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Extrapolation Methods for Random Approximations of π

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Abstract. *For a random polygon generated by n independent points uniformly distributed on a unit circle in* $\mathbb{R}^2$*, it is known that both its semiperimeter* $\mathcal{S}_n$ *and area* $\mathcal{A}_n$ *converge to* π *with probability* 1 *as* $n \to \infty$ *with* $\mathbb{E}(\mathcal{S}_n) = \pi - \pi^3/n^2 + O(n^{-3})$, $\mathbb{E}(\mathcal{A}_n) = \pi - 4\pi^3/n^2 + O(n^{-3})$*, and* $n^{5/2}(\mathcal{S}_n - (\pi - \pi^3/n^2)) \xrightarrow{\mathcal{L}} N(0, 10\pi^6)$, $n^{5/2}(\mathcal{A}_n - (\pi - 4\pi^3/n^2)) \xrightarrow{\mathcal{L}} N(0, 160\pi^6)$, *where the notation* $\xrightarrow{\mathcal{L}}$ *means convergence in distribution as* $n \to \infty$*. In this work, we apply various extrapolation methods to obtain several improved convergence estimates by combining the semiperimeter* $\mathcal{S}_n$ *and the area* $\mathcal{A}_n$ *of such a random n-sided polygon, and also the semiperimeter and the area of a suitably constructed* 2*n-sided random polygon inscribed in the unit circle.*

Key words : Random Polygons, Approximations of π, Extrapolation Processes, Asymptotic Convergence, Central Limit Theorems.

AMS Subject Classifications : 60D05, 60F05, 65C50

1. Introduction

For $n \geq 3$, an n-sided regular polygon inscribed in a unit circle in $\mathbb{R}^2$ has semiperimeter $\mathcal{S}_n = n \sin \frac{\pi}{n}$ and area $\mathcal{A}_n = \frac{1}{2} n \sin \frac{2\pi}{n}$, while a similar n-sided regular polygon circumscribed about the circle has both semiperimeter and area $\mathcal{S}'_n = n \tan \frac{\pi}{n}$. The well-known Archimedean approximation of π is essentially based on the fact that $\mathcal{S}_n < \pi < \mathcal{S}'_n$ and $\lim_{n\to\infty} \mathcal{S}_n = \lim_{n\to\infty} \mathcal{S}'_n = \pi$. Additionally, with the doubling of the sides of the polygons, the following harmonic-geometric-mean relations for $\mathcal{S}_n$ and $\mathcal{S}'_n$

$$1/\mathcal{S}_n + 1/\mathcal{S}'_n = 2/\mathcal{S}'_{2n}, \quad \mathcal{S}_n \mathcal{S}'_{2n} = \mathcal{S}^2_{2n}$$

made it possible for Archimedes to actually compute $\mathcal{S}_n$ and $\mathcal{S}'_n$ for $n = 6, 12, 24, 48, 96$ and obtain in particular the famous bounds $223/71 < \pi < 22/7$. Note that while it is also true that $\lim_{n\to\infty} \mathcal{A}_n = \pi$, we have $\mathcal{S}_n = \mathcal{A}_{2n}$ and $\mathcal{A}_n < \mathcal{S}_n < \pi$. Thus for any fixed n, the semiperimeter $\mathcal{S}_n$ provides a better approximation than the area $\mathcal{A}_n$. Following Archimedes, many later mathematicians (over many centuries) tried hard with larger values of n to obtain more accurate estimates of π [1]. On the other hand, instead of computing $\mathcal{S}_n$ and $\mathcal{S}'_n$ for extremely large values of n, modern extrapolation techniques [7], [9] may be used to obtain significantly more accurate estimates of π based on $\mathcal{S}_n$ and $\mathcal{S}'_n$ for even relatively small values of n. In fact, from the Taylor series expansion for $\sin\theta$ and $\tan\theta$, it is easy to see that

$$\mathcal{S}_n = \pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} + O(n^{-6}), \quad \mathcal{S}'_n = \pi + \frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + O(n^{-6}).$$

Thus, to cancel the leading source of approximation errors, we may combine, for example, $\mathcal{S}_n$ and $\mathcal{S}'_n$, or $\mathcal{S}_n$ and $\mathcal{S}_{2n}$ to obtain,

$$\tfrac{2}{3}\mathcal{S}_n + \tfrac{1}{3}\mathcal{S}'_n = \pi + \frac{\pi^5}{20n^4} + O(n^{-6}), \quad \tfrac{4}{3}\mathcal{S}_{2n} - \tfrac{1}{3}\mathcal{S}_n = \pi - \frac{\pi^5}{480n^4} + O(n^{-6}).$$

Such improvements go back to Snellius and Huygens in the 17th century [1], and provide significantly more accurate estimates for π than either $\mathcal{S}_n$, $\mathcal{S}'_n$, or the simple average $\frac{1}{2}\mathcal{S}_n + \frac{1}{2}\mathcal{S}'_n$ as originally suggested by Archimedes.

More recently, in [2], Bélisle has considered the case of a random polygon generated by n independent points uniformly distributed on the unit circle and has shown that the area $\mathcal{A}_n$ of such a random n-gon converges to π *in probability* with $\mathbb{E}(\mathcal{A}_n) = \pi - 4\pi^3/n^2 + o(n^{-5/2})$, and $n^{5/2}(\mathcal{A}_n - (\pi - 4\pi^3/n^2)) \xrightarrow{\mathcal{L}} N(0, 160\pi^6)$ as $n \to \infty$ where the notation $\xrightarrow{\mathcal{L}}$ means *convergence in distribution* [3], [5], [8]. (Similar results also hold true for the semiperimeter $\mathcal{S}_n$.) Separately in [9], the author has also studied the related problem of approximating π using the semiperimeter or area of such a random n-gon inscribed in a unit circle. By using elementary analysis but with some more refined estimates, it has been shown that both $\mathcal{S}_n$ and $\mathcal{A}_n$ actually converge to π *with probability* 1 as $n \to \infty$ with

$$\mathbb{E}(\mathcal{S}_n) = \pi + \sum_{k=1}^{\infty}(-1)^k \frac{n!}{(n+2k)!}\pi^{2k+1} = \pi - \frac{\pi^3}{n^2} + O(n^{-3}),$$

$$\mathbb{E}(\mathcal{A}_n) = \pi + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k \frac{n!}{(n+2k)!}(2\pi)^{2k+1} = \pi - \frac{4\pi^3}{n^2} + O(n^{-3}).$$

In this work, we extend similar extrapolation analysis for the deterministic Archimedean polygon case to the above mentioned random approximations of π to obtain improved convergence estimates based on various combinations of the semiperimeter and area of random polygons inscribed in the unit circle. One simple such example is the combination $\mathcal{X}_n = \frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n$ which clearly also converges to π with probability 1 as $n \to \infty$ with

$$\mathbb{E}(\mathcal{X}_n) = \pi - \sum_{k=2}^{\infty}(-1)^k \frac{4^k - 4}{3}\frac{n!}{(n+2k)!}\pi^{2k+1} = \pi - \frac{4\pi^5}{n^4} + O(n^{-5}), \tag{1}$$

and in Section 3 we will show that the distribution of $\mathcal{X}_n$ is also asymptotically normal in the sense that

$$n^{9/2}(\mathcal{X}_n - (\pi - 4\pi^5/n^4)) = \sqrt{n}\,(n^4(\mathcal{X}_n - \pi) + 4\pi^5) \xrightarrow{\mathcal{L}} N(0, 3616\pi^{10}), \tag{2}$$

as $n \to \infty$. However, we note that a random polygon circumscribed about the circle is not always well-defined (when all random points fall on a semicircle), thus as in [2], [9], we consider only random polygons inscribed in the unit circle. Additionally, even for an inscribed random polygon, unlike the Archimedean regular polygon case, the process of doubling of the sides of the random polygon can be much more complicated and the way this is done can affect the optimal construction of the extrapolation methods. In particular, we will consider three different approaches in sections 4, 5 and 6 for independent doubling, equal bisection and random bisection of the random polygons respectively and establish the corresponding optimal extrapolation estimates.

2. Preliminary Results

2.1. Some inequalities for the sine function

Lemma 2.1. *Let* $\theta > 0$. *Then* $\sin\theta < \theta$, $\sin\theta > \theta - \frac{1}{3!}\theta^3$, $\sin\theta < \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5$, $\sin\theta > \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7$.

These inequalities correspond to the partial sums of the Taylor series for the sine function. The first one, $\sin\theta < \theta$, is well-known. By integrating this inequality over the interval $[0,\theta]$, we obtain $\cos\theta > 1 - \frac{1}{2!}\theta^2$. Further integrating $\cos\theta > 1 - \frac{1}{2!}\theta^2$ then gives $\sin\theta > \theta - \frac{1}{3!}\theta^3$. Continuing this process, we may further obtain $\cos\theta < 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4$, $\sin\theta < \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5$, etc. for all $\theta > 0$.

2.2. Random divisions of the unit interval (0,1)

Given $n \geq 1$, let $0 = X_0 < X_1 < X_2 < \cdots < X_{n-1} < X_n = 1$ be the order statistics of $n-1$ independent and uniformly distributed points on the unit interval. It is known [4] that the lengths of the resulting n segments (spacings) $X_i - X_{i-1}$ are all identically distributed with the probability density function $f(x) = (n-1)(1-x)^{n-2}$ for $0 < x < 1$, and for any $i \neq j$, the joint probability density function of $X_i - X_{i-1}$ and $X_j - X_{j-1}$ is given by $f(x,y) = (n-1)(n-2)(1-x-y)^{n-3}$ for $x > 0$, $y > 0$ and $x + y < 1$. Furthermore, the joint distribution of all $X_i - X_{i-1}$ is the same as that of $V_1/\sum_{j=1}^n V_j$, $V_2/\sum_{j=1}^n V_j$, $\cdots$, $V_n/\sum_{j=1}^n V_j$, where $V_1, V_2, \cdots, V_n$ are independent and identically distributed exponential random variables. For any positive integer k, it can be checked that for any $i \neq j$, we have

$$\mathbb{E}(|X_i - X_{i-1}|^k) = \frac{k!(n-1)!}{(n+k-1)!}, \quad \mathbb{E}(|X_i - X_{i-1}|^k |X_j - X_{j-1}|^k) = \frac{(k!)^2(n-1)!}{(n+2k-1)!},$$

$$\mathbb{V}\mathrm{ar}(|X_i - X_{i-1}|^k) = \frac{(2k)!(n-1)!}{(n+2k-1)!} - \left(\frac{k!(n-1)!}{(n+k-1)!}\right)^2 \approx \frac{(2k)! - (k!)^2}{n^{2k}} \quad \text{for large } n,$$

$$\mathbb{C}\mathrm{ov}(|X_i - X_{i-1}|^k, |X_j - X_{j-1}|^k) = \frac{(k!)^2(n-1)!}{(n+2k-1)!} - \left(\frac{k!(n-1)!}{(n+k-1)!}\right)^2 \approx \frac{k^2\,(k!)^2}{n^{2k+1}}.$$

For convenience, we denote

$$\mathcal{D}_{n,k} = \sum_{i=1}^{n} |X_i - X_{i-1}|^k. \tag{3}$$

The next three lemmata contain some asymptotic estimates for $\mathcal{D}_{n,k}$ that will be used frequently in later sections.

Lemma 2.2. [9] *For any positive integers n and k, it always holds that* $n^{1-k} \le \mathcal{D}_{n,k} \le 1$, *and for large n, we have* $\mathbb{E}(\mathcal{D}_{n,k}) = k!n!/(n+k-1)! \approx k!/n^{k-1}$ *and* $\mathbb{V}ar(\mathcal{D}_{n,k}) \approx \{(2k)! - (1+k^2)(k!)^2\}/n^{2k-1}$.

Lemma 2.3. [2] *For any* $k \ge 1$, *we have*

$$\sqrt{n}\,(n^{k-1}\mathcal{D}_{n,k} - k!) = n^{k-1/2}(\mathcal{D}_{n,k} - k!/n^{k-1}) \xrightarrow{\mathcal{L}} N(0, (2k)! - (1+k^2)(k!)^2) \quad \text{as } n \to \infty.$$

Lemma 2.4. *For any* $\delta > 0$, *we have* $n^{k-2-\delta}\mathcal{D}_{n,k} \to 0$ *with probability 1 as* $n \to \infty$.

The proof of Lemma 2.2 is straightforward and can be found in [9]. Lemma 2.3 is proved in [2] by using Cramér's theorem (for all values of $k > -1/2$). The strong convergence in Lemma 2.4 may be proved by using Markov's inequality and Borel-Cantelli lemma as in [9]. Note that for any $\varepsilon > 0$, we have by Markov's inequality $\mathbb{P}(n^{k-2-\delta}\mathcal{D}_{n,k} > \varepsilon) \le \varepsilon^{-1}\mathbb{E}(n^{k-2-\delta}\mathcal{D}_{n,k}) \le \varepsilon^{-1}n^{k-2-\delta}k!/n^{k-1} = \varepsilon^{-1}k!/n^{1+\delta}$ with $\sum_{n=1}^{\infty}\mathbb{P}(n^{k-2-\delta}\mathcal{D}_{n,k} > \varepsilon) \le \sum_{n=1}^{\infty}\varepsilon^{-1}k!/n^{1+\delta} < \infty$. Finally, Borel-Cantelli lemma implies the almost sure convergence of $n^{k-2-\delta}\mathcal{D}_{n,k} \to 0$ for any $\delta > 0$ as $n \to \infty$.

3. Convergence of $\frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n$

We now turn to the proof of relation (2). Given a random polygon generated by n independent points uniformly distributed on the unit circle, we will label its vertices $P_0, P_1, \ldots, P_{n-1}, P_n$ in counterclockwise direction with P_n representing the same point as P_0 on the circle (after one cycle). Let θ_i be the length of the arc from the fixed reference point $(1,0)$ to P_i with $\theta_0 < \theta_1 < \cdots < \theta_{n-1} < \theta_n = \theta_0 + 2\pi$. Then $\theta_{i+1} - \theta_i$ gives the length of the arc $\overset{\frown}{P_iP_{i+1}}$ (or the angle $\angle P_iOP_{i+1}$ measured in radian).

Without loss of generality, we assume $\theta_0 = 0$. A further rescaling $\theta_i = 2\pi X_i$ then yields $0 = X_0 < X_1 < X_2 < \cdots < X_{n-1} < X_n = 1$, which corresponds to a random division of the unit interval by $n-1$ uniformly distributed random points. The semiperimeter $\mathcal{S}_n$ and area $\mathcal{A}_n$ of the n-gon can now be expressed as

$$\mathcal{S}_n = \sum_{i=1}^{n} \sin\frac{\theta_i - \theta_{i-1}}{2} = \sum_{i=1}^{n} \sin\pi(X_i - X_{i-1}),$$

$$\mathcal{A}_n = \frac{1}{2}\sum_{i=1}^{n} \sin(\theta_i - \theta_{i-1}) = \frac{1}{2}\sum_{i=1}^{n} \sin 2\pi(X_i - X_{i-1}).$$

To prove (2), first we note, by Lemma 2.1, that

$$\left|\mathcal{S}_n - \left\{\pi - \frac{\pi^3}{3!}\mathcal{D}_{n,3} + \frac{\pi^5}{5!}\mathcal{D}_{n,5}\right\}\right| \le \frac{\pi^7}{7!}\mathcal{D}_{n,7}. \tag{4}$$

Next, by Lemma 2.4, we have $n^{5-\delta}\mathcal{D}_{n,7} \to 0$ with probability 1 for all $\delta > 0$. For simplicity, we rewrite (4) as

$$\mathcal{S}_n = \pi - \frac{\pi^3}{3!}\mathcal{D}_{n,3} + \frac{\pi^5}{5!}\mathcal{D}_{n,5} + n^{-5+\delta}o(1), \tag{5}$$

where $o(1)$ represents a bounded random variable that converges to 0 with probability 1 as $n \to \infty$.

Similarly, for $\mathcal{A}_n$, we have

$$\mathcal{A}_n = \pi - \frac{4\pi^3}{3!}\mathcal{D}_{n,3} + \frac{16\pi^5}{5!}\mathcal{D}_{n,5} + n^{-5+\delta}o(1), \tag{6}$$

and hence for $\mathcal{X}_n = \frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n$,

$$\mathcal{X}_n = \pi - \frac{4\pi^5}{5!}\mathcal{D}_{n,5} + n^{-5+\delta}o(1), \tag{7}$$

$$n^{9/2}(\mathcal{X}_n - (\pi - 4\pi^5/n^4)) = -\frac{4\pi^5}{5!}n^{9/2}(\mathcal{D}_{n,5} - 5!/n^4) + n^{-1/2+\delta}o(1).$$

Note that from Lemma 2.3 we have $n^{9/2}(\mathcal{D}_{n,5} - 5!/n^4) \xrightarrow{\mathcal{L}} N(0, 10! - 26 \cdot (5!)^2)$. Furthermore, by using $0 < \delta \le 1/2$ and Slutsky's theorem, we then obtain the desired asymptotic convergence estimate (2). The following theorem summarizes our asymptotic results for $\mathcal{X}_n = \frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n$.

Theorem 3.1. *Let $n \ge 3$ and $\mathcal{X}_n = \frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n$. Then $\mathcal{X}_n \to \pi$ with probability 1 as $n \to \infty$ and*

$$\mathbb{E}(\mathcal{X}_n) = \pi - \sum_{k=2}^{\infty}(-1)^k \frac{4^k - 4}{3}\frac{n!}{(n+2k)!}\pi^{2k+1} = \pi - \frac{4\pi^5}{n^4} + O(n^{-5}),$$

$$n^{9/2}(\mathcal{X}_n - (\pi - 4\pi^5/n^4)) = \sqrt{n}\,(n^4(\mathcal{X}_n - \pi) + 4\pi^5) \xrightarrow{\mathcal{L}} N(0, 3616\pi^{10}) \quad \text{as } n \to \infty.$$

4. Independent Doubling of Random Polygons

To study extrapolation methods based on the doubling of the sides of random polygons inscribed in the unit circle, we will first consider the relatively easy case of a pair of *independently* constructed random n-gon and $2n$-gon. This means we have two independent random divisions of the unit interval, one by $n-1$, and the other by $2n-1$, independent and uniformly distributed points on $(0,1)$. Denote these by $0 = X_0 < X_1 < \cdots < X_{n-1} < X_n = 1$ and $0 = Y_0 < Y_1 < \cdots < Y_{2n-1} < Y_{2n} = 1$. Then note that the two sets of order statistics $\{X_1, X_2, \cdots X_{n-1}\}$ and $\{Y_1, Y_2, \cdots Y_{2n-1}\}$ are assumed to be independent.

It is clear that the semiperimeters

$$\mathcal{S}_n = \sum_{i=1}^{n}\sin\pi(X_i - X_{i-1}) \text{ and } \mathcal{S}_{2n} = \sum_{i=1}^{2n}\sin\pi(Y_i - Y_{i-1})$$

of these random polygons satisfy

$$\mathbb{E}(\mathcal{S}_n) = \pi - \frac{\pi^3}{(n+1)(n+2)} + O(n^{-4}) \text{ and } \mathbb{E}(\mathcal{S}_{2n}) = \pi - \frac{\pi^3}{(2n+1)(2n+2)} + O(n^{-4}),$$

respectively. Thus, to eliminate the leading source of approximation errors, we should again use a similar combination $\mathcal{Y}_n = \frac{4}{3}\mathcal{S}_{2n} - \frac{1}{3}\mathcal{S}_n$ with $\mathbb{E}(\mathcal{Y}_n) = \pi - \pi^3/(2n^3) + O(n^{-4})$.

To determine the asymptotic distribution of $\mathcal{Y}_n = \frac{4}{3}\mathcal{S}_{2n} - \frac{1}{3}\mathcal{S}_n$, we note by Lemma 2.1, that

$$\mathcal{S}_n = \pi - \frac{\pi^3}{3!}\mathcal{D}_{n,3} + n^{-3+\delta}o(1),$$

$$\mathcal{S}_{2n} = \pi - \frac{\pi^3}{3!}\mathcal{D}_{2n,3} + n^{-3+\delta}o(1),$$

$$\mathcal{Y}_n = \pi - \frac{\pi^3}{3!}\mathcal{D}^*_{n,3} + n^{-3+\delta}o(1),$$

where $\mathcal{D}_{n,3} = \sum_{i=1}^{n}|X_i - X_{i-1}|^k$, $\mathcal{D}_{2n,3} = \sum_{i=1}^{2n}|Y_i - Y_{i-1}|^3$, and $\mathcal{D}^*_{n,3} = \frac{4}{3}\mathcal{D}_{2n,3} - \frac{1}{3}\mathcal{D}_{n,3}$.
Next, by Lemma 2.3, we have $n^{5/2}(\mathcal{D}_{n,3} - 3!/n^2) \xrightarrow{\mathcal{L}} N(0, 360)$ and $n^{5/2}(\mathcal{D}_{2n,3} - 3!/(2n)^2) \xrightarrow{\mathcal{L}} N(0, 45/4)$. Since $\mathcal{D}_{n,3}$ and $\mathcal{D}_{2n,3}$ are independent, then by Lemma 2.3, it follows that

$$n^{5/2}\mathcal{D}^*_{n,3} = \frac{4}{3}n^{5/2}\left(\mathcal{D}_{2n,3} - \frac{3!}{4n^2}\right) - \frac{1}{3}n^{5/2}\left(\mathcal{D}_{n,3} - \frac{3!}{n^2}\right) \xrightarrow{\mathcal{L}} N(0, 60),$$

and thus by using Slutsky's theorem, when $0 < \delta \leq 1/2$, we have

$$n^{5/2}(\mathcal{Y}_n - \pi) = -\frac{\pi^3}{3!}n^{5/2}\mathcal{D}^*_{n,3} + n^{-1/2+\delta}o(1) \xrightarrow{\mathcal{L}} N(0, 5\pi^6/3).$$

Theorem 4.1 below summarizes the above asymptotic estimates for $\frac{4}{3}\mathcal{S}_{2n} - \frac{1}{3}\mathcal{S}_n$ and similar results for $\frac{4}{3}\mathcal{A}_{2n} - \frac{1}{3}\mathcal{A}_n$. Note that these estimates improve over $\mathcal{S}_n$ or $\mathcal{S}_{2n}$, but are weaker than those for $\frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n$ of Theorem 3.1.

Theorem 4.1. *Let $\mathcal{Y}_n = \frac{4}{3}\mathcal{S}_{2n} - \frac{1}{3}\mathcal{S}_n$ and $\mathcal{Z}_n = \frac{4}{3}\mathcal{A}_{2n} - \frac{1}{3}\mathcal{A}_n$. Then both $\mathcal{Y}_n$ and $\mathcal{Z}_n$ converge to π with probability 1 as $n \to \infty$ and*

$$\mathbb{E}(\mathcal{Y}_n) = \pi - \pi^3/(2n^3) + O(n^{-4}), \quad n^{5/2}(\mathcal{Y}_n - \pi) \xrightarrow{\mathcal{L}} N(0, 5\pi^6/3), \quad \text{as } n \to \infty,$$

$$\mathbb{E}(\mathcal{Z}_n) = \pi - 2\pi^3/n^3 + O(n^{-4}), \quad n^{5/2}(\mathcal{Z}_n - \pi) \xrightarrow{\mathcal{L}} N(0, 80\pi^6/3), \quad \text{as } n \to \infty.$$

5. Bisection of the Random n-gon

The marked difference between Theorems 3.1 and 4.1 suggests that extrapolation methods based on independent doubling of the sides of random polygons described in the previous section is probably not the best approach. Given a random n-gon generated by n independent points $P_0, P_1, \cdots, P_{n-1}$ (and $P_n = P_0$) uniformly distributed on the unit circle, perhaps a more natural (and less chaotic) approach to generate a random $2n$-gon is simply to add n new vertices by equally bisecting each arc (or angle) so that between any two consecutive points P_i and P_{i-1}, a new point $P_{i-1/2}$ is added half way between P_i and P_{i-1} on the unit circle. Note that the new $2n$-gon constructed this way does not behave the same as a random $2n$-gon generated directly by $2n$ independent points uniformly distributed on the unit circle. For this reason, we will use slightly different notation $\hat{\mathcal{S}}_{2n}$ and $\hat{\mathcal{A}}_{2n}$ to represent its semiperimeter and area.

Recall that for the random n-gon, we have $\mathcal{S}_n = \sum_{i=1}^n \sin \pi(X_i - X_{i-1})$, $\mathcal{A}_n = \frac{1}{2}\sum_{i=1}^n \sin 2\pi(X_i - X_{i-1})$ where $0 = X_0 < X_1 < \cdots < X_{n-1} < X_n = 1$ defines a random division of the unit interval by $n-1$ independent and uniformly distributed points on $(0,1)$. Then the above bisection procedure yields $X_{i-1/2} = (X_i + X_{i-1})/2$ for vertex $P_{i-1/2}$ and the semiperimeter and area of the newly constructed $2n$-gon are now given by

$$\hat{\mathcal{S}}_{2n} = 2\sum_{i=1}^{n} \sin \frac{\pi(X_i - X_{i-1})}{2}, \quad \hat{\mathcal{A}}_{2n} = \sum_{i=1}^{n} \sin \pi(X_i - X_{i-1}) = \mathcal{S}_n.$$

Theorem 5.1. *Let* $\hat{\mathcal{Y}}_n = \frac{4}{3}\hat{\mathcal{S}}_{2n} - \frac{1}{3}\mathcal{S}_n$ *and* $\hat{\mathcal{Z}}_n = \frac{4}{3}\hat{\mathcal{A}}_{2n} - \frac{1}{3}\mathcal{A}_n$ *. Then both* $\hat{\mathcal{Y}}_n$ *and* $\hat{\mathcal{Z}}_n$ *converge to* π *with probability 1 as* $n \to \infty$ *and*

$$\mathbb{E}(\hat{\mathcal{Y}}_n) = \pi - \pi^5/(4n^4) + O(n^{-5}), \quad n^{9/2}(\hat{\mathcal{Y}}_n - \pi^5/4) \xrightarrow{\mathcal{L}} N(0, 113\pi^{10}/8), \quad \text{as } n \to \infty,$$

$$\mathbb{E}(\hat{\mathcal{Z}}_n) = \pi - 4\pi^5/n^4 + O(n^{-5}), \quad n^{9/2}(\hat{\mathcal{Z}}_n - \{\pi - 4\pi^5/n^4\}) \xrightarrow{\mathcal{L}} N(0, 3616\pi^{10}), \quad \text{as } n \to \infty.$$

Proof. Since $\hat{\mathcal{A}}_{2n} = \mathcal{S}_n$, we have $\hat{\mathcal{Z}}_n = \frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n = \mathcal{X}_n$ (a similar combination of $\hat{\mathcal{S}}_{2n}$ and $\hat{\mathcal{A}}_{2n}$ produces $\hat{\mathcal{X}}_{2n} = \frac{4}{3}\hat{\mathcal{S}}_{2n} - \frac{1}{3}\hat{\mathcal{A}}_{2n} = \hat{\mathcal{Y}}_n$). The desired asymptotic estimates for $\hat{\mathcal{Z}}_n$ now follows directly from those in Theorem 3.1. On the other hand, for $\hat{\mathcal{Y}}_n$, by slightly modifying the same analysis in section 3, we have $\hat{\mathcal{Y}}_n = \pi - \frac{\pi^5}{4 \cdot 5!}\mathcal{D}_{n,5} + n^{-5+\delta}o(1)$. By using Lemmas 2.2, 2.3, 2.4 and Slutsky's theorem, with $0 < \delta \le 1/2$, we then obtain similar estimates for the stated $\hat{\mathcal{Y}}_n$. ■

It is worth mentioning that in this case, we also have $\hat{\mathcal{S}}_{2n} = \pi - \frac{\pi^3}{4 \cdot 3!}\mathcal{D}_{n,3} + n^{-3+\delta}o(1)$ with $\mathbb{E}(\hat{\mathcal{S}}_{2n}) = \pi - \pi^3/(4n^2) + O(n^{-3})$ and $n^{5/2}(\hat{\mathcal{S}}_{2n} - \pi^3/4) \xrightarrow{\mathcal{L}} N(0, 5\pi^6/8)$ as $n \to \infty$. Additionally, by combining all three estimates $\mathcal{S}_n$, $\mathcal{A}_n$ and $\hat{\mathcal{S}}_{2n}$, we may further obtain the following optimal extrapolation improvement.

Theorem 5.2. *Let* $\delta > 0$ *and* $\hat{\mathcal{W}}_n = -\frac{4}{9}\mathcal{S}_n + \frac{1}{45}\mathcal{A}_n + \frac{64}{45}\hat{\mathcal{S}}_{2n}$. *Then we have*

$$\hat{\mathcal{W}}_n = \pi - \frac{\pi^7}{7!}D_{n,7} + n^{-7+\delta}o(1), \ \mathbb{E}(\hat{\mathcal{W}}_n) = \pi - \pi^7/n^6 + O(n^{-7}),$$

and

$$n^{13/2}(\hat{\mathcal{W}}_n - \{\pi - \pi^7/n^6\}) \xrightarrow{\mathcal{L}} N(0, 3382\pi^{14}), \quad \text{as } n \to \infty.$$

6. Random Bisection

Finally we consider a variation to the above bisection procedure to allow each newly added vertex $P_{i-1/2}$ between the two consecutive vertices P_i and P_{i-1} of the original random n-gon to be uniformly randomly selected on the arc $\overset{\frown}{P_iP_{i-1}}$ (with each $P_{i-1/2}$ also selected independently of the others). Note that even though the n new vertices of the $2n$-gon are also randomly selected, the distribution of all $2n$ vertices of the new random $2n$-gon constructed this way is different from the order statistics of $2n$ independent points uniformly distributed on the unit circle. To reflect this distinction, we use here $\tilde{\mathcal{S}}_{2n}$ and $\tilde{\mathcal{A}}_{2n}$ for its semiperimeter and area respectively.

Again, let $0 = X_0 < X_1 < \cdots < X_{n-1} < X_n = 1$ be the random division of the unit interval by $n-1$ independent and uniformly distributed points on $(0,1)$ so that the semiperimeter and area of the random n-gon are given by $\mathcal{S}_n = \sum_{i=1}^{n} \sin\pi(X_i - X_{i-1})$, $\mathcal{A}_n = \frac{1}{2}\sum_{i=1}^{n} \sin 2\pi(X_i - X_{i-1})$. For each newly added point $P_{i-1/2}$, the rescaled $X_{i-1/2}$ can now be written as $X_{i-1/2} = X_{i-1} + (X_i - X_{i-1})U_i$ where $U_1, U_2, \cdots, U_n$ are independent and uniformly distributed over $(0,1)$ and independent of $X_1, X_2, \cdots, X_{n-1}$. Then it can be easily checked that the semiperimeter and area of the newly constructed $2n$-gon are now given by

$$\tilde{\mathcal{S}}_{2n} = \sum_{i=1}^{n} \sin\{\pi(X_i - X_{i-1})U_i\} + \sin\{\pi(X_i - X_{i-1})(1-U_i)\},$$

$$\tilde{\mathcal{A}}_{2n} = \frac{1}{2}\sum_{i=1}^{n} \sin\{2\pi(X_i - X_{i-1})U_i\} + \sin\{2\pi(X_i - X_{i-1})(1-U_i)\}.$$

Similar to (3), we now define

$$\mathcal{M}_{n,k} = \sum_{i=1}^{n} |X_i - X_{i-1}|^k \{U_i^k + (1-U_i)^k\}. \tag{8}$$

Note that since U_i is uniformly randomly distributed over $(0,1)$, each pair $(U_i, 1-U_i)$ also defines a random division of $(0,1)$. Hence $U_i^k + (1-U_i)^k$ follows the same distribution as $\mathcal{D}_{2,k}$. In particular, since $U_i^k + (1-U_i)^k \le 1$, it follows that $\mathcal{M}_{n,k} \le \mathcal{D}_{n,k}$. Thus by Lemma 2.4, we can state the following lemmata.

Lemma 6.1. *For any $\delta > 0$ and positive integers n, k, we have $n^{k-2-\delta}\mathcal{M}_{n,k} \to 0$ with probability 1 as $n \to \infty$. More importantly, similar to Lemma 2.3, we also have, for any $k \ge 1$,*

$$\mathbb{E}(\mathcal{M}_{n,k}) = \mathbb{E}(\mathcal{D}_{2,k})\mathbb{E}(\mathcal{D}_{n,k}) \approx k!\,\mathbb{E}(\mathcal{D}_{2,k})/n^{k-1}, \quad \textit{for large } n,$$

$$\mathbb{V}\mathrm{ar}(\mathcal{M}_{n,k}) = \mathbb{E}(\mathcal{D}_{n,2k})\mathbb{V}\mathrm{ar}(\mathcal{D}_{2,k}) + \mathbb{E}(\mathcal{D}_{2,k})^2\,\mathbb{V}\mathrm{ar}(\mathcal{D}_{n,k}) \approx \sigma_k^2/n^{2k-1}, \quad \textit{for large } n,$$

$$\sqrt{n}\,(n^{k-1}\mathcal{M}_{n,k} - k!\,\mathbb{E}(\mathcal{D}_{2,k})) = n^{k-1/2}(\mathcal{M}_{n,k} - k!\,\mathbb{E}(\mathcal{D}_{2,k})/n^{k-1}) \xrightarrow{\mathcal{L}} N(0,\sigma_k^2), \quad \textit{as } n \to \infty$$

where $\sigma_k^2 = (2k)!\,\mathbb{E}(\mathcal{D}_{2,k}^2) - (1+k^2)(k!)^2\,\mathbb{E}(\mathcal{D}_{2,k})^2$, $\mathbb{E}(\mathcal{D}_{2,k}) = \frac{2}{k+1}$, and

$$\mathbb{E}(\mathcal{D}_{2,k}^2) = \frac{2}{2k+1}\left(1 + \frac{(k!)^2}{(2k)!}\right).$$

Lemma 6.2. Let $\mathcal{M}_{n,k}^* = \mathcal{M}_{n,k} - \mathbb{E}(\mathcal{D}_{2,k})\mathcal{D}_{n,k}$. *We then have $\mathbb{E}(\mathcal{M}_{n,k}^*) = 0$ and*

$$n^{k-1/2}\mathcal{M}_{n,k}^* = n^{k-1/2}(\mathcal{M}_{n,k} - \mathbb{E}(\mathcal{D}_{2,k})\mathcal{D}_{n,k}) \xrightarrow{\mathcal{L}} N(0,(2k)!\,\mathbb{V}\mathrm{ar}(\mathcal{D}_{2,k})), \quad \text{as } n \to \infty.$$

The proofs of Lemmas 6.1 and 6.2 are rather lengthy and will be deferred to the next section. It is easy to check however that, for $k = 3$ and $k = 5$, we have in particular,

$$n^{5/2}(\mathcal{M}_{n,3} - 3/n^2) \xrightarrow{\mathcal{L}} N(0,126), \quad n^{9/2}(\mathcal{M}_{n,5} - 40/n^4) \xrightarrow{\mathcal{L}} N(0,620800),$$

$$n^{5/2}\mathcal{M}_{n,3}^* = n^{5/2}(\mathcal{M}_{n,3} - \mathbb{E}(\mathcal{D}_{2,3})\mathcal{D}_{n,3}) \xrightarrow{\mathcal{L}} N(0,36)$$

We now return to $\tilde{\mathcal{S}}_{2n}$ and $\tilde{\mathcal{A}}_{2n}$. Using $\mathcal{M}_{n,k}$ and in view of Lemmas 2.1 and 6.1 we can rewrite $\tilde{\mathcal{S}}_{2n}$ and $\tilde{\mathcal{A}}_{2n}$ as

$$\tilde{\mathcal{S}}_{2n} = \pi - \frac{\pi^3}{3!}\mathcal{M}_{n,3} + \frac{\pi^5}{5!}\mathcal{M}_{n,5} + n^{-5+\delta}o(1), \quad \tilde{\mathcal{A}}_{2n} = \pi - \frac{4\pi^3}{3!}\mathcal{M}_{n,3} + \frac{16\pi^5}{5!}\mathcal{M}_{n,5} + n^{-5+\delta}o(1).$$

Then, by using Lemma 6.1 and Slutsky's theorem, we arrive at the theorem that follows.

Theorem 6.1. *Both $\tilde{\mathcal{S}}_{2n}$ and $\tilde{\mathcal{A}}_{2n}$ converge to π with probability 1 as $n \to \infty$ with*

$$\mathbb{E}(\tilde{\mathcal{S}}_{2n}) = \pi - \pi^3/(2n^2) + O(n^{-3}), \quad n^{5/2}(\tilde{\mathcal{S}}_{2n} - \{\pi - \pi^3/(2n^2)\}) \xrightarrow{\mathcal{L}} N(0, 7\pi^6/2),$$

$$\mathbb{E}(\tilde{\mathcal{A}}_{2n}) = \pi - 2\pi^3/n^2 + O(n^{-3}), \quad n^{5/2}(\tilde{\mathcal{A}}_{2n} - \{\pi - 2\pi^3/n^2\}) \xrightarrow{\mathcal{L}} N(0, 56\pi^6).$$

Next we note that by comparing, for example, $\mathbb{E}(\tilde{\mathcal{S}}_{2n})$ with $\mathbb{E}(\mathcal{S}_n) = \pi - \pi^3/n^2 + O(n^{-3})$, it is possible to kill the leading approximation error term. A new combination $\tilde{\mathcal{Y}}_n = 2\tilde{\mathcal{S}}_{2n} - \mathcal{S}_n$ should now be used. This leads to

$$\tilde{\mathcal{Y}}_n = \pi - \frac{\pi^3}{3!}(2\mathcal{M}_{n,3} - \mathcal{D}_{n,3}) + \frac{\pi^5}{5!}(2\mathcal{M}_{n,5} - \mathcal{D}_{n,5}) + n^{-5+\delta}o(1) = \pi - \frac{2\pi^3}{3!}\mathcal{M}^*_{n,3} + n^{-3+\delta}o(1).$$

Hence by Lemma 6.1, we have

$$\mathbb{E}(\tilde{\mathcal{Y}}_n) = \pi - \pi^5/(3n^4) + O(n^{-5}), \quad n^{5/2}(\tilde{\mathcal{Y}}_n - \pi) \xrightarrow{\mathcal{L}} N(0, 4\pi^6).$$

Similarly, we have, for $\tilde{\mathcal{Z}}_n = 2\tilde{\mathcal{A}}_{2n} - \mathcal{A}_n$,

$$\tilde{\mathcal{Z}}_n = \pi - \frac{8\pi^3}{3!}\mathcal{M}^*_{n,3} + n^{-3+\delta}o(1),$$

$$\mathbb{E}(\tilde{\mathcal{Z}}_n) = \pi - 16\pi^5/(3n^4) + O(n^{-5}),$$

$$n^{5/2}(\tilde{\mathcal{Z}}_n - \pi) \xrightarrow{\mathcal{L}} N(0, 64\pi^6).$$

Finally we note that in this case, similar to $\mathcal{X}_n = \frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n$, the combination $\tilde{\mathcal{X}}_{2n} = \frac{4}{3}\tilde{\mathcal{S}}_{2n} - \frac{1}{3}\tilde{\mathcal{A}}_{2n}$ satisfies $\tilde{\mathcal{X}}_{2n} = \pi - \frac{4\pi^5}{5!}\mathcal{M}_{n,5} + n^{-5+\delta}o(1)$ with $n^{9/2}\left(\tilde{\mathcal{X}}_{2n} - \left\{\pi - \frac{4\pi^5}{3n^4}\right\}\right) \xrightarrow{\mathcal{L}} N(0, 6208\pi^{10}/9)$.

All this can be summarized in the following theorem.

Theorem 6.2. *Let $\tilde{\mathcal{X}}_{2n} = \frac{4}{3}\tilde{\mathcal{S}}_{2n} - \frac{1}{3}\tilde{\mathcal{A}}_{2n}$, $\tilde{\mathcal{Y}}_n = 2\tilde{\mathcal{S}}_{2n} - \mathcal{S}_n$, $\tilde{\mathcal{Z}}_n = 2\tilde{\mathcal{A}}_{2n} - \mathcal{A}_n$, then $\tilde{\mathcal{X}}_{2n}$, $\tilde{\mathcal{Y}}_n$ and $\tilde{\mathcal{Z}}_n$ all converge to π with probability 1 as $n \to \infty$ with*

$$\mathbb{E}(\tilde{\mathcal{X}}_{2n}) = \pi - \frac{4\pi^5}{3n^4} + O(n^{-5}), \quad n^{9/2}\left(\tilde{\mathcal{X}}_{2n} - \left\{\pi - \frac{4\pi^5}{3n^4}\right\}\right) \xrightarrow{\mathcal{L}} N(0, 6208\pi^{10}/9),$$

$$\mathbb{E}(\tilde{\mathcal{Y}}_n) = \pi - \frac{\pi^5}{3n^4} + O(n^{-5}), \quad n^{5/2}(\tilde{\mathcal{Y}}_n - \pi) \xrightarrow{\mathcal{L}} N(0, 4\pi^6),$$

$$\mathbb{E}(\tilde{\mathcal{Z}}_n) = \pi - \frac{16\pi^5}{3n^4} + O(n^{-5}), \quad n^{5/2}(\tilde{\mathcal{Z}}_n - \pi) \xrightarrow{\mathcal{L}} N(0, 64\pi^6).$$

7. Proofs for Lemmas 6.1 and 6.2

The proofs reported here are based on similar ideas as in [2]. First, by using the equivalent representation

$$(X_1, X_2 - X_1, \cdots, 1 - X_{n-1}) = \left(\frac{V_1}{\sum_{j=1}^n V_j}, \frac{V_2}{\sum_{j=1}^n V_j}, \cdots, \frac{V_n}{\sum_{j=1}^n V_j} \right), \tag{9}$$

we may write

$$\mathcal{M}_{n,k} = \frac{\sum_{i=1}^n V_i^k \{U_i^k + (1-U_i)^k\}}{\left(\sum_{j=1}^n V_j\right)^k},$$

where $V_1, V_2, \cdots V_n$ are independent exponential random variables with a common density function $f_V(v) = \lambda e^{-\lambda v}$ for $v > 0$ and $\lambda > 0$. Since (9) is invariant under a rescaling of $V_i \mapsto \lambda V_i$, then for simplicity we shall assume below that $\lambda = 1$. With the above reformulation of $\mathcal{M}_{n,k}$, we may now consider the joint asymptotic distribution of the two sums in the numerator and the denominator separately by using the multivariate central limit theorem. For this purpose, we calculate that

$$\mu_1 = \mathbb{E}(V_i^k \{U_i^k + (1 - U_i)^k\}) = \mathbb{E}(V_i^k)\mathbb{E}(U_i^k + (1 - U_i)^k) = k!\, \mathbb{E}(\mathcal{D}_{2,k}), \quad \mu_2 = \mathbb{E}(V_i) = 1,$$

$$\mathbb{V}\mathrm{ar}(V_i^k \{U_i^k + (1 - U_i)^k\}) = (2k)!\, \mathbb{E}(\mathcal{D}_{2,k}^2) - (k!)^2 \mathbb{E}(\mathcal{D}_{2,k})^2, \quad \mathbb{V}\mathrm{ar}(V_i) = 1,$$

$$\mathbb{C}\mathrm{ov}(V_i^k \{U_i^k + (1 - U_i)^k\}, V_i) = k \bullet k!\, \mathbb{E}(\mathcal{D}_{2,k}).$$

Thus the covariance matrix Σ for each pair $(V_i^k \{U_i^k + (1 - U_i)^k\}, V_i)$ is given by

$$\Sigma = \begin{bmatrix} (2k)!\, \mathbb{E}(\mathcal{D}_{2,k}^2) - (k!)^2\, \mathbb{E}(\mathcal{D}_{2,k})^2 & k \bullet k!\, \mathbb{E}(\mathcal{D}_{2,k}) \\ k \bullet k!\, \mathbb{E}(\mathcal{D}_{2,k}) & 1 \end{bmatrix}.$$

And by the multivariate central limit theorem, we now obtain, as $n \to \infty$,

$$\sqrt{n} \left(\begin{pmatrix} \frac{1}{n} \sum_{i=1}^n V_i^k \{U_i^k + (1 - U_i)^k\} \\ \frac{1}{n} \sum_{i=1}^n V_i \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right) \xrightarrow{\mathcal{L}} N(\mathbf{0}, \Sigma).$$

Next, we apply Cramér's theorem, [6], to determine the asymptotic distribution of $\mathcal{M}_{n,k}$. For this purpose, we choose $g(x,y) = xy^{-k}$ with $\frac{\partial g}{\partial x} = y^{-k}$, $\frac{\partial g}{\partial y} = -kxy^{-k-1}$. Then we have

$$g\left(\frac{1}{n} \sum_{i=1}^n V_i^k \{U_i^k + (1 - U_i)^k\}, \frac{1}{n} \sum_{i=1}^n V_i \right) = n^{k-1} \mathcal{M}_{n,k}$$

with $g(\mu_1, \mu_2) = k!\, \mathbb{E}(\mathcal{D}_{2,k})$, $\frac{\partial g}{\partial x}(\mu_1, \mu_2) = 1$ and $\frac{\partial g}{\partial y}(\mu_1, \mu_2) = -k \bullet k!\, \mathbb{E}(\mathcal{D}_{2,k})$. Thus by Cramér's theorem, we have, as $n \to \infty$,

$$\sqrt{n}\, (n^{k-1} \mathcal{M}_{n,k} - k!\, \mathbb{E}(\mathcal{D}_{2,k})) = n^{k-1/2} (\mathcal{M}_{n,k} - k!\, \mathbb{E}(\mathcal{D}_{2,k})/n^{k-1}) \xrightarrow{\mathcal{L}} N(0, \sigma_k^2),$$

where

$$\sigma_k^2 = (1, -k \cdot k!\,\mathbb{E}(\mathcal{D}_{2,k}))\Sigma\begin{pmatrix} 1 \\ -k \cdot k!\,\mathbb{E}(\mathcal{D}_{2,k}) \end{pmatrix} = (2k)!\,\mathbb{E}(\mathcal{D}_{2,k}^2) - (1+k^2)(k!)^2\,\mathbb{E}(\mathcal{D}_{2,k})^2.$$

This finishes the proof of Lemma 6.1.

Similarly, to prove Lemma 6.2, we note that the same substitution (9) now yields

$$\mathcal{M}_{n,k}^* = \mathcal{M}_{n,k} - \mathbb{E}(\mathcal{D}_{2,k})\mathcal{D}_{n,k} = \frac{\sum_{i=1}^n V_i^k\{U_i^k + (1-U_i)^k - \mathbb{E}(\mathcal{D}_{2,k})\}}{\left(\sum_{j=1}^n V_j\right)^k}.$$

Next, we calculate that

$$\mu_1^* = \mathbb{E}(V_i^k\{U_i^k + (1-U_i)^k - \mathbb{E}(\mathcal{D}_{2,k})\}) = 0, \quad \mu_2^* = \mathbb{E}(V_i) = 1,$$

$$\mathbb{V}\mathrm{ar}(V_i^k\{U_i^k + (1-U_i)^k - \mathbb{E}(\mathcal{D}_{2,k})\}) = (2k)!\,\mathbb{V}\mathrm{ar}(\mathcal{D}_{2,k}), \quad \mathbb{V}\mathrm{ar}(V_i) = 1,$$

$$\mathbb{C}\mathrm{ov}(V_i^k\{U_i^k + (1-U_i)^k - \mathbb{E}(\mathcal{D}_{2,k})\}, V_i) = 0,$$

with the corresponding covariance matrix for each pair now given by

$$\Sigma^* = \begin{bmatrix} (2k)!\,\mathbb{V}\mathrm{ar}(\mathcal{D}_{2,k}) & 0 \\ 0 & 1 \end{bmatrix}.$$

The use of the multivariate central limit theorem happens to yield

$$\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n V_i^k\{U_i^k + (1-U_i)^k - \mathbb{E}(\mathcal{D}_{2,k})\} \\ \frac{1}{n}\sum_{i=1}^n V_i \end{pmatrix} - \begin{pmatrix} \mu_1^* \\ \mu_2^* \end{pmatrix}\right) \xrightarrow{\mathcal{L}} N(\mathbf{0}, \Sigma^*).$$

Again, with $g(x,y) = xy^{-k}$, we have

$$g\left(\frac{1}{n}\sum_{i=1}^n V_i^k\{U_i^k + (1-U_i)^k - \mathbb{E}(\mathcal{D}_{2,k})\}, \frac{1}{n}\sum_{i=1}^n V_i\right) = n^{k-1}\mathcal{M}_{n,k}^*,$$

with $g(\mu_1^*, \mu_2^*) = 0$, $\frac{\partial g}{\partial x}(\mu_1^*, \mu_2^*) = 1$, $\frac{\partial g}{\partial y}(\mu_1^*, \mu_2^*) = 0$, and $(1,0)\Sigma^*(1,0)^T = (2k)!\,\mathbb{V}\mathrm{ar}(\mathcal{D}_{2,k})$. Finally, by Cramér’s theorem, we obtain $n^{k-1/2}\mathcal{M}_{n,k}^* \xrightarrow{\mathcal{L}} N(0, (2k)!\,\mathbb{V}\mathrm{ar}(\mathcal{D}_{2,k}))$ as $n \to \infty$. This completes the proof.

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