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The Maximal Nonnegative Solution to the Discrete Time Algebraic Riccati Equation

Dedicated to professor Milko Petkov on the occasion of his 80th birthday.

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Abstract. *We consider a class of algebraic discrete time Riccati equations arising in the study of positive linear systems. A sequence of Stein algebraic Riccati equations is constructed with nonnegative coefficients, whose solutions are nonnegative and converge to the solution of an algebraic discrete time Riccati equation of the optimal control problem in the infinite horizon for positive systems. The obtained solution is nonnegative maximal leading to the nonnegative state trajectories of the system. A sufficient condition is established for the positivity of the linear quadratic (LQ)-optimal closed-loop system. A numerical example is presented to illustrate the theoretical results.*

Key words : Positive Linear Systems, Discrete-Time Riccati Equation, Nonnegative Solution.

AMS Subject Classifications : 15A24, 15A45, 65F35, 91B55

1. Introduction

Consider the following discrete time (time invariant) linear system

$$x_{i+1} = Ax_i + Bu_i, \qquad \mathrm{i} = 0, 1, 2, \dots \tag{1}$$

where the state x_i and the control u_i are real vectors, A and B are $n \times n$ and $n \times m$ real matrices, respectively, and x_0 is given.
The recurrence relation for the state x_i is known [8]to be

$$x_i = A^i x_0 + \sum_{r=0}^{i-1} A^{i-r-1} B u_r, \quad i = 1, 2, 3, \dots$$

The cost functional associated with the above discrete time system is

$$J(x_0) = \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i \,, \tag{2}$$

where R is an $m \times m$ real symmetric positive definite matrix, and $Q = Q^T$ is an $n \times n$ real positive semidefinite matrix.

In case where $i = 0, \dots N-1$ in (2) and (1), the above model is called a linear quadratic finite discrete time control model. And we shall investigate the infinite horizon case.

The optimal cost in x_0 is given by

$\tilde{J}(x_0) = \inf_u J(x_0)$.

Very interesting and actual problems are considered in the framework of the linear system (1) where the state and output variables should remain nonnegative for any nonnegative initial conditions and input functions. Such type systems are coined as positive systems according to the following definition.

Definition 1.1. The system (1) is said to be positive if for all initial nonnegative x_0 and for nonnegative controls u_i for all values of i, then the state trajectories x_i are nonnegative for all values of i.

There are many examples and applications for the positive systems. Most notable forms of these positive systems arise in economics [1, 6] and/or financial modelling [7]. The above definition can be specified for the finite horizon case ($i = 0, \dots, N-1$) and for the infinite horizon case ($i = 0, \dots, \infty$) and remains valid for these two cases. Moreover, the following property of positive systems is well known (see e.g. [4, 5], Proposition 5.1, [2]).

Proposition 1.1. The system (1) is positive if and only if A and B are nonnegative matrices.

The finite horizon linear quadratic control problems for positive linear systems in discrete time is studied in [2, 3]. Necessary and sufficient conditions for their solvability can be derived by using the maximum principle. Much of the theory and applications of positive systems can be found in [4, 5].

This paper is a study the linear quadratic infinite discrete time control problem. The infinite horizon positive LQ problem in discrete time consists of minimizing the quadratic functional (2) for a given positive linear system described by (1), where the initial state x_0 is nonnegative. We assume that the matrix coefficients A, B, Q are nonnegative matrices and R is a nonnegative. In addition R is a positive definite and Q is a positive semidefinite matrix. It is well known that the equilibrium point for this problem can be obtained, [8], as a maximal solution to the discrete time algebraic Riccati equation

$$\begin{aligned} P = \mathcal{M}(P) \; := \; & A^T P A + Q \\ & - A^T P B (R + B^T P B)^{-1} B^T P A \,, \end{aligned} \tag{3}$$

where

$P \in Dom\,\mathcal{M} = \{P \text{ is nonnegative}, \; det(R + B^T P B) \neq 0,\}.$

Moreover, based on the standard problem [8], the optimal control is given by

$$u_i = -(R + B^T \tilde{P} B)^{-1} B^T \tilde{P} A x_i$$

and

$$x_{i+1} = (A - B(R + B^T \tilde{P} B)^{-1} B^T \tilde{P} A) x_i, \quad i = 0, 1, 2, \dots,$$

where $\tilde{P}$ is the maximal symmetric solution to $P = \mathcal{M}(P)$ and $P \in Dom\,\mathcal{M}$. We shall define the controller as a matrix $K_P = (R + B^T P B)^{-1} B^T P A$. Thus $x_{i+1} = (A - BK_{\tilde{P}}) x_i$.

In this paper we will investigate a special class of positive systems, where the state trajectories x_i are nonnegative for all values of i, i.e. we will interest for the case when $(A - BK_{\tilde{P}})$ is a nonnegative matrix! For this purpose we consider a recurrence equation for computing the maximal symmetric nonnegative solution $\tilde{P}$ to discrete-time algebraic Riccati equation (3). A user friendly sufficient condition is proposed for convergence of the pertaining recurrence equation. A numerical example is presented where the theoretical results are illustrated.

As a matter of notation, the inequality $A \geq 0$ $(A > 0)$ means that all elements of the matrix (or vector) A are real nonnegative (positive). Here the matrix A is said to be nonnegative (positive). For the matrices $A = (a_{ij})$ and $B = (b_{ij})$, we write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ hold for all indexes i and j. The spectral matrix norm is represented by $\|.\| = \sqrt{\rho(AA^T)}$, where $\rho(.)$ stands for a spectral radius of A.

2. Discrete Time Riccati Equation With a Nonnegative Solution

As it is possible to rewrite equation (3) in the form

$$P = \mathcal{M}(P) \;:=\; (A - BK_P)^T P (A - BK_P) + Q + K_P^T R K_P \tag{4}$$

or

$$P = \tilde{A}_P^T P \tilde{A}_P + \tilde{Q}_P,$$

with $\tilde{A}_P = A - BK_P$ and $\tilde{Q}_P = Q + K_P^T R K_P$, we note that the $\tilde{Q}_P$ is a positive semidefine matrix because R is positive definite and Q is positive semidefinite.

It is easy to derive the following properties of $\mathcal{M}(P)$ for symmetricY, Z matrices :

$$\mathcal{M}_Z(Y) = \tilde{A}_Z^T Y \tilde{A}_Z + \tilde{Q}_Z - W_Z(Y), \tag{5}$$

and

$$\mathcal{M}_Z(Z) - \mathcal{M}_Z(Y) = \tilde{A}_Z^T (Z - Y) \tilde{A}_Z + W_Z(Y), \tag{6}$$

where

$$W_Z(Y) = (K_Z - K_Y)^T (R + B^T Y B)(K_Z - K_Y).$$

A solution $\tilde{\mathbf{X}}$ of (3) is called maximal if $\tilde{\mathbf{X}} \geq \mathbf{X}$ for any solution $\mathbf{X}$, and this will be investigated the next section.

3. An Iteration With Positivity Preserving Property

Let us introduce the iteration

$$P_{i+1} = \tilde{A}^T_{P_i} P_i \tilde{A}_{P_i} + \tilde{Q}_{P_i}, \quad i = 0, 1, 2, \dots \tag{7}$$

where the initial point $P_0 = P_0^T$ is chosen such that $K_{P_0} \geq 0$, and $\tilde{A}_{P_0} \geq 0$ and the inequality $P_0 - \mathcal{M}(P_0) \geq 0$ holds. The convergence properties of matrix sequences defined by (7) are derived under the following assumption.

Assumption 3.1. The inequality $K_{P_0} - K_{P_i} \geq 0$ where P_i is computed via (7) is satisfied for all integers i.

Our method can be applied under the main assumption that there exists an initial point P_0 with the special properties noted above. These properties happen to be essential for the proof of the convergence of (7) in the theorem that follows. As for the choice of the matrix P_0 in the illustrative example, it is assumed that $P_0 = \alpha e e^T$ with $e^T = (1, 1, \dots, 1)$. The convergence result is given by the theorem that follows.

Theorem 3.1. *Letting that there is a positive semidefinite matrix P_0 such that: $(a) P_0 \in Dom M$, (b) $K_{P_0} \geq 0$, $(c) P_0 - M(P_0) \geq 0$ and $(d) \tilde{A}_{P_0} \geq 0$. Then for the matrix sequences $\{P_i\}_{i=1}^{\infty}$, defined by (7), the following properties are satisfied.*
(i) $P_i \in Dom \mathcal{M}$,
(ii) $P_0 \geq P_i$, $i = 1, 2, \dots$.
(iii) $\tilde{A}_{P_i} \geq 0$, $i = 1, 2, \dots$.

Proof. We compute P_1 via (7) as

$$P_1 = \tilde{A}^T_{P_0} P_0 \tilde{A}_{P_0} + \tilde{Q}_{P_0}.$$

Note that $\tilde{Q}_{P_0}$ and $\tilde{A}_{P_0}$ are nonnegative matrices and hence $P_1 \geq 0$. In addition P_1 is a positive semidefinite matrix because it is a sum of two positive semidefinite matrices. Consequently $R + B^T P_1 B - R$ is a positive definite matrix and hence $R + B^T P_1 B$ is nonsingular. Therefore $P_1 \in Dom \mathcal{M}$.
Moreover, since $P_0 - \mathcal{M}(P_0) \geq 0$ then

$$P_0 = \mathcal{M}(P_0) + N_0,$$

where $N_0 \geq 0$. Using the recurrence equation (7) and matrix identity (5) we have

$$P_{r+1} = \mathcal{M}(P_r) = \mathcal{M}_{P_0}(P_r) = \tilde{A}^T_{P_0} P_r \tilde{A}_{P_0} + \tilde{Q}_{P_0} - W_{P_0}(P_r), \; r = 0, 1, \dots.$$

Thus

$$P_0 - P_{r+1} = \tilde{A}^T_{P_0} (P_0 - P_r) \tilde{A}_{P_0} + N_0 + W_{P_0}(P_r). \tag{8}$$

According to assumption 3.1 we have $K_{P_0} - K_{P_r} \geq 0$ and then

$$W_{P_0}(P_r) = (K_{P_0} - K_{P_r})^T (R + B^T P_r B)(K_{P_0} - K_{P_r}).$$

Also it follows that $W_{P_0}(P_1) \geq 0$. Thus $P_0 - P_1 = N_0 + W_{P_0}(P_1) \geq 0$. Moreover,

$$\tilde{A}_{P_1} - \tilde{A}_{P_0} = B(K_{P_0} - K_{P_1}) \geq 0, \ (B \geq 0).$$

Since $\tilde{A}_{P_0} \geq 0$, it follows that $\tilde{A}_{P_1} \geq \tilde{A}_{P_0} \geq 0$.

Now, assume there exists a natural number r and the matrix sequence $P_0, P_1, \ldots, P_r$ is computed with the properties $P_j \in Dom\,\mathcal{M}$, $P_0 \geq P_j$ and $\tilde{A}_{P_j} \geq 0$ for $j = 1, \ldots, r$. Next, we are ready to compute the matrix P_{r+1} using (7) and we will prove properties (i), (ii) and (iii) for $i = r+1$.

The matrices $\tilde{A}_{P_r} + \tilde{Q}_{P_r}$ are nonnegative in (7), hence $P_{r+1} \geq 0$ and $R + B^T P_{r+1} B$ is nonsingular. We may then conclude that $P_j \in Dom\,\mathcal{M}$ and the right-hand side of (8) is nonnegative because $P_0 - P_r \geq 0$. Then $P_0 - P_{r+1} \geq 0$.

With regard to the nonnegativity of $\tilde{A}_{P_{r+1}}$, we have

$$\tilde{A}_{P_{r+1}} - \tilde{A}_{P_0} = B(K_{P_0} - K_{P_{r+1}}).$$

Moreover, assumption 3.1 and $B \geq 0$ lead to $\tilde{A}_{P_i} \geq \tilde{A}_{P_0} \geq 0$. Hence

$$\tilde{A}_{P_{r+1}} \geq 0.$$

Applying (7) for computing P_{r+2} , we establish that P_{r+2} is nonnegative. Therefore the computed, via (7), matrix sequence $\{P_i\}_{i=1}^{\infty}$ is bounded. ■

In next theorem we prove that the iteration (7) constructs a matrix Cauchy sequence that converges to a solution of (3) with a linear rate of convergence.

Theorem 3.2. *Assume that the conditions $(a), (b), (c)$ and (d) of theorem 3.1 are satisfied for a nonnegative matrix $P_0 \in Dom\,M$,*

$$a = \|A\|^2, \ \ b = \|B\|^2, \ \ p_0 = \|P_0\|, \ \ r_2 = \|R^{-1}\|, \ \ r_0 = \|R + B^T P_0 B\|,$$

and that

(i) $\tilde{a} = a(1 + b p_0 r_2)^2 < 1,$

$(ii) \beta = \tilde{a}(1 + 2 b p_0 r_0 r_2^2) \leq 1.$

Then the matrix sequence $\{P_i\}_{i=1}^{\infty}$ defined by (7) converges to a nonnegative symmetric matrix $\tilde{P}$ approximation to (3) with a rate of convergence $\beta = \frac{1-\tilde{a}}{\tilde{b}}$ and $0 \leq \tilde{P} \leq P_0$, or

$$\|P_{s+q} - P_s\| \leq \beta^s \|P_q - P_0\| < \frac{\beta^s}{1-\beta} \|P_1 - P_0\|$$

for all nonnegative integers s and q.

Proof. Following the course of the proof of theorem 3.1, it has been shown that $0 \leq P_i \leq P_0$ for all $i = 1, 2, \ldots$. Therefore, we have $(B \geq 0)$ and

$$R \leq R + B^T P_i B \leq R + B^T P_0 B$$

for $i = 1, 2, \ldots$. Thus

$$\|R + B^T P_i B\| \leq \|R + B^T P_0 B\| \quad and \quad \|R^{-1}\| \geq \|(R + B^T P_i B)^{-1}\|$$

for $i = 1, 2, \ldots$.

For integers s and $q > s$ we obtain:

$$P_{s+q} - P_s = \mathcal{M}(P_{s+q-1}) - \mathcal{M}(P_{s-1})$$
$$= \tilde{A}^T_{P_{S-1}} (P_{s+q-1} - P_{s-1}) \tilde{A}_{P_{S-1}} - W_{P_{S-1}}(P_{s+q-1}) .$$

Using

$$\tilde{A}_{P_{S-1}} = A - B(R + B^T P_{S-1} B)^{-1} B^T P_{S-1} A,$$

leads to the estimate

$$\|\tilde{A}_{P_{S-1}}\| \leq \|A\| \, (1 + \|B\|^2 \, \|P_{s-1}\| \, \|(R + B^T P_{s-1} B)^{-1}\| \,)$$
$$\leq \|A\| \, (1 + \|B\|^2 \, \|P_0\| \, \|R^{-1}\| \,) = \|A\| \, (1 + b p_0 \, r_2) \, .$$

If Y and Z are matrices from the sequence $\{P_i\}, i = 1, 2, \ldots$, then in order to estimate $\|W_Z(Y)\|$ we consider the difference $K_z - K_Y$ and following [8] we obtain

$$K_z - K_Y = (R + B^T Z B)^{-1} B^T (Z - Y) A$$
$$+ (R + B^T Y B)^{-1} B^T (Y - Z) B (R + B^T Z B)^{-1} B^T Y A \, .$$

Thus

$$\|K_z - K_Y\| \leq \|(R + B^T Z B)^{-1}\| \, \|B\| \, \|Z - Y\| \, \|A\|$$
$$+ \|(R + B^T Y B)^{-1}\| \, \|(R + B^T Z B)^{-1}\| \, \|B\|^3 \, \|Z - Y\| \, \|A\| \, \|Y\|$$
$$\leq \|R^{-1}\| \, \|B\| \, \|Z - Y\| \, \|A\| + \|R^{-1}\|^2 \, \|B\|^3 \, \|Z - Y\| \, \|A\| \, \|P_0\|$$
$$= \|R^{-1}\| \, \|B\| \, \|Z - Y\| \, \|A\| (1 + \|R^{-1}\| \, \|B\|^2 \, \|P_0\| \,) \, .$$

Further on, we conclude that

$$\|W_Z(Y)\| = \|K_Z - K_Y)^T (R + B^T Y B) (K_Z - K_Y)\|$$
$$\leq \|K_Z - K_Y\|^2 \, \|R + B^T P_0 B\|$$
$$\leq r_0 \, \|R^{-1}\|^2 \, \|B\|^2 \, \|Z - Y\|^2 \, \|A\|^2 (1 + \|R^{-1}\| \, \|B\|^2 \, \|P_0\| \,)^2$$
$$= r_0 \, r_2^2 \, b \, \tilde{a} \, \|Z - Y\|^2 \, .$$

Now we are ready to estimate $P_{s+q} - P_s$. Indeed

$$\|P_{s+q} - P_s\| \leq \|\tilde{A}_{P_{S-1}}\|^2 \, \|P_{s+q-1} - P_{s-1}\| + \|W_{P_{S-1}}(P_{s+q-1})\|$$
$$\leq \|\tilde{A}_{P_{S-1}}\|^2 \, \|P_{s+q-1} - P_{s-1}\| + r_0 \, r_2^2 \, b \, \tilde{a} \, \|P_{s+q-1} - P_{s-1}\|^2$$
$$\leq \|A\|^2 \, (1 + b p_0 \, r_2)^2 \, \|P_{s+q-1} - P_{s-1}\| + r_0 \, r_2^2 \, b \, \tilde{a} \, \|P_{s+q-1} - P_{s-1}\|^2$$
$$= \tilde{a} \, \|P_{s+q-1} - P_{s-1}\| + r_0 \, r_2^2 \, b \, \tilde{a} \, \|P_{s+q-1} - P_{s-1}\|^2$$
$$\leq \tilde{a} \, \|P_{s+q-1} - P_{s-1}\| + r_0 \, r_2^2 \, b \, \tilde{a} \, 2 \, \|P_0\| \, \|P_{s+q-1} - P_{s-1}\|$$
$$= (\tilde{a} + 2 \, r_0 \, r_2^2 \, b \, \tilde{a} \, p_0) \, \|P_{s+q-1} - P_{s-1}\|$$
$$\leq \beta \, \|P_{s+q-1} - P_{s-1}\| \leq \ldots \leq \beta^s \, \|P_q - P_0\| < \frac{\beta^s}{1-\beta} \, \|P_1 - P_0\| \, .$$

Thus the sequence $\{P_i\}_{i=1}^{\infty}$ is a Cauchy sequence, defined in $[0, P_0]$. Hence this sequence has a nonnegative limit $\tilde{P}$ and it is easy to check that $\tilde{P}$ is a solution to (3). The proof of the theorem is therefore complete. ■

4. A Numerical Example

Here we investigate the numerical behavior of the considered iteration (7) for finding the maximal solution to the discrete time nonnegative Riccati equation $P = \mathcal{M}(P),\ P \in Dom\,\mathcal{M}$. Based on this solution the optimal control for linear quadratic infinite discrete time control problem is obtained. An experiment will be carried out where we show how the iteration (7) works and we will apply theorem 3.2 to compute the convergence rate β defined in this theorem.

Real nonnegative matrices are used as coefficients of Riccati equation (3). All reported results are obtained by using the MATLAB platform. In order to apply the introduced iteration we have to choose the initial matrix P_0 such that the conditions of theorem 3.1 are satisfied. Our experience with similar iterative approaches motivates taking a matrix P_0 of the form $P_0 = \alpha e e^T$. The error of each iteration step is denoted by $Error_i = \|\mathcal{M}(P_i) - P_i\|$. The computations stop as soon as the inequality $Error_{It} \le tol = 1.e - 12$ holds true. This inequality is in fact used as a practical stopping criterion.

Example 4.1. The coefficient matrices A, B, R and C are:

$$A = \begin{bmatrix} 0.08 & 0.05 & 0.045 \\ 0.4 & 0.08 & 0.035 \\ 0.05 & 0.5 & 0.75 \end{bmatrix},\ B = \begin{bmatrix} 0.04 & 0.025 \\ 0.02 & 0.01 \\ 0.04 & 0.03 \end{bmatrix},$$

$$R = \begin{bmatrix} 1.05 & 0 \\ 0 & 1.0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\quad Q = C' * C.$$

Note that the matrices A and B are nonnegative, then apply iteration (7) for computing the maximal nonnegative solution to (3). Take further $P_0 = 2.85 e e^T$ to obtain

$$K_{P_0} = (R + B^T P_0 B)^{-1} B^T P_0 A = \begin{bmatrix} 0.1384 & 0.1646 & 0.2168 \\ 0.0945 & 0.1123 & 0.1480 \end{bmatrix} \ge 0.$$

In addition $P_0 - M(P_0) \ge 0$ and $\tilde{A}_{P_0} \ge 0$. The conditions of theorem 3.1 are fulfilled. We begin the computations with iteration (7). Error propagation is presented in Table 1. After 55 iteration steps, the computed nonnegative solution is

$$\tilde{P} = \begin{bmatrix} 2.3956 & 0.4373 & 1.5737 \\ 0.4373 & 1.8278 & 1.1522 \\ 1.5737 & 1.1522 & 2.6115 \end{bmatrix},$$

and the optimal nonnegative control is

$$\tilde{A}_{\tilde{P}} = A - B(R + B^T\tilde{P}B)^{-1}B^T\tilde{P}A = \begin{bmatrix} 0.0767 & 0.0440 & 0.0367 \\ 0.3984 & 0.0772 & 0.0311 \\ 0.0465 & 0.4936 & 0.7412 \end{bmatrix}.$$

According to theorem 3.2 for the rate of convergence we obtain $\beta = 0.9951$.

Table 1. Error propagation for Example 1.

Iteration	Error
1	1.0876e+0
5	0.0487e+0
10	0.0034e+0
20	1.6615e-5
30	8.1355e-8
40	3.9835e-10
52	6.7314e-13

5. Conclusion

We have illustrated that the maximal nonnegative solution of (3) can be found by using the proposed standard Newton iteration (7). A sufficient condition for obtaining a nonnegative symmetric solution has been derived. This leads to the nonnegative optimal control to the LQ optimal control problem for positive systems. The linear rate of convergence to (7) has been proved.

The pivot point of theorem 3.1 has been assumption 3.1. Note that in this work if there exists an integer s for which the inequality $K_{P_0} - K_{P_s} \geq 0$ is impaired, then the iteration stops.

The matrix P_{s-1} can be considered as an approximation to the maximal solution $\tilde{P}$. A prove of the statement in assumption 3.1 is however left as a subject for a future work.

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