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Semilinear Stochastic Functional Differential Equations of Fractional Order With State-Dependent Delay

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Abstract. *In this paper, we establish sufficient conditions for the existence and uniqueness of mild solutions for semilinear stochastic functional differential equations with state-dependent delay. Our approach is based on the C_0 semigroups theory combined with some suitable fixed point theorems.*

Key words : Semilinear Stochastic Functional Differential Equations, Densely Defined Operator, State-Dependent Delay, Fractional Derivative, Fractional Integral, Fixed Point, C_0 Semigroup, Mild Solutions.

AMS Subject Classifications : 60H10, 93E20

1. Introduction

This work is concerned with existence of mild solutions to the following stochastic functional differential equations with state dependent delay

$$D^{\alpha}y(t) = Ay(t) + f(t, y(t-\rho(y(t)))) + \sigma(t, y(t-\rho(y(t))))dw(t), \ t \in J = [0,b], 0 < \alpha \leq 1, \quad (1)$$

$$y(t) = \phi(t), \quad t \in [-r,0], \quad (2)$$

where D^{α} is the standard Riemman-Liouville fractional derivative. E and K are separable Hilbert spaces with the norms $\|.\|_E$ and $\|.\|_K$ respectively. $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on E, $\phi : [-r,0] \to E$ a given continuous function with $\phi(0) = 0$. For any function y defined on $[-r,b]$ and any $t \in [0,b]$ we denote by y_t the element of $C([-r,0],E)$ defined by:

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0].$$

The functions $f : J \times C([-r,0], \ E) \to E$ and $\sigma : J \times C([-r,0],E) \to L_2^0(K,E)$ are appropriate mappings that are specified later. $L_2^0(K,E)$ denotes the space of all Q–Hilbert-Schmidt operators from K into E, while ρ is a positive bounded continuous function on E. Letter r is the maximal delay defined by

$$r = \sup_{x \in E} \rho(x).$$

Stochastic functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years, see for instance [1, 5, 6, 8,13, 14, 22]. Also the study of partial stochastic differential equations with state-dependent delay has recently been initiated.

Stochastic differential equations of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics and other fields can be described with the help of fractional differential equations, see [15, 19, 24, 25]. The theory of stochastic differential equations of fractional order has recently received a lot of attention and now constitutes a solid branch of mathematical analysis. In fact numerous research papers and monographs have appeared devoted to fractional differential equations, for example see [3, 7, 9, 16, 17, 20, 26] and [5, 22, 23].

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non integer order. The subject is as old as the differential calculus, and goes back to time when Leibnitz and Newton invented differential calculus. The idea of fractional calculus has been a subject of interest not only among mathematicians but also among physicists and engineers. For some recent advances on fractional calculus, differential equations and stochastic differential equations, the reader is referred to [2, 4, 5, 22, 23].

This paper is organized as follows. In Section 2 we introduce notations, definitions, and preliminary facts which are used in the sequel. In Section 3 we give our main existence and uniqueness results for problem (1)-(2).

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all P-null sets. Let $w = (w_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $e_k, k \geq 1$ in K, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k,\ k = 1, 2, \ldots,$ and a sequence $\hat{\beta}_k$ of independent Brownian motions such that

$$(w(t), e)_k = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,(e_k, e)\hat{\beta}_k, \qquad e \in K, t \geq 0.$$

Let $L_2^0 = L_2(Q^{\frac{1}{2}}K, E)$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}K$ to E with the inner product $< \varphi, \bar{\phi} >_{L_2^0} = tr[\varphi Q \bar{\phi}_*]$. $C(J, E)$ is the Banach space of all continuous functions from J to E with the norm

$\|y\|_\infty := \sup\{\|y(t)\| : t \in J\}.$

For $\psi \in C([-r,0],E)$ the norm of ψ is defined by

$\|\psi\|_C := \sup\{\|\psi(\theta)\| : \theta \in [-r,0]\},$

while for $\phi \in C([-r,0],E)$ the norm of ϕ is defined by

$\|\phi\|_{\mathfrak{D}} := \sup\{\|\phi(\theta)\| : \theta \in [-r,b]\}.$

Definition 2.1. A semigroup of class C_0 is a one parameter family $\{T(t)\}_{t\geq 0} \subset \mathfrak{L}(E)$ satisfying the conditions

(i) $T(0) = I$, (I is the identity operator on X),

(ii) $T(t+s) = T(t)T(s)$, $\forall$ s,t ≥ 0,

(iii) $\lim_{t\searrow o} T(t)x = x$, $\forall$ x in E,

when $\mathfrak{L}(E)$ denotes the Banach space of bounded linear operators from E to E, and the corresponding norm is denoted by $\|.\|$.

Definition 2.2. The infinitesimal generator $A : D(A) \subset E \to E$ of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space E is the operator

$$Ax = \lim_{t\searrow o} \frac{T(t)x - x}{t}, \quad for \quad x \in D(A)$$

defined for every x in its domain

$$D(A) = \{x \in E \mid \lim_{t\searrow o} \frac{T(t)x - x}{t} \; exists\}$$

Theorem 2.1. *Let* $T(t)_{t\geq 0}$ *be a semigroup of class* C_0 *on* E *and let* A *be its infinitesimal generator. Then*

(i) $t \to \|T(t)\|_{\mathfrak{L}(E)}$ *is bounded on every compact interval* $[0,T]$,

(ii) *For all* $x \in E$, *the function* $t \to T(t)x$ *is continues on* R^+,

(iii) $T(t)$ *is exponentially bounded ; i.e. there exist constants* $w \in R$ *and* $M \geq 1$ *such that*

$$\|T(t)\| \leq M \exp(wt), \quad \forall \quad t \geq 0.$$

Proposition 2.1. *Let* $T(t)_{t\geq 0}$ *be a* C_0 *semigroup and let A be its infinitesimal generator. Then*

(i) *For every* $x \in E$ *and* $t \geq 0$ *there holds*

$$\lim_{t\searrow o} \frac{1}{h} \int_t^{t+h} T(s)x\, ds = T(t)x,$$

(ii) *For every* $x \in E$ *and* $\int_0^t T(s)x\, ds \in D(A)$ *there holds*

$$A\int_0^t T(s)x\, ds = T(t)x - x,$$

(iii) *If* $x \in D(A)$, *then* $T(t)x \in D(A)$ *and*

$$\frac{d}{dt} T(t)x = T(t)Ax = AT(t)x,$$

(iv) *For every* $x \in D(A)$, $t \geq 0$ *and* $s \geq 0$ *there holds*

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax\, d\tau = \int_s^t AT(\tau)x\, d\tau.$$

Remark 2.1. If $T(t)_{t\geq 0}$ is a semigroup of class C_0 of bounded linear operators with an infinitesimal generator A, then it is unique.

Theorem 2.2. *Let A be an infinitesimal generator of a C_0 semigroup $T(t)_{t\geq 0}$ then*
(i) $\overline{D(A)} = E$,
(ii) *A is a closed linear operator.*

Definition 2.3. Let A be a closed linear operator in E. The resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e. $(\lambda I - A)^{-1}$ is a bounded linear operator in E . The family $R(\lambda;A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of a bounded linear operator is called the resolvent of A.

Theorem 2.3. *A linear operator A is the infinitesimal generator of a C_0 semigroup $T(t)_{t\geq 0}$ if and only if:*
(i) *A is a closed operator and $\overline{D(A)} = E$;*
(ii) *There exist constants $w \in R$ and $M \geq 1$ such that $A_w \subset \rho(A)$ and for $\lambda \in A_w$, we have*

$$|R(\lambda,A)^n|_{\mathfrak{L}(E)} \leq \frac{M}{(R(\lambda)-w)^n}, \quad \forall n \in \mathbb{N}^*.$$

For more details on the theory of C_0 semigroups , we refer the reader to the books of Goldstein [10] and Pazy [21]. Moreover, throughout this paper we adopt the following definitions of fractional primitive and fractional derivative.

Definition [24, 20] **2.4**. The Riemann Liouville fractional integral of order $\alpha \in R^+$of a function $h : (0,b] \to E$ is defined by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds,$$

provided that the right hand side exists pointwise on $(0,b]$, where Γ is the gamma function.

Definition [24, 20] **2.5**. The Riemann Liouville fractional derivative of order $0 < \alpha < 1$ of a function $h : (0,b] \to E$ is defined by

$$D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}h(s)ds = \frac{d}{dt}I_0^{1-\alpha}h(t).$$

Theorem [11] **2.4**. (The Banach contraction principle) *Let E a Banach space, and $f : E \longrightarrow E$ be contractive. Then f has a unique fixed point.*

Theorem [11] **2.5**. (The nonlinear alternative of Leray-Schauder) *Let E be a Banach space and $C \subset E$ be convex with $0 \in C$. Let $F : C \longrightarrow C$ be a completely continuous operator. Then either*

(a) F has a fixed point,
or
(b) the set $\varepsilon = \{x \in C : x = \lambda F(x), 0 < \lambda < 1\}$ *is unbounded.*

3. Main Result

This section reports on our main existence results for problem (1)-(2). Before stating and proving these results, we give the definition of the mild solution.

Definition 3.1. We say that a continuous function $y : [-r,b] \to E$ is a mild solution of problem (1)-(2) if $y(t) = \phi(t),\ t \in [-r,0]$, and

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s)\, f(s, y(s-\rho(y(s))))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s)\, \sigma(s, y(s-\rho(y(s))))dw(s),\ t \in J.$$

In order to establish the result, we need the following assumptions.

(A1) $\|T(t)\| \leq \widetilde{M}$, where $\widetilde{M} = \sup_{t \in J} \|T(t)\|$.

(A2)The function $f : J \times C([-r,0],E) \to E$ satisfies the following properties.
(i) For each $t \in J$ the function $f(t,.) : C([-r,0],E) \to E$ is continuous.
(ii) For each $x \in C$ the function $f(.,x) : J \to E$ is strongly measurable.
(iii) There exists a non negative constant k such that

$$\|f(t,u) - f(t,v)\|_E \leq k\|u-v\|_C, \quad \textit{for } t \in J \quad \textit{and every} \quad u,v \in C([-r,0],E).$$

(A3)The function $\sigma : J \times C([-r,0],E) \to L_2^0(K,E)$ satisfies the following properties.
(i) For each $t \in J$ the function $\sigma(t,.) : C([-r,0],E) \to L(E,K)$ is continuous.
(ii) For each $x \in C$ the function $f(.,x) : J \to L(E,K)$ is strongly measurable.
(iii) There exists a non negative constant M_σ such that

$$\|\sigma(t,u) - \sigma(t,v)\|_{L_2^0} \leq M_\sigma\|u-v\|_C, \quad \textit{for } t \in J \quad \textit{and every} \quad u,v \in C([-r,0],E).$$

Our first existence result for problem (1)-(2) is based on the Banach contraction principle.

Theorem 3.1. *Let the assumptions (A1)-(A2)-(A3) hold. Then there exists a unique mild solution of problem (1)-(2) on* $[-r,b]$.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator $\mathcal{F} : C([-r,-b],E) \to C([-r,-b],\ E)$ defined by

$$\mathcal{F}(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y(s-\rho(y(s))))ds + & \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)\sigma(s,y(s-\rho(y(s))))dw(s), & \text{if } t \in [0,b]. \end{cases} \tag{3}$$

Let us define the iterates of operator $\mathcal{F}$ by

$$\mathcal{F}^1 = \mathcal{F}, \quad \mathcal{F}^{n+1} = \mathcal{F}o\mathcal{F}^n, \tag{4}$$

and it will be sufficient to prove that $\mathcal{F}^n$ is a contraction operator for n sufficiently large. For every $x,y \in C([-r,b],E)$ we have

$$\|\mathcal{F}^n(y)(t) - \mathcal{F}^n(z)(t)\| \leq \frac{(k\widetilde{M})^n + (M_\sigma\widetilde{M})^n}{\Gamma(n\alpha+1)} t^{n\alpha}\|y-z\|_\infty. \tag{5}$$

Indeed,

$$\begin{aligned} \|\mathcal{F}(y)(t)-\mathcal{F}(z)(t)\|_E &\leq \frac{\widetilde{M}}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}\ \|f(s,\ y(s-\rho(y(s)))) - f(s,\ z(s-\rho(z(s))))\|_E\, ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\|\sigma(s,y(s-\rho(y(s)))) - \sigma(s,z(s-\rho(z(s))))\|_{L_2^0} ds\Big) \\ &\leq \frac{k\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s-\rho(y(s)))-z(s-\rho(z(s)))\|_C ds \\ &\quad + \frac{\widetilde{M}M_\sigma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s-\rho(y(s)))-z(s-\rho(z(s)))\|_C ds \\ &\leq \frac{(k+M_\sigma)\widetilde{M}}{\alpha\,\Gamma(\alpha)} t^\alpha\|y-z\|_\infty = \frac{(k+M_\sigma)\widetilde{M}}{\Gamma(\alpha+1)} t^\alpha\|y-z\|_\infty. \end{aligned}$$

Therefore (5) is proved for $n = 1$. Assuming by induction that (5) is valid for n, then

$$\begin{aligned} \|\mathcal{F}^{n+1}(y)(t) - \mathcal{F}^{n+1}(z)(t)\| &\leq \frac{k\widetilde{M}}{\Gamma(\alpha)}\frac{(k\widetilde{M})^n}{\Gamma(n\alpha+1)}\|y-z\|_\infty\int_0^t (t-s)^{\alpha-1}s^{n\alpha}ds \\ &\quad + \frac{(\widetilde{M}M_\sigma)^{n+1}}{\Gamma(\alpha)\Gamma(n\alpha+1)}\|y-z\|_\infty\int_0^t (t-s)^{\alpha-1}s^{n\alpha}ds \\ &= \frac{(k\widetilde{M})^{n+1} + (\widetilde{M}M_\sigma)^{n+1}}{\Gamma([n+1]\alpha+1)} t^{(n+1)\alpha}\|y-z\|_\infty \end{aligned}$$

and then (5) follows for $n+1$.

Now, taking n sufficiently large in (5) such that $\frac{(k\widetilde{M})^n + (M_\sigma\widetilde{M})^n}{\Gamma(n\alpha+1)} t^{n\alpha} < 1$ illustrates the contraction of operator $\mathcal{F}^n$. Consequently $\mathcal{F}$ has a unique fixed point by the Banach contraction principle, which gives rise to a unique mild solution to the problem (1)-(2). ■

Next we give an existence result based upon the following nonlinear alternative of Leray-Schauder applied to completely continuous operators. Here the following lemma is essential to state and prove our main result.

Lemma [12] **3.1**. *Suppose $b \geq 0$, $\beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t \leq T$ (some $T \leq \infty$), and suppose that $u(t)$ is nonnegative and locally integrable on $0 \leq t \leq T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1}u(s)ds, \quad 0 \leq t \leq T.$$

Then

$$u(t) \leq a(t) + \int_0^t \sum_{j=0}^{\infty} \frac{(b\Gamma(\beta))^j}{\Gamma(j\beta)}(t-s)^{j\beta-1}a(s)ds, \quad 0 \leq t \leq T, \tag{6}$$

and if $a(t) \equiv a$, constant on $0 \leq t \leq T$, then the inequality (6) reduces to

$$u(t) \leq aE_\beta(b\Gamma(\beta)t^\beta).$$

Proof. Let $B\phi(t) = b\int_0^t (t-s)^{\beta-1}\phi(s)ds, \quad t \geq 0$ for locally integrable functions ϕ. Then $u(t) \leq a(t) + Bu(t)$ implies that $u(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n u(t)$ and

$B^n u(t) = \int_0^t \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)ds \to 0$ as $n \to +\infty$ for each $t \in [0,T)$.

Then for all $t \in [0,T)$

$$u(t) \leq \lim_{n\to\infty}\left(\sum_{k=0}^{n-1} B^k a(t) + B^n u(t)\right) \leq \sum_{k=0}^{n-1} B^k a(t) \leq a(t) + \sum_{k=1}^{\infty} B^k a(t)$$

$$\leq a(t) + \int_0^t \sum_{j=0}^{\infty} \frac{(b\Gamma(\beta))^j}{\Gamma(j\beta)}(t-s)^{j\beta-1}a(s)ds.$$

If $a(t) \equiv a$, a constant, then

$$u(t) \leq a + a\int_0^t \sum_{j=0}^{\infty} \frac{(b\Gamma(\beta))^j}{\Gamma(j\beta)}(t-s)^{j\beta-1}ds$$

$$\leq a\left[1 + \int_0^t \sum_{j=0}^{\infty} \frac{(b\Gamma(\beta))^j}{\Gamma(j\beta)}(t-s)^{j\beta-1}\right]ds$$

$$\leq a\sum_{n=0}^{\infty} \frac{(b\Gamma(\beta)t^\beta)^n}{\Gamma(n\beta+1)} \leq aE_\beta(b\Gamma(\beta)t^\beta),$$

where E_β is the function of Mittag-Leffler. ■

Our main result reads as follows.

Theorem 3.2. *Let $f : J \times C([-r,0],E) \to E$ and $\sigma : J \times C([-r,0],\ E) \to L_2^0$ are continuous. Assume that:*

(1)$\|T(t)\| \le \widetilde{M}$, where $\widetilde{M} = \sup_{t\in J} \|T(t)\|$.

(2) The semigroup $\{T(t)\}_{t\in J}$ is compact for $t > 0$.

(3) There exist functions $p,q \in C(J,R_+)$ such that

$$\|f(t,u)\|_E \le p(t) + q(t)\|u\|_C, \quad \text{for each} \quad t \in J, \quad \text{and each} \quad u \in C([-r,b],E).$$

(4) There exist functions $\acute{p},\acute{q} \in C(J,R_+)$ such that

$$\|\sigma(t,u)\|_{L_2^0} \le \acute{p}(t) + \acute{q}(t)\|u\|_C, \quad \text{for each} \quad t \in J, \quad \text{and each} \quad u \in C([-r,b],E).$$

Then the problem (1)-(2) has at least one mild solution on $[-r,b]$.

Proof. Transform the problem (1)-(2) in to a fixed point problem. Consider the operator $\mathcal{F} : C([-r,b],\ E) \to C([-r,b],\ E)$, defined in the proof of theorem 3.1, to develop this proof in four steps.

Step 1: $\mathcal{F}$ is continuous.

Let $\{y_n\}$ be a sequence such that y in $C([-r,b],\ E)$.Then

$$\begin{aligned}
\|\mathcal{F}(y_n)(t) - \mathcal{F}(y)(t)\|_E &= \Bigg\| \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1} T(t-s)\, f(s,\ y_n(s-\rho(y_n(s)))ds\right.\\
&\quad \left. + \int_0^t (t-s)^{\alpha-1} T(t-s)\sigma(s,y_n(s-\rho(y_n(s)))dw(s)\right)\\
&\quad - \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1} T(t-s) f(s,y(s-\rho(y(s))))ds\right.\\
&\quad \left. + \int_0^t (t-s)^{\alpha-1} T(t-s)\sigma(s,y(s-\rho(y(s))))dw(s)\right)\Bigg\|_E
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1}\|T(t-s)\|\,\|f(s,y_n(s-\rho(y_n(s)))) - f(s,y(s-\rho(y(s))))\|_E\, ds\right.\\
&\left. + \int_0^t (t-s)^{\alpha-1}\|T(t-s)\|\,\|\sigma(s,y_n(s-\rho(y_n(s)))) - \sigma(s,y(s-\rho(y(s))))\|_{L_2^0}\right)ds\\
&\le \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y_n(s-\rho(y_n(s)))) - f(s,y(s-\rho(y(s))))\\
&+ f(s,y(s-\rho(y_n(s)))) - f(s,y(s-\rho(y_n(s))))\|_E\, ds\\
&+ \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\sigma(s,y_n(s-\rho(y_n(s)))) - \sigma(s,y(s-\rho(y(s))))\\
&+ \sigma(s,y(s-\rho(y_n(s)))) - \sigma(s,y(s-\rho(y_n(s))))\|_{L_2^0} ds
\end{aligned}$$

$$\leq \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y_n(s-\rho(y_n(s)))) - f(s,y(s-\rho(y_n(s))))\|ds$$
$$+ \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s-\rho(y_n(s)))) - f(s,y(s-\rho(y(s))))\|ds$$
$$+ \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\sigma(s,y_n(s-\rho(y_n(s)))) - \sigma(s,y(s-\rho(y_n(s))))\|_{L_2^0}ds$$
$$+ \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\sigma(s,y(s-\rho(y_n(s)))) - \sigma(s,y(s-\rho(y(s))))\|_{L_2^0}ds$$

Since f and σ are a continuous functions, the continuity of $\rho,$ and by the Lebesgue dominated convergence theorem the second hand side of the above inequality tends to zero as $n \longrightarrow \infty$ Thus
$\|\mathcal{F}(y_n) - \mathcal{F}(y)\|_{\mathfrak{D}} \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$

Step 2: $\mathcal{F}$ maps bounded sets into bounded sets.

It is enough to show that for any $\ell > 0$, there exists a positive constant δ such that for each $y \in B_\ell = \{y \in C([-r,b],E) : \|y\|_{\mathfrak{D}} \leq \ell\}$, we have $\mathcal{F}(y) \in B_\delta$. For each $t \in J$ we have

$$\begin{aligned}
\|\mathcal{F}(y)(t)\|_E &= \left\| \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y(s-\rho(y(s))))ds \right.\\
&\left. + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)\sigma(s,y(s-\rho(y(s))))dw(s)\right\|\\
&\leq \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s-\rho(y(s))))\|_E ds\\
&+ \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\sigma(s,y(s-\rho(y(s))))\|_{L_2^0}ds\\
&\leq \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(p(s)+q(s)\|y(s-\rho(y(s))))\|_C)ds\\
&+ \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\acute{p}(s)+\acute{q}(s)\|y(s-\rho(y(s))))\|_C)ds\\
&\leq \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)ds + \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ell q(s)ds\\
&+ \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\acute{p}(s)ds + \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ell\acute{q}(s)ds\\
&\leq \frac{\widetilde{M}}{\Gamma(\alpha)}\|p\|_\infty\int_0^t (t-s)^{\alpha-1}ds + \frac{\widetilde{M}}{\Gamma(\alpha)}\ell\|q\|_\infty\int_0^t (t-s)^{\alpha-1}ds\\
&+ \frac{\widetilde{M}}{\Gamma(\alpha)}\|\acute{p}\|_\infty\int_0^t (t-s)^{\alpha-1}ds + \frac{\widetilde{M}}{\Gamma(\alpha)}\ell\|\acute{q}\|_\infty\int_0^t (t-s)^{\alpha-1}ds
\end{aligned}$$

$$\leq \frac{\widetilde{M}}{\Gamma(\alpha)}(\|p\|_\infty + \ell\|q\|_\infty)\int_0^t (\text{t-s})^{\alpha-1}\text{ds} + \frac{\widetilde{M}}{\Gamma(\alpha)}(\|\acute{p}\|_\infty + \ell\|\acute{q}\|_\infty)\int_0^t (\text{t-s})^{\alpha-1}\text{ds}$$

$$\leq \frac{\widetilde{M}b^{\alpha}}{\Gamma(\alpha+1)}(\|p\|_{\infty}+\ell\|q\|_{\infty})+\frac{\widetilde{M}b^{\alpha}}{\Gamma(\alpha+1)}(\|\acute{p}\|_{\infty}+\ell\|\acute{q}\|_{\infty})$$
$$\leq \frac{\widetilde{M}b^{\alpha}}{\Gamma(\alpha+1)}((\|p\|_{\infty}+\|\acute{p}\|_{\infty})+\ell(\|q\|_{\infty}+\|\acute{q}\|_{\infty})) =:\delta<\infty.$$

Step 3: $\mathcal{F}$maps bounded sets into equicontinuous sets in $C([-r,b],E)$.

We consider $B_\rho$as in Step 2. Let $\tau_1,\tau_2 \in J$and $\tau_1 < \tau_2$. Thus if $\epsilon > 0$and $\epsilon < \tau_1 < \tau_2$we have

$$\begin{aligned}\|\mathcal{F}(y)(\tau_2)-\mathcal{F}(y)(\tau_1)\| = {} & \|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}[(\tau_2-s)^{\alpha-1}T(\tau_2-s)f(s,y(s-\rho(y(s))))ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}(\tau_2-s)^{\alpha-1}T(\tau_2-s)\sigma(s,y(s-\rho(y(s))))dw(s) \\ & - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}T(\tau_1-s)f(s,y(s-\rho(y(s))))ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}T(\tau_1-s)\sigma(s,y(s-\rho(y(s))))dw(s)\| \\ \leq {} & \frac{1}{\Gamma(\alpha)}(\int_0^{\tau_1-\epsilon}\|[(\tau_2-s)^{\alpha-1}T(\tau_2-s)-(\tau_1-s)^{\alpha-1}T(\tau_1-s)]\|\end{aligned}$$

$$\begin{aligned}&\|f(s,y(s-\rho(y(s))))\|_E ds+\int_{\tau_1-\epsilon}^{\tau_1}\|[(\tau_2-s)^{\alpha-1}T(\tau_2-s)-(\tau_1-s)^{\alpha-1}T(\tau_1-s)]\| \\ &\|f(s,y(s-\rho(y(s))))\|_E ds+\int_{\tau_1}^{\tau_2}\|(\tau_2-s)^{\alpha-1}T(\tau_2-s)\|\|f(s,y(s-\rho(y(s))))\|_E ds) \\ &+\frac{1}{\Gamma(\alpha)}(\int_0^{\tau_1-\epsilon}\|[(\tau_2-s)^{\alpha-1}T(\tau_2-s)-(\tau_1-s)^{\alpha-1}T(\tau_1-s)]\| \\ &\|\sigma(s,y(s-\rho(y(s))))\|_{L_2^0} ds+\int_{\tau_1-\epsilon}^{\tau_1}\|[(\tau_2-s)^{\alpha-1}T(\tau_2-s)-(\tau_1-s)^{\alpha-1}T(\tau_1-s)]\| \\ &\|\sigma(s,y(s-\rho(y(s))))\|_{L_2^0} ds+\int_{\tau_1}^{\tau_2}\|(\tau_2-s)^{\alpha-1}T(\tau_2-s)\|\|\sigma(s,y(s-\rho(y(s))))\|_{L_2^0}ds)\end{aligned}$$

Using the following semigroup identities

$$T(\tau_2-s) = T(\tau_2-\tau_1+\epsilon)T(\tau_1-\epsilon-s),$$
$$T(\tau_1-s) = T(\tau_1-\epsilon-s)T(\epsilon),$$

we get

$$\|\mathcal{F}(y)(\tau_2)-\mathcal{F}(y)(\tau_1)\|_E \leq \frac{((\|p\|_{\infty}+\|\acute{p}\|_{\infty})+\rho(\|q\|_{\infty}+\|\acute{q}\|_{\infty}))}{\Gamma(\alpha)}$$

$$(\int_0^{\tau_1-\epsilon}\|[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]T(\tau_1-s)\|ds$$
$$+\int_0^{\tau_1-\epsilon}\|(\tau_2-s)^{\alpha-1}T(\tau_1-s-\epsilon)[T(\tau_2-\tau_1+\epsilon)-T(\epsilon)]\|ds$$
$$+\int_{\tau_1-\epsilon}^{\tau_1}\|[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]T(\tau_1-s)\|ds$$
$$+\int_{\tau_1-\epsilon}^{\tau_1}\|(\tau_2-s)^{\alpha-1}T(\tau_1-\epsilon-s)[T(\tau_2-\tau_1+\epsilon)-T(\epsilon)]\|ds$$
$$+\int_{\tau_1}^{\tau_2}\|(\tau_2-s)^{\alpha-1}T(\tau_2-s)\|ds)$$
$$\leq \frac{((\|p\|_\infty+\|\acute{p}\|_\infty)+\rho(\|q\|_\infty+\|\acute{q}\|_\infty))\widetilde{M}}{\Gamma(\alpha)}\ (\int_0^{\tau_1-\epsilon}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]ds$$
$$+\|T(\tau_2-\tau_1+\epsilon)-T(\epsilon)\|_{\mathfrak{L}(E)}\int_0^{\tau_1-\epsilon}(\tau_2-s)^{\alpha-1}ds$$
$$+\int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]ds$$
$$+\|T(\tau_2-\tau_1+\epsilon)-T(\epsilon)\|_{\mathfrak{L}(E)}\int_{\tau_1-\epsilon}^{\tau_1}(\tau_2-s)^{\alpha-1}ds\ +\ \int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\ )$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. Since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology (see [21]). By the Arzela-Ascoli theorem it suffices to show that $\mathcal{F}$ maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$, we define

$$\mathcal{F}_\epsilon(y)t = \frac{T(\epsilon)}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)f(s,y(s-\rho(y(s))))ds$$
$$+\frac{T(\epsilon)}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)\sigma(s,y(s-\rho(y(s))))dw(s) \tag{7}$$

Since $T(t)$ is a compact operator for $t > 0$ the set

$$Y_\epsilon(t) = \{\mathcal{F}_\epsilon(y)(t) : y \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\|\mathcal{F}(y)(t)-\mathcal{F}_\epsilon(y)(t)\|_E = \|\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T(t-s)f(s,y(s-\rho(y(s))))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T(t-s)\sigma(s,y(s-\rho(y(s))))dw(s)$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)f(s,y(s-\rho(y(s))))ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)\sigma(s,y(s-\rho(y(s))))dw(s)\|$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\|\int_0^{t-\epsilon}(t-s)^{\alpha-1}T(t-s)f(s,y(s-\rho(y(s))))ds\\
&+\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}T(t-s)f(s,y(s-\rho(y(s))))ds\\
&+\int_0^{t-\epsilon}(t-s)^{\alpha-1}T(t-s)\sigma(s,y(s-\rho(y(s))))dw(s)\\
&+\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}T(t-s)\sigma(s,y(s-\rho(y(s))))dw(s)\\
&-\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)f(s,y(s-\rho(y(s))))ds\\
&-\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)\sigma(s,y(s-\rho(y(s))))dw(s)\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}(\int_0^{t-\epsilon}\|[(t-s)^{\alpha-1}T(t-s)-(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)]\|\\
&\quad \|f(s,y(s-\rho(y(s))))\|ds+\int_{t-\epsilon}^{t}\|(t-s)^{\alpha-1}T(t-s)\|\,\|f(s,y(s-\rho(y(s))))\|ds)\\
&+\frac{1}{\Gamma(\alpha)}(\int_0^{t-\epsilon}\|[(t-s)^{\alpha-1}T(t-s)-(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)]\|\\
&\times\|\sigma(s,y(s-\rho(y(s))))\|_{L_2^0}\,ds+\int_{t-\epsilon}^{t}\|(t-s)^{\alpha-1}T(t-s)\|\,\|\sigma(s,y(s-\rho(y(s))))\|_{L_2^0}ds)\\
&\leq \frac{\widetilde{M}\,((\|p\|_\infty+\|\acute{p}\|_\infty)+\rho(\|q\|_\infty+\|\acute{q}\|_\infty))}{\Gamma(\alpha)}\ (\ \int_0^{t-\epsilon}[(t-s)^{\alpha-1}-(t-s-\epsilon)^{\alpha-1}]ds\\
&+\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}ds\ )\\
&\leq \frac{\widetilde{M}\,((\|p\|_\infty+\|\acute{p}\|_\infty)+\rho(\|q\|_\infty+\|\acute{q}\|_\infty))}{\Gamma(\alpha+1)}(t^\alpha-(t-\epsilon)^\alpha).
\end{aligned}
$$

Therefore, the set $Y(t)=\{\mathcal{F}(y)(t):y\in B_\rho\}$ is precompact in E. Hence the operator $\mathcal{F}$ is completely continuous.

Step 4: Now, it remains to show that the set

$$
\mathcal{E}=\{y\in C([-r,b],E):y=\lambda\mathcal{F}(y)\quad \textit{for a certain}\quad 0<\lambda<1\} \tag{8}
$$

is bounded.

Let $y\in\mathcal{E}$ be any element. Then, for each $t\in J$

$$
\begin{aligned}
y(t)&=\lambda\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T(t-s)f(s,y(s-\rho(y(s))))ds\\
&\quad+\lambda\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T(t-s)\sigma(s,y(s-\rho(y(s))))dw(s).
\end{aligned} \tag{9}
$$

Moreover

$$\|y(t)\|_E = \|\lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y(s-\rho(y(s))))ds$$
$$+ \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) \sigma(s, y(s-\rho(y(s))))dw(s)\|$$

$$\leq \frac{\widetilde{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, y(s-\rho(y(s))))\|ds$$
$$+ \frac{\widetilde{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\sigma(s, y(s-\rho(y(s))))\|_{L_2^0} ds$$
$$\leq \frac{\widetilde{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ((p(s) + \acute{p}(s)) + (q(s) + \acute{q}(s)) \|y(s-\rho(y(s))))\|)ds$$
$$\leq \frac{\widetilde{M} b^\alpha (\|P\|_\infty + \|\acute{P}\|_\infty)}{\Gamma(\alpha+1)} + \frac{\widetilde{M}(\|q\|_\infty + \|\acute{q}\|_\infty)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s-\rho(y(s)))\|ds$$

Note that $-r \leq s - \rho(y(s)) \leq s$ for each $s \in J$ and consider the function μ defined by

$$\mu(t) = \max\{\|y(s)\| : -r \leq s \leq t\}, \quad t \in J. \tag{10}$$

For $t \in [0, b]$

$$\mu(t) \leq \frac{\widetilde{M} b^\alpha (\|P\|_\infty + \|\acute{P}\|_\infty)}{\Gamma(\alpha+1)} + \frac{\widetilde{M}(\|q\|_\infty + \|\acute{q}\|_\infty)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mu(s) ds. \tag{11}$$

An application of Lemma 3.1 in (11) yields

$$\mu(t) \leq \frac{\widetilde{M} b^\alpha (\|P\|_\infty + \|\acute{P}\|_\infty)}{\Gamma(\alpha+1)} E_\alpha\left(\frac{\widetilde{M}(\|q\|_\infty + \|\acute{q}\|_\infty)}{\Gamma(\alpha)} \Gamma(\alpha) t^\alpha\right) \leq \Lambda$$

Hence

$$\|y\|_\infty \leq \max\{\|\phi\|_C, \Lambda\}, \quad \forall y \in \mathcal{E},$$

and the set $\mathcal{E}$ is bounded. As a consequence of theorem 2.5, we deduce that the operator $\mathcal{F}$ has a fixed point which is a mild solution of the problem (1)-(2). ■

Next we give a uniqueness result for solutions of problem (1)-(2).

Theorem 3.3. *Assume that the hypotheses of theorem 3.2 hold. Suppose moreover that there exists a nonnegative constant K and M_σ such that*

- $\|f(t,u) - f(t,v)\|_E \leq K\|u - v\|_C$, *for* $t \in J$, *and every* $u, v \in C([-r, 0], E)$,
- $\|\sigma(t,u) - \sigma(t,v)\|_{L_2^0} \leq M_\sigma \|u - v\|_C$, *for* $t \in J$, *and every* $u, v \in C([-r, 0], E)$.

Then the problem (1)-(2) is uniquely solvable on $[-r, b]$.

Proof. The existence of at least one integral solution $y(t)$ of problem (1)-(2) is insured by theorem 3.2. To prove the uniqueness of $y(t)$, let $z(t)$ be another solution of problem (1)-(2). Then $y(t) = z(t) = \phi(t), t \in [-r, 0]$, and, for each $t \in J$ we have

$$
\begin{aligned}
\|y(t)-z(t)\|_E = {} & \Big\| \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y(s-\rho(y(s))))dt \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)\sigma(s,y(s-\rho(y(s))))dw(t) \\
& - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,z(s-\rho(z(s))))dt \\
& - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)\sigma(s,z(s-\rho(z(s))))dw(t)\Big\| \\
& \leq \frac{1}{\Gamma(\alpha)}\Big\|\int_0^t (t-s)^{\alpha-1}T(t-s)[f(s,y(s-\rho(y(s)))) - f(s,z(s-\rho(z(s))))]ds \\
& + \int_0^t (t-s)^{\alpha-1}T(t-s)[\sigma(s,y(s-\rho(y(s)))) - \sigma(s,z(s-\rho(z(s))))]dw(s)\Big\| \\
& \leq \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s-\rho(y(s)))) - f(s,z(s-\rho(z(s))))\|ds \\
& + \frac{\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\sigma(s,y(s-\rho(y(s)))) - \sigma(s,z(s-\rho(z(s))))\|_{L_2^0}ds \\
& \leq \frac{(K+M_\sigma)\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s-\rho(y(s))) - z(s-\rho(z(s)))\|_C ds
\end{aligned}
$$

Now, using lemma 3.1 with an $a(t) \equiv 0$ yields the uniqueness of $y(t)$. ■

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