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An Optimal Series Expansion of Sub-Mixed Fractional Brownian Motion

M. ZILI

Department of Mathematics, Faculty of Sciences of Monastir, Avenue de l'environnement, 5000 Monastir, Tunisia,

E-mail: Mounir.Zili@fsm.rnu.tn

Abstract. *We present an explicit series expansion of the sub-mixed fractional Brownian motion and study its rate of convergence. We show that the obtained expansion is rate-optimal in the sense that the expected uniform norm of the truncated series vanishes at optimal rate as the truncation point tends to infinity. As an application of this result, we present a computer generation of sample paths for sub-mixed fractional Brownian motion.*

Key words : Explicit Series Expansion, Sub-Mixed Fractional Brownian motion, Rate of Convergence, Optimality, Computer Generation of Sample Paths.

AMS Subject Classifications : 60G15, 60G17, 65C20, 33C10

1. Introduction

Let $\{B_t^H, t \in \mathbb{R}\}$ be a fractional Brownian motion (fBm) with Hurst index $0 < H < 1$, defined on a probability space $(\Omega, F, \mathbb{P})$; i.e. a centered Gaussian process with stationary increments satisfying $B_0^H = 0$, with probability 1, and $\mathbb{E}(B_t^H)^2 = |t|^{2H}$, $t \in \mathbb{R}$. The covariance function of this process is given by

$$Cov\left(B_t^H, B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad (s,t) \in \mathbb{R}^2. \tag{1}$$

Note that if the Hurst index equals 1/2, the fBm is simply the ordinary standard Brownian motion (Bm).

A sub-fractional Brownian motion(sfBm) is another extension of a Bm, that preserves most of the properties of a fBm, but not the stationarity of the increments. It is the stochastic process $\xi^H = \{\xi_t^H; t \geq 0\}$, defined on $(\Omega, F, \mathbb{P})$ by:

$$\xi_t^H = \frac{B_t^H + B_{-t}^H}{\sqrt{2}}, \quad t \in \mathbb{R}_+, \tag{2}$$

It is obvious that, when $H = 1/2$, $\xi^{1/2}$ is a Bm. We refer to [2-4, 11] for further information on this process.

In [7] the authors introduced a new Gaussian process that they called the sub-mixed fractional Brownian motion (smfBm). A smfBm of parameters a, b and H, is a process

$X^H = \{X_t^H(a,b);\ t \geq 0\} = \{X_t^H;\ t \geq 0\}$,

defined on $(\Omega, F, \mathbb{P})$ by:

$$X_t^H = X_t^H(a,b) = a\ \xi_t + b\ \xi_t^H, \quad t \in \mathbb{R}_+, \tag{3}$$

where ξ^H is a fBm, with Hurst index H, and ξ is an independent Bm (both defined on the same probability space $(\Omega, F, \mathbb{P})$), a and b are two real constants such that $(a, b) \neq (0, 0)$.

So a smfBm is clearly an other extension, not only of a Bm, but also of a sfBm. This is a first interest in this process. In [7], it was proved that the parameters H, a and b can be chosen so that $\{X_t^H(a,b); t \geq 0\}$ yields a good model, taking not only the sign (like in the case of a fBm or a sfBm), but also the level of the increments correlation of the phenomenon of interest into account. This is another main interest of this process. The covariance function of a smfBm, of parameters a, b and H, is given by:

$$Cov\left(X_t^H(a,b),\ X_s^H(a,b)\right) = a^2\ (s \wedge t) + b^2\ \left(t^{2H} + s^{2H} - \tfrac{1}{2}\left((s+t)^{2H} + |\ t-s\ |^{2H}\ \right)\right), \tag{4}$$

for every $s \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$.

A smfBm does not have stationary increments but has

$$E(X_t^H(a,b) - X_s^H(a,b))^2 \leq a^2\ |\ t-s\ | + b^2 v\ |\ t-s\ |^{2H} \quad (s,t) \in \mathbb{R}_+^2\ , \tag{5}$$

where

$$v = \begin{cases} 1 & if \quad H \geq \frac{1}{2} \\ 2 - 2^{2H-1} & if \quad H < \frac{1}{2} \end{cases}\ . \tag{6}$$

Moreover, from [7], we know that smfBm satisfies the *mixed-self-similarity**; i.e. for any $h > 0$, the processes $\{X_{ht}^H(a,b)\}_{t \in \mathbb{R}_+}$ and $\left\{X_t^H\left(ah^{1/2},\ bh^H\right)\right\}_{t \in \mathbb{R}_+}$ have the same law. Then, without loss of generality, we can restrict the time parameter t to the interval $[0, 1]$.

In this article, we present an explicit series expansion of the smfBm and prove its rate-optimality in sense that the expected uniform norm of the truncated series vanishes at the optimal rate as the truncation point tends to infinity. Optimality is not only desirable feature if the expansion is to be used for the simulation of sample paths, but is also important in connection with estimates of small ball probabilities (see[9], [10], and [8]).

The technique used to obtain our series expansion is the same as the one applied in the papers [5] and [6], where the authors obtained a rate optimal explicit series expansion of the fBm. Our proof is based on the expansion of the fBm even part covariance gotten in [5]. Let us

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*The notion of mixed-self-similarity was introduced in [14].

specify that in [1], the authors showed the optimality of a wavelet-type expansion of the fBm. This expansion is of a different type than ours.

As an application of our result, we generate some computer sample paths of $S^H(a,b)$ for different values of H, a and b. In particular we present some simulation results of some trajectories of the sfBm.

The remainder of the paper is organized as follows. In the next section we obtain an explicit series expansion of the smfBm. In section 3 we present some results on the rates of convergence of the obtained series expansion. Then, in section 4, we show that our expansion achieves an optimal rate in the sense of Kühn and Linde [8]. The final section reports on some computer generated smfBm sample paths.

2. Explicit Series Expansion of the smfBm

Recall that for $\nu \neq -1,-2,\dots$ the Bessel function J_ν of the first kind of order ν can be defined on the region $\{z \in \mathbb{C};\ |\ arg(z)\ \ |< \pi\}$ as the absolutely convergent sum

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu+2n}}{\Gamma(n+1)\Gamma(\nu+n+1)}.$$

It is well known that for $\nu > -1$, the function J_ν has a countable number of real, positive, simple zeros (see [13], chapter 15). These zeros $x_1 < x_2 < \dots$ can be arranged in ascending order of magnitude and they become arbitrarily large. The following lemma follows from the asymptotic properties of the Bessel function and its positive zeros (see [13]) and will be useful throughout this paper.

Lemma 2.1. *For $\nu > -1$, let J_ν be the Bessel function of the first kind of order ν and let $z_1 < z_2 < \dots$ be its positive zeros. Then,*

1. $z_n \sim n\pi$, *and* $J^2_{1+\nu}(z_n) \sim 2/n\pi^2$ *as* $n \to \infty$,
2. $J_{\nu-1}(z_n) + J_{\nu+1}(z_n) = 0$, *for every* $n = 1,2,\dots$

Let $H \in (0,1)$ and $x_{H,1} < x_{H,2} < \dots$ be the positive real zeros of the Bessel function J_{1-H}, and for $k \in N$, define

$$(\tau_{H,k})^2 = \frac{2c_H^2}{x_{H,k}^{2H} J^2_{-H}(x_{H,k})}, \tag{7}$$

where

$$c_H^2 = \frac{\Gamma(1+2H)\sin \pi H}{\pi}, \tag{8}$$

and Γ is Euler's gamma function.

The following lemma will be useful to prove the main theorem of this section.

Lemma 2.2. *For the partial sum process X^N, defined by*

$$X_t^N = a\sum_{n=1}^{N} \frac{1-\cos(x_{1/2,n}t)}{x_{1/2,n}} Y_n + b\sum_{n=1}^{N} \frac{1-\cos(x_{H,n}t)}{x_{H,n}} Y_{H,n}\ , \tag{9}$$

the finite dimensional distributions (fdd's) converge weakly to the fdd's of the smfBm.

Proof. In [5], the authors showed that if we denote by $B^{H,e}$ the even part of the fBm B^H, then for all $s,t \in [0,1]$ we have

$$\mathbb{E}(B_s^{H,e}B_t^{H,e}) = \sum_{n=1}^{\infty} \frac{\big(1-\cos(x_{H,n}s)\big)\big(1-\cos(x_{H,n}t)\big)}{x_{H,n}^2}\tau_{H,n}^2, \tag{10}$$

where the series converges absolutely and uniformly in $(s,t) \in [0,1]\times[0,1]$.

By the independence of the Gaussian processes ξ and ξ^H, and since $\xi^H = B^{H,e}\times\sqrt{2}$, we get

$$\begin{aligned}\mathbb{E}(X_s^H X_t^H) &= 2a^2 \sum_{n=1}^{\infty} \frac{\big(1-\cos(x_{1/2,n}s)\big)\big(1-\cos(x_{1/2,n}t)\big)}{x_{1/2,n}^2}\tau_{1/2,n}^2 \\ &+ 2b^2 \sum_{n=1}^{\infty} \frac{\big(1-\cos(x_{H,n}s)\big)\big(1-\cos(x_{H,n}t)\big)}{x_{H,n}^2}\tau_{H,n}^2,\end{aligned}$$

for every $s,t \in [0,1]$, where the series converges absolutely and uniformly in $(s,t) \in [0,1]\times[0,1]$. So the lemma follows immediately. ■

The main result in this section is the following theorem.

Theorem 2.1. *Let $Y_1, Y_2, \ldots$ and $Y_{H,1}, Y_{H,2}, \ldots$ be independent sequences of independent, centered Gaussian random variables on a common probability space, with $Var(Y_n) = \tau_{1/2,n}^2$ and $Var(Y_{H,n}) = \tau_{H,n}^2$. Then, for $(a,b) \in R^2 \setminus \{(0,0)\}$, the stochastic process $X^H(a,b) = (X_t^H(a,b))_{t\in[0,1]}$ given by*

$$X_t^H(a,b) = a\sum_{n=1}^{\infty} \frac{1-\cos(x_{1/2,n}t)}{x_{1/2,n}}Y_n + b\sum_{n=1}^{\infty} \frac{1-\cos(x_{H,n}t)}{x_{H,n}}Y_{H,n}, \tag{11}$$

is well defined and with probability 1, and both series converge absolutely and uniformly in $t \in [0,1]$. The process $X^H(a,b)$ is a smfBm.

Proof. Let $C[0,1]$ be the space of continuous functions on $[0,1]$, endowed with the supremum metric. By the Lévy-Ito-Nisio Theorem (see [12], p. 431), the processes X^N converge in $C[0,1]$ with probability 1 if and only if they have a weak limit in $C[0,1]$. By lemma 2.2, it remains to show that the sequence X^N is asymptotically tight in $C([0,1])$. To prove this, we first consider the terms

$$X_{1,t}^N = a\sum_{n=1}^{N} \frac{1-\cos(x_{1/2,n}t)}{x_{1/2,n}}Y_n, \text{ and } X_{2,t}^N = b\sum_{n=1}^{N} \frac{1-\cos(x_{H,n}t)}{x_{H,n}}Y_{H,n},$$

and we will just treat the second one. The tightness of the first term can be shown exactly in the same manner.

Since $X^N_{2,\cdot}$ is Gaussian, it is a sub-Gaussian process for the standard deviation semimetric d_N defined by:

$$d_N^2(s,t) = \mathbb{V}(X^N_{2,t} - X^N_{2,s}) = b^2 \sum_{n=1}^{N} \frac{\left(\cos(x_{H,n}t) - \cos(x_{H,n}s)\right)^2}{x_{H,n}^2} \tau_{H,n}^2.$$

According to the maximal inequality for sub-Gaussian processes (see [12], p. 101), we have

$$\mathbb{E} \sup_{d_N(s,t)\le\delta} | X^N_{2,t} - X^N_{2,s} | \le Cte \int_0^\delta \sqrt{LogN(\epsilon,[0,1],d_N)}\, d\epsilon \,, \tag{12}$$

for every $\delta > 0$, where Cte is a universal constant and $N(\epsilon,[0,1],d_N)$ is the $\epsilon-$ covering number of the semimetric space $([0,1],d_N)$ (i.e. the minimal number of balls of radius ϵ needed to cover $[0,1]$). By equations (5) and (6), we get

$$d_N(s,t) \le M \times | \, t-s \, |^{1/2\wedge H} \,,$$

where $M = \sqrt{a^2 + b^2 v}$, and consequently,

$$N(\epsilon,[0,1],d_N) \le N(\epsilon,[0,1],M \, |.|^{1/2\wedge H}).$$

Together with (12), this implies that

$$\begin{aligned} \mathbb{E} \sup_{d_N(s,t)\le\delta} | X^N_{2,t} - X^N_{2,s} | &\le Cte \int_0^\delta \sqrt{LogN(\epsilon,[0,1],M \, |.|^{1/2\wedge H})}\, d\epsilon \\ &\le Cte \, (2 \vee H^{-1}) \int_0^\delta \sqrt{Log\left(\tfrac{M}{\epsilon}\right)}\, d\epsilon. \end{aligned}$$

Here it is clear that the number of balls of $M \times |.|^{1/2\wedge H}$ - radius ϵ that are needed to cover $[0,1]$ is bounded by $(\frac{\epsilon}{M})^{-(2\vee H^{-1})}$.

The integral on the right hand side converges to 0 as $\delta \searrow 0$, so it follows that the processes $X^N_{2,\cdot}$ are uniformly equicontinuous in probability, hence $X^N_{2,\cdot}$ is tight in $C([0,1])$ (see [12], p. 37). Thus, we have that partial sums of X^N converge weakly in $C[0,1]$. ■

3. Rates of Convergence

With the notations already introduced in theorem 2.1 and in lemma 2.2, in this section we investigate the rate at which the partial sum process X^N approaches the process X^H. First we consider the covariance functions, for which we have the following result.

Proposition 3.1. *For all $H \in (0,1)$ we have*

$$\limsup_{N\to\infty} N^{2H\wedge 1} \sup_{s,t\in[0,1]} |\mathbb{E}(X^H_s X^H_t) - \mathbb{E}(X^N_s X^N_t)| \le 8a^2 \frac{c_{1/2}^2}{\pi} + 4b^2 \frac{c_H^2}{H\pi^{2H}},$$

where c_H^2 is defined by (8).

Proof. By theorem 2.1 and (9) we have

$$\sup_{s,t\in[0,1]} |\mathbb{E}(X_s^H X_t^H) - \mathbb{E}(X_s^N X_t^N)| \leq 8\left(a^2 \sum_{n>N} \frac{\tau_{1/2,n}^2}{x_{1/2,n}^2} + b^2 \sum_{n>N} \frac{\tau_{H,n}^2}{x_{H,n}^2} \right). \tag{13}$$

On the other hand, by lemma 2.1 we have

$$\sum_{n>N} \frac{\tau_{H,n}^2}{x_{H,n}^2} \sim \frac{c_H^2}{\pi^{2H}} \sum_{n>N} \frac{1}{n^{1+2H}} \quad \textit{and} \quad \sum_{n>N} \frac{\tau_{1/2,n}^2}{x_{1/2,n}^2} \sim \frac{c_{1/2}^2}{\pi} \sum_{n>N} \frac{1}{n^2} \ \textit{as}\ N \to \infty. \tag{14}$$

Furthermore, as for every $H \in (0,1)$,

$$\sum_{n>N} \frac{1}{n^{1+2H}} \sim \int_N^\infty \frac{1}{n^{1+2H}} dx = \frac{1}{2H\, N^{2H}}, \tag{15}$$

we get

$$a^2 \sum_{n>N} \frac{\tau_{1/2,n}^2}{x_{1/2,n}^2} + b^2 \sum_{n>N} \frac{\tau_{H,n}^2}{x_{H,n}^2} \sim a^2 \frac{c_{1/2}^2}{\pi} \frac{1}{N} + b^2 \frac{c_H^2}{2H\,\pi^{2H}} \frac{1}{N^{2H}}, \tag{16}$$

as $N \to \infty$. Here the proof completes. ■

In the following theorem we specify the rate of convergence of the sample paths of the partial sum process X^N to the sample paths of the smfBm X^H.

Theorem 3.1. *For all* $H \in (0,1)$ *we have*

(i) $\limsup_{N\to\infty} N^{2H\wedge 1} \sup_{t\in[0,1]} \mathbb{E}(X_t^N - X_t^H)^2 \leq 4a^2 \frac{c_{1/2}^2}{\pi} + 2b^2 \frac{c_H^2}{H\,\pi^{2H}}$,

(ii) $\limsup_{N\to\infty} N^{2H\wedge 1}\, \mathbb{E} \parallel X^N - X^H \parallel^2_{L^2[0,1]} \leq 4a^2 \frac{c_{1/2}^2}{\pi} + 2b^2 \frac{c_H^2}{H\,\pi^{2H}}$,

(iii) $\limsup_{N\to\infty} N^{(1-\epsilon)(H\wedge 1/2)}\, \mathbb{E} \sup_{t\in[0,1]} |\ X_t^N - X_t^H\ | < \infty,\ \forall \epsilon > 0$,

where c_H^2 *is defined by (8).*

Proof. Lets us prove (i) and (ii) in the same time. Observe that we have

$$X_t^H - X_t^N = a \sum_{n>N} \frac{1-\cos(x_{1/2,n}t)}{x_{1/2,n}} Y_n + b \sum_{n>N} \frac{1-\cos(x_{H,n}t)}{x_{H,n}} Y_{H,n}\ .$$

Hence

$$\mathbb{E}(X_t^H - X_t^N)^2 \leq 4a^2 \sum_{n>N} \frac{\tau_{1/2,n}^2}{x_{1/2,n}^2} + 4b^2 \sum_{n>N} \frac{\tau_{H,n}^2}{x_{H,n}^2}\ .$$

So by equation (16), we obtain the statements (i) and (ii).

For the proof of (iii), let us denote by

$$R_t^{N,H} = \sum_{n>N} \frac{1-\cos(x_{H,n}t)}{x_{H,n}} Y_{H,n} \quad and \quad R_t^N = \sum_{n>N} \frac{1-\cos(x_{1/2,n}t)}{x_{1/2,n}} Y_n \,,$$

to be able to write

$$X_t^H - X_t^N = aR_t^N + bR_t^{N,H}.$$

Since $R^{N,H}$ is a Gaussian process, then according to the maximal inequality for sub-Gaussian processes (see [12], p. 101), we have for every $\delta > 0$ the inequality

$$\mathbb{E} \sup_{d_N(s,t) \le \delta} | \; R_t^{N,H} - R_s^{N,H} \; | \le Cte \int_0^\delta \sqrt{LogN(\epsilon,[0,1],d_N)}\, d\epsilon, \tag{17}$$

where $N(\epsilon,[0,1],d_N)$ is the $\epsilon-$ covering number of the semimetric space $([0,1],d_N)$ and d_N is the standard deviation semimetric of $R^{N,H}$. Now observe that for all $p,q > 0$ we have

$$\begin{aligned} d_N(s,t)^{2(p+q)} &= \left| \sum_{n>N} \frac{\left(\cos(x_{H,n}t) - \cos(x_{H,n}s)\right)^2}{x_{H,n}^2} \tau_{H,n}^2 \right|^{p+q} \\ &\le \left| \sum_{n>N} \frac{\tau_{H,n}^2}{x_{H,n}} \right|^p |\mathbb{E}(X_t^H - X_s^H)^2|^q. \end{aligned}$$

Let us denote further $a(N) = \sum_{n>N} \frac{\tau_{H,n}^2}{x_{H,n}}$, and by equations (5) and (6) we get

$$\mathbb{E}(X_t^H - X_s^H)^2 \le (a^2 + b^2 v) \times | \; t-s \; |^{1 \wedge 2H} \,.$$

Then

$$d_N(s,t) \le \alpha \times | \; t-s \; |^{\beta} \;\;, \tag{18}$$

where

$$\alpha = (a^2+b^2 v)^{\frac{q}{2(p+q)}} \times | \; a(N) \; |^{\frac{p}{2(p+q)}} \quad and \quad \beta = (1/2 \wedge H)\frac{q}{p+q}.$$

Therefore,

$$N(\epsilon,[0,1],d_N) \le N\Big((\tfrac{\epsilon}{\alpha})^{1/\beta}, [0,1], |.| \Big) \le (\tfrac{\alpha}{\epsilon})^{1/\beta}. \tag{19}$$

It follows from (17) and (19) that

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,\,1]} | \; R_t^{N,H} \; | &\le \mathbb{E} \sup_{|t-s|\le 1} | \; R_t^{N,H} - R_s^{N,H} \; | \\ &\le \mathbb{E} \sup_{d_N(s,t) \le \alpha} | \; R_t^{N,H} - R_s^{N,H} \; | \\ &\le Cte \int_0^\alpha \sqrt{LogN(\epsilon,[0,1],d_N)}\, d\epsilon \\ &\le Cte \int_0^\alpha \sqrt{\tfrac{1}{\beta} Log \tfrac{\alpha}{\epsilon}}\, d\epsilon. \end{aligned}$$

Hence, a change of variables shows that

$$\mathbb{E} \sup_{t\in[0,\,1]} | \; R_t^{N,H} \; | \le Cte \frac{\alpha}{\sqrt{\beta}} \int_0^1 Log \frac{1}{x} dx.$$

By equations (14) and (15) we have $a(N) \sim \frac{c_H}{2} 2H\pi^{2H} N^{-2H}$ as $N \to \infty$, and consequently,

$$\limsup_{N\to\infty} N^{\frac{p}{p+q}H}\, \mathbb{E} \sup_{t\in[0,1]} \mid R_t^{N,H} \mid < \infty . \tag{20}$$

By the same technique we get

$$\limsup_{N\to\infty} N^{\frac{p}{2(p+q)}}\, \mathbb{E} \sup_{t\in[0,1]} \mid R_t^{N} \mid < \infty . \tag{21}$$

And equations (20) and (21) yield

$$\limsup_{N\to\infty} N^{\frac{p}{p+q}(1/2\wedge H)}\, \mathbb{E} \sup_{t\in[0,t]} \mid X_t^N - X_t^H \mid < \infty,$$

for every arbitrary positive numbers p and q, which completes the proof. ■

Remark 3.1. In theorem 3.1, we have proved that the sample paths of the partial sum process X^N tend to the sample paths of the smfBm X^H at the rate $N^{H\wedge 1/2}$. The dependence of this rate on $H \wedge 1/2$ can be explained by the nature of the sample paths of the smfBm. When $H \wedge 1/2$ gets smaller, the sample paths of the smfBm fluctuate more widely. Hence, we should expect that for smaller $H \wedge 1/2$ we need more terms in the series (9) to achieve a given level of accuracy of the approximation.

4. Optimality of the smfBm Explicit Series Expansion

In this section, we will prove that the optimal rate of uniform convergence of our series expansion of the smfBm obtained in theorem2.1 is $N^{-(H\wedge 1/2)}\sqrt{LogN}$. Note that assertion (iii) of theorem 3.1 states that for any $\epsilon > 0$, the rate of uniform convergence is faster than $N^{-(1-\epsilon)(H\wedge 1/2)}$. By considering the partial sum $S_N^H(t)$ defined by

$$S_N^H(t) = \sum_{2^{N-1} < k \leq 2^N} \frac{1-\cos(x_{H,k}t)}{x_{H,k}} Y_{H,k}\ ,\ t \in [0,1], \tag{22}$$

let us first prove the following lemma.

Lemma 4.1. *The inequality* $\mathbb{E} \sup_{t\in[0,1]} \mid S_N^H(t) \mid \leq Cte \times \sqrt{N}\, 2^{-NH}$ *is true.*

Proof. For a given $\epsilon > 0$, cover the interval $[0,1]$ with $n \leq Cte\ \epsilon^{-1}$ sub-intervals of length 2ϵ. Call the sub-intervals I_i and their centers t_i, for $i = 1,\dots,n$. Then we have

$$\mathbb{E} \sup_{t\in[0,1]} \mid S_N^H(t) \mid \leq \mathbb{E} \sup_{1\leq i\leq n} \mid S_N^H(t_i) \mid + \mathbb{E} \sup_{1\leq i\leq n} \sup_{t,u\in I_i} \mid S_N^H(t) - S_N^H(u) \mid . \tag{23}$$

Let us first estimate the first term on the right-hand side. By a standard maximal inequality for Gaussian sequences (see [10], Lemma 2.2.2.) the first term is bounded by a positive constant times

$$\sqrt{1+Log\ n}\ \sup_{1\le i\le n}\sqrt{\mathbb{E}(S_N^H(t_i))^2}\,.$$

But $\mathbb{V}ar(Y_{H,k}) = \tau_{H,k}^2$; so

$$\mathbb{E}(S_N^H(t_i))^2 \le 4\sum_{2^{N-1}< k \le 2^N}\frac{\tau_{H,k}^2}{x_{H,k}^2}. \tag{24}$$

Now, by lemma 2.1, we get

$$\mathbb{E}(S_n^H(t_i))^2 \le Cte \times \sum_{2^{N-1}< k \le 2^N} k^{-1-2H}.$$

Since the number of terms in the sum is bounded by a constant times 2^N, we obtain

$$\mathbb{E}(S_N^H(t_i))^2 \le Cte \times 2^{-2NH}.$$

All together, we find the bound

$$\mathbb{E}\sup_{1\le i\le n}|\ S_N^H(t_i)\ | \le Cte \times 2^{-NH}\sqrt{1+Log\ n} \tag{25}$$

for the first term.

To estimate the second term we first write

$$\begin{aligned}
&\mathbb{E}\sup_{1\le i\le n}\ \sup_{t,u\in I_i}|\ S_N^H(t)-S_N^H(u)\ |\\
&\le \mathbb{E}\sup_{1\le i\le n}\ \sup_{t,u\in I_i}\sum_{2^{N-1}< k \le 2^N}|\ Y_{H,k}\ |\ \left|\frac{\cos(x_{H,k}t)-\cos(x_{H,k}u)}{x_{H,n}}\right|\\
&\le \mathbb{E}\sup_{1\le i\le n}\ \sup_{t,u\in I_i}\sum_{2^{N-1}< k \le 2^N}|\ Y_{H,k}\ |\ \left|\int_t^u \sin(x_{H,k}v)dv\right|\\
&\le \mathbb{E}\sup_{1\le i\le n}\ \sup_{t,u\in I_i}\sum_{2^{N-1}< k \le 2^N}|\ Y_{H,k}\ ||\ u-t\ |
\end{aligned}$$

By lemma 2.1, we know that $\tau_{H,k}$ is of the order $k^{1/2-H}$. Hence,

$$\begin{aligned}
\mathbb{E}\sup_{1\le i\le n}\ \sup_{t,u\in I_i}|\ S_N^H(t)-S_N(u)\ | &\le Cte \times \epsilon \times \sum_{2^{N-1}< k \le 2^N}|\ \tau_{H,k}\ |\\
\le Cte \times \epsilon \times \sum_{2^{N-1}< k \le 2^N} k^{1/2-H} &\le Cte \times \epsilon \times 2^{N(\frac{3}{2}-H)}.
\end{aligned}$$

This, in combination with the estimation (25) for the first term, yields the inequality

$$\mathbb{E}\sup_{t\in[0,\,1]}|\ S_N(t)\ | \le Cte\left(2^{-NH}\sqrt{1+Log\ n} + \epsilon \times 2^{N(\frac{3}{2}-H)}\right).$$

For $\epsilon = 4^{-N}$, by the fact that $n \le Cte \times \epsilon^{-1}$, the first term is bounded by a constant times $\sqrt{N}\,2^{-NH}$ and the second one is of lower order. ■

Now we can present the main result of this section.

Theorem 4.1. *The expansion in theorem* 2.1 *of the smfBm is rate-optimal,and the inequality*

$$\mathbb{E}\sup_{t\in[0,1]}\left|a\sum_{k>N}\frac{1-\cos(x_{1/2,k}t)}{x_{1/2,k}}Y_k+b\sum_{k>N}\frac{1-\cos(x_{H,k}t)}{x_{H,k}}Y_{H,k}\right|\le Cte\times N^{-(H\wedge 1/2)}\sqrt{Log\,N}.$$

always holds.

Proof. To arrive at this result, it suffices to show that

$$\mathbb{E}\sup_{t\in[0,1]}\left|\sum_{k>N}\frac{1-\cos(x_{H,k}t)}{x_{H,k}}Y_{H,k}\right|\le Cte\times N^{-H}\sqrt{Log\,N},$$

and

$$\mathbb{E}\sup_{t\in[0,1]}\left|\sum_{k>N}\frac{1-\cos(x_{1/2,k}t)}{x_{1/2,k}}Y_k\right|\le Cte\times N^{-1/2}\sqrt{Log\,N},$$

for every $H\in(0,1)$. This ends the proof of the first statement of this theorem.

The relation's proof is similar. Let l be the positive integer such that $2^{l-1}<N\le 2^l$. Then by the triangle inequality,

$$\begin{aligned}\left|\sum_{k>N}\frac{1-\cos(x_{H,k}t)}{x_{H,k}}Y_{H,k}\right| &= \left|\sum_{k>2^l}\frac{1-\cos(x_{H,k}t)}{x_{H,k}}Y_{H,k}+\sum_{N<k\le 2^l}\frac{1-\cos(x_{H,k}t)}{x_{H,k}}Y_{H,k}\right|\\ &\le \sum_{j>l}|S_j^H(t)|+\left|\sum_{N<k\le 2^l}\frac{1-\cos(x_{H,k}t)}{x_{H,k}}Y_{H,k}\right|\end{aligned}$$

where S_j^H is defined as in (22). We remark that, for every $a>1$, $p>0$ and $n\in\{1,2,\ldots\}$, we have

$$\sum_{k>n}k^p a^{-k}\le Cte\times n^p a^{-n}.$$

This, with lemma 4.1, and the fact that $2^{l-1}<N\le 2^l$, yield

$$\begin{aligned}\sum_{j>l}\mathbb{E}\sup_{t\in[0,1]}|S_j^H(t)| &\le Cte\times\sum_{j>l}\sqrt{j}\,2^{-jH}\\ &\le Cte\times\sqrt{l-1}\times 2^{-(l-1)H}\le Cte\times N^{-H}\sqrt{Log\,N}.\end{aligned}$$

The arguments in the proof of lemma 4.1 show that since $2^{l-1}<N\le 2^l$, we also have

$$\mathbb{E} \sup_{t\in[0,1]} \left| \sum_{N<k\leq 2^l} \frac{1-\cos(x_{H,k}t)}{x_{H,k}} Y_{H,k} \right| \leq Cte \times \sqrt{l} \times 2^{-lH} \leq Cte \times N^{-H} \times \sqrt{Log\, N}.$$

Therefore the desired relation holds and the proof of this theorem is complete. ■

5. Computer Generation of smfBm Sample Paths

The main application of the previous expansion, is the simulation of the *smfBm* sample paths. We have first truncated the expansion obtained in theorem 2.1 at the level $N = 2000$. Then, by Matlab software packages we have inserted numerical values of the positive real zeros of the Bessel function J_{1-H}, for different Hurst parameters $H = 0.25$, $H = 0.5$ and $H = 0.75$.

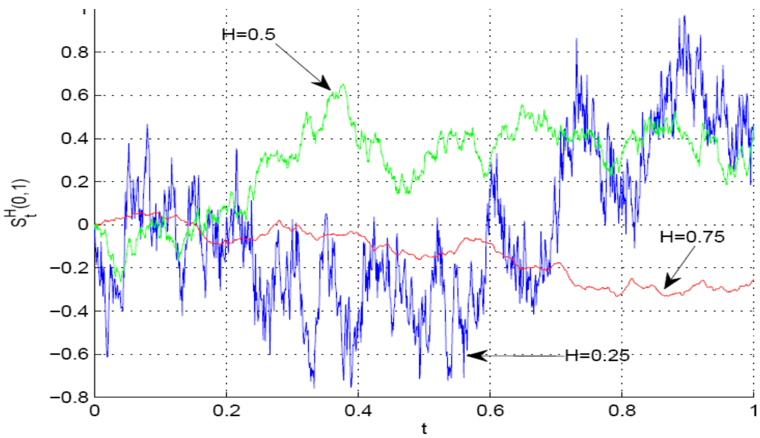

Figure 1: Generated sfBm sample paths for different values of the Hurst parameter $(a = 0.3, b = 0)$

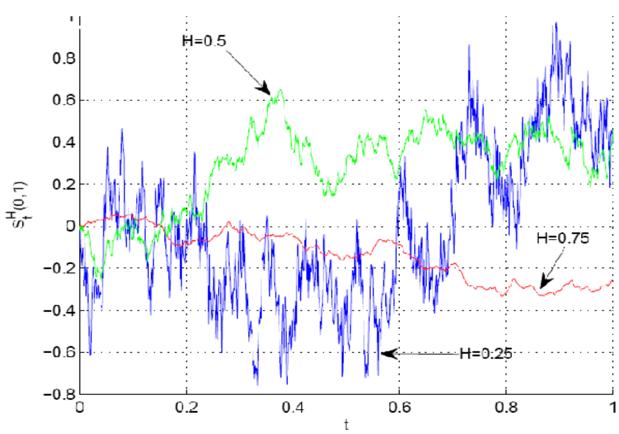

Figure 2: Generated sfBm sample paths for different values of the Hurst parameter $(a = 0, b = 1)$

Finally we simulate the smfBm trajectories and display them in Figures 1 and 2.

The simulation results with different values of H, a and b illustrate the main property of smfBm: a large value of H corresponds to smoother sample paths. In other words, for smaller values of H, the sample paths of a smfBm fluctuate more wildly. This is particularly illustrated in Figure 2 for the generated sample paths of the sub-fractional Brownian motion (sfBm), corresponding to the sub-mixed fractional Brownian motion (smfBm) with $a = 0$ and $b = 1$.

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