\quad + \int_E[\varphi(r,x^*(r)+\beta^*(r,e)) - \varphi(r,x^*(r)) - \varphi_x(r,x^*(r))\beta^*_x(r,e)]\nu(de)dr\Big\}\Big]\\ &= \mathbb{E}[\mathcal{G}(t,x^*(t),u^*(t)) - \psi(t)b^*_x(t) - \tfrac{1}{2}\sigma^*_x(t)^T\Psi(t)\sigma^*_x(t)\\ &\quad + \int_E[\varphi(t,x^*(t)+\beta^*(t,e)) - \varphi(t,x^*(t)) + \psi(t)\beta^*_x(t,e)]\nu(de)\Big].\end{aligned}$$

Consideration of (40) and (41) then division of both by h, and letting $h \to 0$ leads to

$$\mathbb{E}[\mathcal{G}(t,x^*(t),u^*(t))] \leq \mathbb{E}\Big[G\Big(t,x^*(t),u^*(t),-\varphi(t,x^*(t)),\psi(t),\Psi(t)\Big)\Big],\ a.e.\ t \in [s,T].$$

This completes the proof.

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