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Zero Delay Convergence of the Solution to a Singular Stochastic Delay Differential Equation Driven by Fractional Brownian Motion

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Abstract. *In this note we prove an existence and uniqueness result of the solution for a singular stochastic differential delay equation with hereditary drift driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Then we show, when the delay goes to zero, that the solution to this equation converges to the solution for the equation without delay.*

Key words : Singular Stochastic Delay Differential Equations, Fractional Brownian Motion, Lebesgue Integral.

AMS Subject Classifications : 60H10, 93E20

1. Introduction

Consider the following stochastic differential equation,

$$\begin{aligned} X^r(t) &= \eta(0) + \int_0^t b(s, X^r)ds + B_t^H, \qquad t \in (0, T] \\ X^r(t) &= \eta(t), \qquad t \in [-r, 0], \end{aligned} \tag{1}$$

driven by an additive fractional Brownian motion (fBm) B^H with Hurst parameter H. Here r denotes a strictly positive time delay, $(B_t^H, t \geq 0)$ is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The hereditary term $b(s, X^r)$ depends on the path $\{X^r(u), -r \leq u \leq s\}$, while $\eta : [-r, 0] \to \mathbb{R}^d$ is a smooth function. We call (1) a delay differential equation with hereditary drift driven by a fractional Brownian motion; and to the best of our knowledge this problem has not been considered before in the wide literature on stochastic differential equations. Conceiving $\int_0^t b(s, X^r)ds$ as a Lebesgue integral, Nualart and Rascanu [6] have studied the existence and uniqueness of the solution for a class of such integral equations without delay and they also proved that the solution is bounded on a finite interval.

In our paper, using also the Lebesgue integral, we will first prove the existence and uniqueness of a solution to equation (1), in a way that extends the results of Nualart and Rascanu[6]. Then we will study the convergence of the solution to this equation when the delay r tends to zero and the drift coefficient b depends on $(s,\ X_r(s))$. As occurs for the Brownian motion case, we are able to prove that the solution to the delay equation converges, almost surely and in L^p, to the solution of the equation without delay. Throughout the paper, we shall prove first our results for deterministic equations and then we will easily apply them pathwise to fractional Brownian motion equations.

The structure of the paper is as follows. In the next section we state the main results of our paper. In Section 2 we give some useful estimate for Lebesgue integrals. Section 3 is devoted to the existence, uniqueness and boundedness of the solution to deterministic equations. In section 4, we study the convergence of the deterministic equations. Finally, in Section 5 we apply the results of the previous sections to stochastic equations driven by fractional Brownian motion and we give the proof of our main theorem.

2. Main Results

Let r denotes a strictly positive time delay($r > 0$) and $\alpha \in (\frac{1}{2}, 1)$. We will also denote by $W_0^{(\alpha,\infty)}([-r,T];\mathbb{R}^d)$ the space of measurable functions $f : [0,T] \to \mathbb{R}^d$ such that

$$||f||_{\alpha,\infty(r)} = \sup_{t\in[-r,T]} \left(|f(t)|+\int_0^t |f(t)-f(s)|(t-s)^{\beta+1}ds\right) < \infty.$$

For any $\lambda \in (0,1]$, we will consider $C^\lambda(-r,T;\mathbb{R}^d)$ as the space of λ-Hölder continuous functions $f : [0,T] \to \mathbb{R}^d$ such that

$$||f||_{\lambda(r)} = ||f||_{\infty(r)} + \sup_{-r\leq s<t\leq T} \frac{|f(t)-f(s)|}{(t-s)^{\lambda}} < \infty,$$

where

$$||f||_{\infty(r)} = \sup_{t\in[-r,T]} |f(t)|.$$

We then consider the spaces $W_0^{(\alpha,\infty)}([-r,T];\mathbb{R}^d)$ and $C^\lambda(-r,T;\mathbb{R}^d)$ with the corresponding norms

$$||\eta||_{\alpha,\infty(-r,0)} = \sup_{t\in[-r,T]} \left(|\eta(t)|+\int_{-r}^t |\eta(t)-\eta(s)|(t-s)^{\beta+1}ds\right),$$

$$||\eta||_{\lambda(-r,0)} = ||\eta||_{\infty(-r,0)} + \sup_{-r\leq s<t\leq T} (|\eta(t)-\eta(s)|(t-s)^{\beta+1}),$$

where

$$||\eta||_{\infty(-r,0)} = \sup_{s\in[-r,0]} |\eta(s)|.$$

Let $b : [0,T]\times C(-r,T;R^d) \to R^d$ is a measurable function such that for every $t > 0$ and $f \in C(-r,T;R^d)$, $b(t,f)$ depend only on $\{f(s); -r \leq s \leq t\}$. Moreover, there exists $b_0 \in L^p(0,T;R^d)$ with $p \geq 2$ and $\forall N \geq 0$ there exists $L_N > 0$.

We will further assume b to satisfy the following hypotheses:

•$(H1)$ $\forall t \in [0,T] \forall x,y,\ \|X\|_{\infty(r)} \leq N,\ \|y\|_{\infty(r)} \leq N$, on a

$$|b(x,t) - b(t,y)| \leq L_N \sup_{-r \leq s \leq t} |x(s) - y(s)|,$$

•$(H2)$ $|b(t,x)| \leq L_0 \sup_{-r \leq s \leq t} |x(s)| + b_0(t), \qquad \forall t \in [0,T],$

•$(H3)$ $|b(x,t) - b(t,y)| \leq L_N |x - y|, \quad \forall x,y, |X| \leq N, |y| \leq N, \forall t \in [0,T],$

•$(H4)$ $|b(t,x)| \leq L_0 |x(t)| + b_0(t), \qquad \forall t \in [0,T].$

3. Estimates for the Integrals

In this section we will obtain some estimates for the Lebesgue integral. In both cases, we will recall some well-known results and obtain some estimates well suited to our equation. In the space $W_0^{(\alpha,\infty)}([-r,T];\mathbb{R}^d)$ we need to introduce a new norm, for any $\lambda \geq 1$, that is

$$\|f\|_{\alpha,\lambda(r)} = \sup_{t \in [-r,T]} \exp(-\lambda t)\left(|f(t)| + \int_0^t |f(t) - f(s)|(t-s)^{\alpha+1} ds\right),$$

and it is easy to check that, for any $\lambda \geq 1$, this norm is equivalent to $\|f\|_{\alpha,\infty(r)}$.

3.1. **The Lebesgue integral**

Let us consider first the ordinary Lebesgue integral. Given $f : [0,T] \to \mathbb{R}^d$ a measurable function, we define

$$F(f)(t) = \int_0^t f(s)ds.$$

Proposition 3.1.[2] *Let* $0 < \alpha < \frac{1}{2}$ *and* $f : [0,T] \to \mathbb{R}^d$ *be a measurable function. If* $F(f)(.) \in W_0^{(\alpha,\infty)}(0,T;\mathbb{R}^d)$ *and*

$$|F(f)(t)| + \int_0^t \frac{|F(f)(t) - F(f)(s)|}{(t-s)^{\alpha+1}} ds < C_{\alpha,T} \int_0^t \frac{|f(s)|}{(t-s)^{\alpha}} ds,$$

then it is possible to define

$$F^b(f)(t) = \int_0^t b(s,f)ds.$$

Proposition 3.2.[2] *Assume that b satisfies* (H_1) *and* (H_2) *with* $\rho = \frac{1}{\alpha}$. *If* $f \in W_0^{(\alpha,\infty)}(-r,T;\mathbb{R}^d)$, *then* $F^b(f)(.) = \int_0^{\cdot} b(s,f)ds \in C^{1-\alpha}(0,T;R^d)$ *and*

•$\|F^{(b)}(f)\|_{1-\alpha} \leq d^1(1 + \|f\|_{\infty(r)})$,

•$\|F^{(b)}(f)\|_{\alpha,\lambda} \leq d^2\left(\frac{1}{\lambda^{1-2\alpha}} + \frac{\|f\|_{\alpha,\lambda(r)}}{\lambda^{1-\alpha}}\right) \leq \frac{d^2}{\lambda^{1-2\alpha}}(1 + \|f\|_{\alpha,\lambda(r)})$,

for all $\lambda \geq 1$ where $d^i, i \in \{1,2\}$ are positive constants depending only on $\alpha,\ T,\ L_0$ and

$B_{0,\alpha} = \|B_0\|_{L^{\frac{1}{\alpha}}}$

If $f, h \in W_0^{(\alpha,\infty)}(0,T;R^d)$ *such that* $\|f\|_{\infty(r)} \leq N,$ and $\|h\|_{\infty(r)} \leq N,$ *then*

$$\|F^b(f) - F^b(h)\|_{\alpha,\lambda} \leq \frac{d_N}{\lambda^{1-\alpha}}\|f-h\|_{\alpha,\infty(r)} ,$$

for all $\lambda \geq 1,$ *where* $d_N = C_{\alpha,T}L_N\Gamma(1-\alpha)$ *depends on* α, T *and* L_N *of* $(H1)$ *and* $(H2)$.

4. Deterministic Integral Equations

Here we will prove a result on existence and uniqueness of the solution to equation (1). We will also obtain a bound for the $\|.\|_{0,\lambda(r)}$ norm of this solution. In order to obtain a bound whose dependence on r could be controlled, we will introduce a new method to compute this estimate. Set $0 < \alpha < \frac{1}{2}$, and $\eta \in W_0^{\alpha,\infty}(-r,0;\mathbb{R}^d) \cap C^{1-\alpha,\infty}(-r,0;\mathbb{R}^d)$, to consider the deterministic stochastic differential equation (1) on $\mathbb{R}^d$.

Adopting the notations introduced in the previous sections, we can give the following other expression for equation (1):

$$X(t) = \eta(0) + F^{(b)}(x)(t) + B_t^H, \qquad t \in (0,T],$$

$$X(t) = \eta(t), \qquad t \in [-r,0].$$

Subsequently the result on existence and uniqueness reads as follows.

Theorem 4.1. *Assume that b satisfies hypothesis* $(H1)$ *with* $\rho = \frac{1}{\alpha}, 0 < \beta,\delta \leq 1$ *and*

$$0 < \alpha < \alpha_0 = \min\{\tfrac{1}{2}, \beta, \frac{\delta}{1+\delta}\}.$$

Then equation (1) *has a unique solution* $x \in W_0^{\alpha,\infty}([-r,T];\mathbb{R}^d) \cap C^{1-\alpha}([-r,T];\mathbb{R}^d)$.

Proof. Step1: For $x \in C^{1-\alpha}([-r,T];\mathbb{R}^d)$, if $x \in W_0^{\alpha,\infty}(-r,T;\mathbb{R}^d)$ is a solution, then $F^b(x) \in C^{1-\alpha}([0,T];\mathbb{R}^d)$(see Proposition 3.2).
Furthermore

$$\|X\|_{1-\alpha(r)} = \|X\|_{\infty(r)} + \sup_{-r\leq s<t\leq T}\frac{|x(t)-x(s)|}{(t-s)^{1-\alpha}}$$

$$\leq \|\eta(0)\|_{\infty(-r,0)} + \|F^{(b)}(x)(t)\|_{\infty} + \sup_{-r\leq s<t\leq T}\frac{|x(t)-x(s)|}{(t-s)^{1-\alpha}} + \|B_t^H\|_{\infty}$$

$$\leq \|\eta(0)\|_{\infty(-r,0)} + \|F^{(b)}(x)(t)\|_{\infty} + \sup_{0<s<t\leq T}\frac{|x(t)-x(s)|}{(t-s)^{1-\alpha}}$$

$$+ \sup_{-r\leq s<t\leq 0}\frac{|\eta(t)-\eta(s)|}{(t-s)^{1-\alpha}} + \sup_{-r\leq s<0\leq t<T}\frac{|x(t)-\eta(s)|}{(t-s)^{1-\alpha}} + \|B_t^H\|_{\infty}$$

$$\leq \|\eta(0)\|_{1-\alpha(-r,0)} + \|F^{(b)}(x)(t)\|_{1-\alpha} + \sup_{-r\leq s<0\leq t<T}\left(\frac{|x(t)-x(0)|}{(t)^{1-\alpha}} + \frac{|x(0)-\eta(s)|}{(-s)^{1-\alpha}}\right) + \|B_t^H\|_{1-\alpha}$$

$$\leq 2(\|\eta(0)\|_{1-\alpha(-r,0)} + \|F^{(b)}(x)(t)\|_{1-\alpha} + \|B_t^H\|_{1-\alpha}) < \infty.$$

Step 2: Uniqueness.

Consider x and x' two solutions such that $\|x\|_{1-\alpha(r)} \leq N$ and $\|x'\|_{1-\alpha(r)} \leq N$, to note that

$$|x(t) - x'(t)| \leq |F^{(b)}(x)(t) - F^{(b)}(x')(t)|,$$

$$\sup_{t\in[-r,T]} e^{-\lambda t}|x(t) - x'(t)| \leq \sup_{t\in[0,T]} e^{-\lambda t}|F^{(b)}(x)(t) - F^{(b)}(x')(t)|,$$

and

$$\sup_{t\in[-r,T]} e^{-\lambda t}\int_{-r}^{t} \frac{|x(t)-x'(t)|-(|x(s)-x'(s)|)}{(t-s)^{\alpha+1}}ds$$

$$= \sup_{t\in[0,T]} e^{-\lambda t}\int_{-r}^{t} \frac{|x(t)-x'(t)|-(|x(s)-x'(s)|)}{(t-s)^{\alpha+1}}ds$$

$$\leq \sup_{t\in[-r,T]} e^{-\lambda t}\int_{0}^{t} \frac{|x(t)-x'(t)|-(|x(s)-x'(s)|)}{(t-s)^{\alpha+1}}ds$$

$$+ \sup_{t\in[0,T]} e^{-\lambda t}\int_{-r}^{0} \frac{|x(t)-x'(t)|-(|x(s)-x'(s)|)}{(t-s)^{\alpha+1}}ds.$$

Therefore

$$\|x - x'\|_{\alpha,\lambda(r)} \leq \|F^{(b)}(x)(t) - F^{(b)}(x')(t)\| + U,$$

where

$$U = \sup_{t\in[0,T]} e^{-\lambda t}\int_{-r}^{0} \frac{|x(t)-x'(t)|}{(t-s)^{\alpha+1}}ds.$$

This U deserves a further study. Indeed

$$U < \sup_{t\in[0,T]} e^{-\lambda t}|x(s) - x'(s)|\frac{1}{\alpha t^{\alpha}} < U_1,$$

with

$$U_1 = \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{\alpha t^{\alpha}}|F^{(b)}(x)(t) - F^{(b)}(x')(t)|.$$

Moreover,

$$U_1 \leq \frac{1}{\alpha} \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^{\alpha}}\left|\int_0^t (b(s,x) - b(s,x'))ds\right|$$

$$\leq \frac{L_N}{\alpha} \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^{\alpha}}\int_0^t \sup_{-r\leq u\leq s} |(x(u) - x'(u)|ds$$

$$\leq \frac{L_N}{\alpha}\left(\sup_{-r\leq u\leq s} \frac{e^{-\lambda u}}{t^{\alpha}}|x(u) - x'(u)|\right) \sup_{t\in[0,T]}\int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\alpha}}ds$$

$$\leq \frac{L_N}{\alpha}\lambda^{\alpha-1}\Gamma(1-\alpha)\|x - x'\|_{\alpha,\lambda(r)}.$$

So

$$\|x-x'\|_{\alpha,\lambda(r)} \leq \|F^{(b)}(x)(t)-F^{(b)}(x')(t)\|+U.$$
$$\leq \|F^{(b)}(x)(t)-F^{(b)}(x')(t)\|_{\alpha,\lambda(r)} + \tfrac{L_N}{\alpha}\lambda^{\alpha-1}\Gamma(1-\alpha)\|x-x'\|_{\alpha,\lambda(r)},$$

with

$$\|F^{(b)}(x)(t)-F^{(b)}(x')(t)\|_{\alpha,\lambda(r)} \leq \frac{d_N}{\lambda^{1-\alpha}}\|x-x'\|_{\alpha,\lambda(r)}.$$

Subsequently

$$\|x-x'\|_{\alpha,\lambda(r)} \leq \left(\frac{d_N}{\lambda^{1-\alpha}} + \frac{L_N}{\alpha\lambda^{1-\alpha}}\Gamma(1-\alpha)\right)\|x-x'\|_{\alpha,\lambda(r)}.$$

Choosing λ large enough so as

$$\left(\frac{d_N}{\lambda^{1-\alpha}} + \frac{L_N}{\alpha\lambda^{1-\alpha}}\Gamma(1-\alpha)\right) \leq \frac{1}{2},$$

leads to

$$\frac{1}{2}\|x-x'\|_{\alpha,\lambda(r)} \leq 0,$$

and obviously $x = x'$.
Step3: Existence.

Let us consider the operator $L : W_0^{\alpha,\infty}([-r,T];\mathbb{R}^d) \to C^{1-\alpha}([-r,T];\mathbb{R}^d)$, defined by

$$L(y)(t) = \eta(0) + F^b(x)(t) + B_t^H, \qquad t \in (0,T] ;$$
$$L(y)(t) = \eta(t), \qquad t \in [-r,0]. \tag{2}$$

In order to prove the existence of $y = L(y)(t)$ we will use a fixed point argument based on the following Lemma with the notation $y^* = L(y)(t)$.

Lemma 4.1. *Let (X,ρ) be a complete metric space and ρ_0,ρ_1,ρ_2 some metrics on X equivalent to ρ. Assume that $l : X \to X$ satisfies the following conditions.*
1. *There exists $\mu_0 > 0$, $X_0 \in X$ such that if $B_0 = \{x \in X : \rho_0(x_0,x) \leq \mu_0\}$ then $l(B_0) \subset B_0$,*
2. *There exist $\varphi : (X,\rho) \to [0,+\infty]$ lower semicontinuous function and some positive constants C_0,K_0 such that when $N_\varphi(a) = \{x \in X : \varphi(x) \leq a\}$,*
(a) $l(B_0) \subset N_\varphi(C_0)$,
(b) $\rho_1(l(x),l(y)) < K_0\rho_1(x,y), \forall x,y \in N_\varphi(C_0) \sqcap B_0$,
3. *There exists $a \in (0,1)$ such that $\rho_2(l(x),l(y)) < a\rho_2(x,y), \forall x,y \in l(B_0)$.*
Then, there exists $x^* \in l(B_0) \subset X$ such that $x^* = l(x^*)$.

Proof. We shall check individually on each of the three conditions of this lemma.
Condition 1: Note that for $t \in [-r,0]$,

$$|y^*(t)| + \int_{-r}^{t}\frac{|y^*(t)-y^*(s)|}{(t-s)^{\alpha+1}}ds = |\eta(t)| + \int_{-r}^{t}\frac{|\eta(t)-\eta(s)|}{(t-s)^{\alpha+1}}ds,$$

and for $t \in (0,T]$,

$$|y^*(t)| + \int_{-r}^{t}\frac{|y^*(t)-y^*(s)|}{(t-s)^{\alpha+1}}ds = |y^*(t)| + \int_{-r}^{0}\frac{|y^*(t)-\eta(s)|}{(t-s)^{\alpha+1}}ds + \int_{0}^{t}\frac{|y^*(t)-y^*(s)|}{(t-s)^{\alpha+1}}ds.$$

Hence

$$\|y^*(t)\|_{\alpha,\lambda(r)} \leq \|\eta\|_{\alpha,\lambda(-r,0)} + \|F^{(b)}(Y)(t)\| + \|B_t^H\|_{\alpha,\lambda} + E,$$

where

$$E = \sup_{t\in[0,T]} e^{-\lambda t} \int_{-r}^{0} \frac{|y^*(t)-\eta(s)|}{(t-s)^{\alpha+1}} ds.$$

Clearly $E \le E_1 + E_2$, with

$$E = \sup_{t\in[0,T]} e^{-\lambda t} \int_{-r}^{0} \frac{|y^*(t)-\eta(s)|}{(t-s)^{\alpha+1}} ds \le C_\alpha \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^\alpha} |y^*(t) - \eta(s)|,$$

$$E_1 = \sup_{t\in[0,T]} e^{-\lambda t} \int_{-r}^{0} \frac{|\eta(0)-\eta(s)|}{(-s)^{\alpha+1}} ds \le \|\eta\|_{\alpha,\lambda(-r,0)}.$$

To study E_1, one can repeat similar arguments to those used to estimate U when we proved uniqueness. However we shall give here only a sketch of these arguments. Observe first that

$$E_1 \le E_{1,1} + E_{1,2},$$

where

$$E_{1,1} = C_\alpha \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^\alpha} \int_0^t |b(s,y)| ds \ ,$$

$$E_{1,2} = C_\alpha \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^\alpha} \int_0^t |B_t^H(y)| ds$$

$$\begin{aligned}
E_{1,1} \le\ & C_\alpha L_0 \left(\sup_{t\in[-r,T]} e^{-\lambda s} |y(s)| \right) \sup_{t\in[0,T]} \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\alpha+1}} ds \\
& + C_\alpha \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^\alpha} \int_0^t |b_0(S)| ds \\
\le\ & C_\alpha L_0 \lambda^{\alpha-1} \Gamma(\alpha - 1) \|Y\|_{\alpha,\lambda(r)} + C_\alpha B_{0,\alpha}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
E_{1,2} =\ & C_\alpha \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^\alpha} |B_t^H| t \le C_\alpha \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^{\alpha-1}} |B_t^H| \\
\le\ & C_\alpha \sup_{t\in[0,T]} e^{-\lambda t} t^{\alpha-1} \sup_{t\in[0,T]} |B_t^H| \\
\le\ & C_\alpha \left(\frac{1-\alpha}{t^\lambda} \right)^{1-\alpha} e^{1-\alpha} \|B_t^H\|_{\infty(0)}.
\end{aligned}$$

So

$$E_1 \le C_\alpha L_0 \lambda^{\alpha-1} \Gamma(\alpha - 1) \|Y\|_{\alpha,\lambda(r)} + C_\alpha B_{0,\alpha} + C \left(\frac{1-\alpha}{t^\lambda} \right)^{1-\alpha} e^{1-\alpha} \|B_t^H\|_{\infty(0)},$$

and

$$\|y^*(t)\|_{\alpha,\lambda(r)} \le M_1(\lambda) + M_2(\lambda)(1 + \|y(t)\|_{\alpha,\lambda}) + M_3(\lambda) \|y(t)\|_{\alpha,\lambda},$$

where

$$M_1(\lambda) = 2\|\eta\|_{\alpha,\lambda} + \left(1 + C_\alpha\left(\frac{1-\alpha}{t^\lambda}\right)^{1-\alpha} e^{1-\alpha}\right)\|B_t^H\|_{\alpha,\lambda} + \frac{d_2}{\lambda^{1-2\alpha}} + C_\alpha B_{0,\alpha},$$

$$M_2(\lambda) = \frac{d_2}{\lambda^{1-2\alpha}}, \quad M_3(\lambda) = C_\alpha L_0 \lambda^{\alpha-1}\Gamma(\alpha-1)\|Y\|_{\alpha,\lambda}.$$

Furthermore, we can choose $\lambda = \lambda_0$ large enough in order to have $M_2(\lambda_0) + M_3(\lambda_0) \le \frac{1}{2}$.

If $\|y\|_{\alpha,\lambda_0(r)} \le 2(1 + M_1(\lambda_0))$, then $\|y^*\|_{\alpha,\lambda_0(r)} \le 2(1 + M_1(\lambda_0))$ and so $L(B_0) \subset B_0$, where $B_0 = \{y \in W_0^{\alpha,\infty}([-r,T];\mathbb{R}^d), \|y\|_{\alpha,\lambda_0(r)} \le 2(1 + M_1(\lambda_0))\}$.

The condition 1 is satisfied as the metric ρ_0 is associated to the norm $\|.\|_{\alpha,\lambda_0(r)}$.

Condition 2: Notice that if $y \in B_0$, then $\|y\|_{\alpha,\infty(r)} \le 2e^{\lambda_0 T}(1 + M_1(\lambda_0)) = N_0$.

Then repetition of the same arguments we have used in step 2 of theorem 4.1 yields that for all $y, y' \in B_0$ and for all $\lambda \ge 1$

$$\|L(y) - L(y')\| \le \left(\frac{d_N}{\lambda^{1-\alpha}} + \frac{L_N}{\alpha\lambda^{1-\alpha}}\Gamma(1-\alpha)\right)\|y - y'\|_{\alpha,\lambda(r)}.$$

Condition 3: From the proof of the above condition we deduce that for all $y, y' \in L(B_0)$

$$\|L(y) - L(y')\| \le \frac{C'}{\lambda^{1-\alpha}}\|y - y'\|_{\alpha,\lambda(r)},$$

with $C' = d_N + \alpha^{-1}L_N\Gamma(1-\alpha)$.

So, it suffices to choose $\lambda = \lambda_0$ such that $\frac{C'}{\lambda^{1-\alpha}} \le \frac{1}{2}$. ■

5. Convergence as the Delay Goes to Zero

Our aim here is to study what happens when the delay r tends to zero. We will assume the validity of the hypotheses $(H1), (H2)$ and $(H3), (H4)$ throughout this section.

Set x^r as the solution of the integral delay equation (1) on $\mathbb{R}^d$ and x as the $\mathbb{R}^d$ solution of the integral equation

$$X(t) = \eta(0) + \int_0^t b(s,X)ds + B_t^H, \qquad t \in (0,T].$$

From the previous sections and the paper of Nualart and Rascanu [6], we know that these solutions exist, they are unique with $x^r \in W_0^{\alpha,\infty}(r,T;\mathbb{R}^d)$ and $x \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$.

The main result of this section is the theorem that follows.

Theorem 5.1. *Assume that b satisfy hypotheses* $(H1), (H2)$ *and* $(H3), (H4)$ *with* $\rho = \frac{1}{\alpha}$, $0 < \beta, \delta \le 1$ *and*

$$0 < \alpha < \alpha_0 := \min\{\frac{1}{2}, \beta, \frac{\delta}{\delta+1}\}$$

then

$$\lim_{r\to 0} \|x^r - x\|_{\alpha,\infty} = 0.$$

Proof. Actually, we will prove that there exists λ_0 such that

$$\lim_{r\to 0} \|x^r - x\|_{\alpha,\lambda_0} = 0.$$

Let us choose N so as $\|x\|_{\alpha,\infty} \leq N$ and $\sup_{0\leq r\leq r_0}\|x^r\|_{\alpha,\infty(r)} \leq N$ to get that

$$\|F^{(b)}(x) - F^{(b)}(x^r)\|_{\alpha,\lambda} \leq \frac{d_N}{\lambda^{1-\alpha}}\|x - x^r\|_{\alpha,\lambda}.$$

This allows for writing

$$\|x - x^r\|_{\alpha,\lambda} \leq \frac{d_N}{\lambda^{1-\alpha}}\|x - x^r\|_{\alpha,\lambda}.$$

Hence, if we choose λ_0 large enough to ensure that

$$\frac{d_N}{\lambda^{1-\alpha}} \leq \frac{1}{2},$$

we arrive at the required result

$$\lim_{r\to 0} \|x^r - x\|_{\alpha,\lambda_0} = 0 \Rightarrow \lim_{r\to 0} \|x - x^r\|_{\alpha,\infty} = 0.$$ ■

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