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Fluid Queues Driven by Rogers-Ramanujan Birth and Death Processes With Catastrophes

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Abstract. *Fluid models are widely used in the performance analysis of Telecommunication Systems. In this paper, a fluid queue driven by a birth and death process with a catastrophe is discussed. We have taken a modest attempt to relate a fluid queue driven by a birth and death process with the Rogers-Ramanujan continued fraction. The stationary solution is expressed through this continued fraction and a solution for the buffer content of a fluid queue driven by an M/M/1 queue is obtained as a particular case. Further, we have analyzed the M/M/1 queue when there are no catastrophes.*

Key words : Fluid Queue, Birth and Death Single Server Queue, Rogers-Ramanujan Continued Fraction, M/M/1 Queue, Catastrophe, Laplace Transform, Inverse Laplace Transform.

AMS Subject Classifications : 60K25, 11Y65

1. Introduction

In this paper we shall study the stationary behavior of the content of a fluid reservoir which receives and releases fluid flows at rates which are determined by the actual state of birth and death process evolving in the background. Fluid models play a significant role in ATM networks since the variations on the cell levels are almost negligible compared to those on the most important burst levels.

One may refer to Elvalid and Mitra [8], Anick et al [2], Simonian and Virtoma [20] for further detailed studies. Sericola [19], Barbot and Sericola [3] and Parathasarathy et al [18] have analyzed the input flow into the fluid queue which is characterized by a Markov modulated input rate process. Fluid models driven by finite state space Markov processes that modulate the input rate in the fluid buffer have been analyzed by many authors (Anick et al [2], Coffman et al [6], Gaver et al [11], Mitra [14], [15], Low and Varaiya [13]). Virtoma and

Norros [22] have analyzed a fluid queue driven by an M/M/1(Markovian arrival process/Markovian service time/Single server) queue and proposed a spectral-decomposition method. Adan and Resing [1] have devised the key of the method, to express the generalized eigenvalues explicitly using the Chebyshev polynomials of the second kind. Explicit expressions for the stationary distribution of the buffer content for fluid queues driven by an M/M/1 queue with constant arrival service rates have been obtained by Van Doorn and Scheinhardt [21]. It was Parthasarathy et al [16] who provided for the first time a continued fraction method to analyze fluid queues. TCP is able to guarantee that each data packet transmitted from a server (computer)and leave it momentarily inactivated until the new arrival occurs, such infected cells may be modeled by catastrophes. Jain and Kumar [12] have obtained the transient solution of the model with correlated arrival queueing with variable capacity and catastrophes for the cell traffic generated by New Broadband Communication Networks in the presence of viruses and noise bursts. Chao [7] has studied a queueing network model with catastrophes. Many of the above studies have dealt with fluid models driven by a finite state space Markov process, also the solutions are either based on recurrence relations or expressed in terms of Laplace transforms and inverted numerically. Now the question arises on what will happen when the service rates, arrival rates and catastrophe rates are not independent. So we have to analyze such fluid models when (i) the above rates are independent and (ii) they are dependent. Rogers-Ramanujan identities arise naturally in the second case. It is well known that the Rogers-Ramanujan Continued Fraction is of the form

$$R(q) = \cfrac{1}{1-\cfrac{q}{1-\cfrac{q^2}{1-\cfrac{q^3}{1-\dots}}}}. \tag{1}$$

In [15], Parthasarathy et al have studied a birth and death process related to the continued fraction $R(q)$ and generalized their results to establish a correspondence between birth and death processes and a larger family of q –continued fractions discussed by Feng et al [9]. One may also refer to Berndt et al [5] for more details regarding the continued fractions.

As, there is no result on a fluid model driven by a birth and death Process (BDP) with a catastrophe wherein the arrival rate, service rate and the catastrophe rate are (i) independent and (ii) dependent, we have made a modest attempt to construct such a fluid model. A birth and death process related to Rogers-Ramanujan continued fraction has been discussed by many authors([4], [9], [16], [17], [23]). We analyze the fluid queue driven by birth-death process with a catastrophe under both independent and dependent rates. We intend to find the stationary distribution of the buffer content for a fluid queue driven by a BDP with a catastrophe and having a general boundary condition. This is obtained by transforming the underlying system of differential equations into a continued fraction and this continued fraction is employed in finding the complete solution. We show that, for a particular case, our results coincide with those of a system without a catastrophe.

The rest of this paper is organized as follows. We describe the model in Section 2. In Section 3, we obtain the stationary solution of this fluid queue driven by a birth and death processes queue with a catastrophe. In Section 4, we discuss the BDP related to Rogers-Ramanujan continued fractions. Some special birth and death processes with catastrophes are examined as particular cases in Section 5. By using a continued fraction we derive in Section 6, the stationary solution of this fluid queue driven by an M/M/1 queue in terms of modified Bessel's function of the first kind . In Section 7, we show, as a particular case, that our result coincides with that of Adan and Resing [1] and further we analyze the M/M/1 queue without catastrophes.

2. Model Description

Let $\{X(t), t \geq 0\}$ denote the background birth and death single-server queueing model taking values in $S \equiv \{0, 1, 2, 3, \dots\}$, where $X(t)$ denotes the state (number of customers in the queue) of the background process at time t. Let λ_n and μ_n denote the arrival and service rates respectively, when there are n customers in the queue. The arrival process is a Poisson process and the service times are exponentially distributed. Catastrophe occurs in the service station at a Poisson rate of ν_n, which annihilates the entire system empty, and we start working on the basis of next arrival. During the busy periods of the server, a fluid commodity which we refer to as credit accumulation in an infinite fluid buffer at a rate $r_j > 0$ when $X(t) = j$. The credit buffer depletes the fluid during the idle periods of the server at a constant rate $r_0 < 0$ as long as the buffer is nonempty. We denote by $C(t)$, the content of buffer at time t. Clearly, the 2-dimensional process $\{(X(t), C(t)), t \geq 0\}$ constitutes a Markov process which possesses a unique stationary distribution under a suitable stability condition. In order that a limit distribution for $C(t)$ exists as $t \to \infty$, the stationary net input rate should be negative, that is,

$$r_0 + r_j \sum_{j=1}^{\infty} \pi_j < 0, \tag{2}$$

where

$$\pi_j = \frac{(\lambda_0 + \nu_0)(\lambda_1 + \nu_1)(\lambda_2 + \nu_2)\dots(\lambda_{j-1} + \nu_{j-1})}{\mu_1 \mu_2 \dots \mu_j} - \nu_0 \frac{(\lambda_1 + \nu_1)(\lambda_2 + \nu_2)\dots(\lambda_{j-1} + \nu_{j-1})}{\mu_2 \dots \mu_j},$$

$j = 1, 2, 3, \dots,$ and $\pi_0 = 1$ are called the potential coefficients.

Remark 2.1. When there is no catastrophe $i.e. \nu_i \to 0$, we get $\pi_j = \frac{\lambda_0 \lambda_1 \lambda_2 \dots \lambda_{j-1}}{\mu_1 \mu_2 \dots \mu_j}$ and $\pi_0 = 1$ which are same as in usual birth and death model.

3. Steady State Equations

Let

$$F_j(t, x) \equiv P\{X(t) = j, C(t) \leq x\}, j \in S, \quad t, x \geq 0.$$

Then $\{F_j(t, x), j \in S, t, x \geq 0\}$ are joint probability distribution functions of the Markov process $(X(t), C(t)), t \geq 0$ at time t. When the process is stable, we write

$F_j(x) = \lim_{t\to\infty} F_j(t,x), j \in S, t, x \geq 0,$

which are independent of the initial state of the process.

Theorem 3.1. *Let a fluid model be driven by Birth and Death Processes with Catastrophe(BDPC) with independent rates λ_n(arrival), μ_n(service) and ν_n(catastrophe). Then the sequence $\{F_0(t,x), F_j(t,x)\}$ of joint distributions of the system $\{X(t), C(t)\}$ satisfies the following system of differential equations:*

$$\frac{\partial F_0(t,x)}{\partial t} = -r_0 \frac{\partial F_0(t,x)}{\partial x} - \lambda_0 F_0(t,x) + \mu_1 F_1(t,x) + \nu_0[1 - F_0(t,x)], \tag{3}$$

$$\frac{\partial F_j(t,x)}{\partial t} = -r_j \frac{\partial F_j(t,x)}{\partial x} + \lambda_{j-1} F_{j-1}(t,x) - (\lambda_j + \mu_j + \nu_j) F_j(t,x) + \mu_{j+1} F_{j+1}(t,x), \tag{4}$$

$j = 1,2,3,\ldots,\ t \geq 0,\ x \geq 0,$ *subject to the initial conditions:*

$$F_0(0,x) = 1,\ F_j(0,x) = 0,\ \textit{for}\ j = 1,2,3,\ldots, \tag{5}$$

and boundary conditions:

$$F_j(t,0) = q_j(t),\ \textit{for}\ j = 0,1,2,3,\ldots. \tag{6}$$

Proof. The proof follows by using standard probabilistic arguments, as in [22]. ■

Remark 3.1. Here $q_j(t)$ represents the probability that at time t the buffer is empty and the state of the background Markov process is j. The condition of the buffer decreases and thereby becomes empty only when the net input rate of the fluid into the buffer is negative. Therefore when the buffer becomes empty at any time t, the background process should necessarily be in state zero corresponding to which the effective input rate is $r_0 < 0$. Hence we have $q_j(t) = 0$ for $j = 1,2,3,\ldots$ and $r_j > 0$ for $j = 1,2,3,\ldots$.

When the process is in equilibrium, $\frac{\partial F_j(t,x)}{\partial t} \equiv 0$, we let $F_j(t,x) \equiv F_j(x)$. Then the above system reduces to a system of ordinary differential equations:

$$\frac{d}{dx} F_0(x) = -\frac{\lambda_0}{r_0} F_0(x) + \frac{\mu_1}{r_0} F_1(x) + \frac{\nu_0}{r_0}[1 - F_0(x)], \tag{7}$$

$$\frac{d}{dx} F_j(x) = \frac{\lambda_{j-1}}{r_j} F_{j-1}(x) - \frac{\lambda_j + \mu_j + \nu_j}{r_j} F_j(x) + \frac{\mu_{j+1}}{r_j} F_{j+1}(x),\ x \geq 0,\ j = 1,2,3,\ldots. \tag{8}$$

When the net input rate of fluid flow into the buffer is positive, the buffer content increases and buffer cannot stay empty. It follows that the solution to (3.1) and (3.2) must satisfy the boundary conditions

$$F_j(0) = 0,\ j = 1,2,3,\ldots \tag{9}$$

$$F_0(0) = a, \quad \text{for some constant } 0 < a < 1. \tag{10}$$

Let $\hat{F}(s)$ denote the Laplace transform of the function $F(t)$ and define

$f_0(s) := \hat{F}_0(s),$

$$f_j(s) := \frac{(-1)^j \mu_1 \mu_2 \dots \mu_j}{r_0 r^{j-1}} \hat{F}_j(s), \text{ where } r_j = r, \; j = 1,2,3,\dots, \tag{11}$$

with

$$\hat{\phi}_j(s) = \cfrac{-\lambda_{j-1}\mu_j/r_j}{s + \cfrac{\lambda_j + \mu_j + \nu_j}{r_j} - \cfrac{\cfrac{\lambda_j \mu_{j+1}}{r_j r_{j+1}}}{s + \cfrac{\lambda_{j+1} + \mu_{j+1} + \nu_{j+1}}{r_{j+1}} - \cfrac{\cfrac{\lambda_{j+1}\mu_{j+2}}{r_{j+1}r_{j+2}}}{s + \cfrac{\lambda_{j+2} + \mu_{j+2} + \nu_{j+2}}{r_{j+2}} - \cdots}}} \quad j = 1,2,\dots, \tag{12}$$

to state the theorem that follows.

Theorem 3.2. *The Laplace transforms of the steady state probability distribution function of the Fluid model driven by BDPC with independent rates* λ_n*(arrival),* μ_n*(service) and* ν_n *(catastrophe) are*

$$f_0(s) = (a + \nu_0/r_0 s) \sum_{k=0}^{\infty} \left(\frac{1}{r_0}\right)^k \frac{(\hat{\phi}_j(s))^k}{\left(s + \frac{\lambda_0 + \nu_0}{r_0}\right)^{k+1}}, \tag{13}$$

and

$$f_j(s) = \prod_{k=1}^{j} \hat{\phi}_k(s)(s) f_0(s), \; j = 1,2,3,\dots. \tag{14}$$

Proof. On taking the Laplace transforms of (7) and (8) , using (9)-(10) we get

$$[r_0 s + (\lambda_0 + \nu_0)]\hat{F}_0(s) - \mu_1 \hat{F}_1(s) = r_0 a + \frac{\nu_0}{s} \tag{15}$$

$$(r_j s + \lambda_j + \mu_j + \nu_j)\hat{F}_j(s) - \lambda_{j-1}\hat{F}_{j-1}(s) - \mu_{j+1}\hat{F}_{j+1}(s) = 0, \; j = 1,2,3,\dots. \tag{16}$$

In terms of the $f_j(s)'s,$ we rewrite equations (15) and (16) as

$$(r_0 s + \lambda_0 + \nu_0) f_0(s) + f_1(s) = r_0 a + \frac{\nu_0}{s} \tag{17}$$

$$(r_1 s + \lambda_1 + \mu_1 + \nu_1) f_1(s) + \lambda_0 \mu_1 f_0(s) + f_2(s) = 0, \tag{18}$$

$$(r_j s + \lambda_j + \mu_j + \nu_j) f_j(s) + \lambda_{j-1}\mu_j f_{j-1}(s) + f_{j+1}(s) = 0, \; j = 2,3,\dots. \tag{19}$$

We again write the above system of equations as

$$f_0(s) = \frac{r_0 a + (\nu_0/s)}{r_0 s + \lambda_0 + \nu_0 + (f_1(s)/f_0(s))} \tag{20}$$

$$\frac{f_1(s)}{f_0(s)} = \frac{-\lambda_0 \mu_1}{r_1 s + \lambda_1 + \mu_1 + \nu_1 + (f_2(s)/f_1(s))} \tag{21}$$

$$\frac{f_j(s)}{f_{j-1}(s)} = \frac{-\lambda_{j-1}\mu_j}{r_j s + \lambda_j + \mu_j + \nu_j + (f_{j+1}(s)/f_j(s))}, \; j = 2,3,\dots. \tag{22}$$

We, thus, represent $f_0(s)$ as a continued fraction:

$$f_0(s) = \frac{r_0 a + \frac{\nu_0}{s}}{(r_0 s + \lambda_0 + \nu_0) -} \frac{\lambda_0 \mu_1}{(r_1 s + \lambda_1 + \mu_1 + \nu_1) - \ldots} \tag{23}$$

and

$$\frac{f_j(s)}{f_{j-1}(s)} = \frac{-\lambda_{j-1}\mu_j}{(r_j s + \lambda_j + \mu_j + \nu_j) -} \frac{\lambda_j \mu_{j+1}}{(r_{j+1} s + \lambda_{j+1} + \mu_{j+1} + \nu_{j+1}) -} \frac{\lambda_{j+1}\mu_{j+2}}{(r_{j+2} s + \lambda_{j+2} + \mu_{j+2} + \nu_{j+2}) - \ldots}, \; j = 1, 2, \ldots \tag{24}$$

$$\frac{f_j(s)}{f_{j-1}(s)} = \frac{-\lambda_{j-1}\mu_j / r_j}{s + \frac{\lambda_j + \mu_j + \nu_j}{r_j} -} \frac{\frac{\lambda_j \mu_{j+1}}{r_j r_{j+1}}}{s + \frac{\lambda_{j+1} + \mu_{j+1} + \nu_{j+1}}{r_{j+1}} -} \frac{\frac{\lambda_{j+1}\mu_{j+2}}{r_{j+1} r_{j+2}}}{s + \frac{\lambda_{j+2} + \mu_{j+2} + \nu_{j+2}}{r_{j+2}} - \cdots}, \; j = 1, 2, \ldots \tag{25}$$

Then from (23)-(24) using (12),we have

$$f_0(s) = \frac{a + \nu_0 / r_0 s}{s + \frac{\lambda_0 + \nu_0}{r_0} - \frac{1}{r_0}\hat{\phi}_1(s)}, \tag{26}$$

$$\frac{f_j(s)}{f_{j-1}(s)} = \hat{\phi}_j(s), \; j = 1, 2, 3, \ldots. \tag{27}$$

After some algebraic simplifications, we obtain (13) and (14). ■

Theorem 3.3. *For every $t \geq 0$ and $x \in [0, r_i t)$, the stationary probabilities of the Fluid model driven by BDPC with independent rates λ_n (arrival), μ_n (service) and ν_n are*

$$\begin{aligned} f_0(x) = {} & a\, e^{-\frac{(\lambda_0 + \nu_0)}{r_0} x} + \frac{\nu_0}{r_0} \int_0^x e^{-\frac{(\lambda_0 + \nu_0)}{r_0} y} \mathrm{d}y \\ & + a \sum_{k=0}^{\infty} \left(\frac{1}{r_0}\right)^k \frac{x^k}{k!} e^{-\frac{(\lambda_0 + \nu_0)}{r_0} x} \phi_1^{*k}(x) \\ & + \sum_{k=0}^{\infty} \left(\frac{1}{r_0}\right)^k \int_0^x \frac{y^k e^{-\frac{(\lambda_0 + \nu_0)}{r_0} y}}{k!} \phi_1^{*k}(x - y) \mathrm{d}y, \end{aligned} \tag{28}$$

and

$$f_0(x) = \phi_1(x) * \phi_2(x) * \phi_3(x) * \cdots * \phi_j(x) * f_0(x),\ j = 1, 2, 3, \ldots \tag{29}$$

Proof. We know that

$$\mathcal{L}_s^{-1}\left(\frac{1}{\left(s + \frac{\lambda_0 + \nu_0}{r_0}\right)^{k+1}} \right) = \frac{x^k}{k!} e^{-\frac{(\lambda_0 + \nu_0)}{r_0} x}. \tag{30}$$

And

$$\mathcal{L}_s^{-1}\left(\frac{1}{s\left(s + \frac{\lambda_0 + \nu_0}{r_0}\right)^{k+1}} \right) = \int_0^x \frac{u^k e^{-\frac{(\lambda_0 + \nu_0)u}{r_0}}}{k!} \mathrm{d}u = h_k(x), \tag{31}$$

say. On inverting (13) and (14) using the above Laplace inverse formulae, we get the results (28) and (29) of the theorem. ■

4. Rogers-Ramanujan Birth and Death Processes

In all cases, in the literature, λ_n and μ_n are independent. In practical situations the service rate μ_n depends on the arrival rate λ_n and catastrophe rate ν_n. This forces us to state the definition that follows.

Definition 4.1. A continuous time stochastic process $\{X(t), t \geq 0\}$ is called Rogers-Ramanujan Birth and Death Process (RRBDP) if the arrival rate λ_n, and the service rates μ_n are not independent and satisfies the usual Kolmogorov system of differential-difference equations.

In this paper we consider RRBDP with catastrophe where the arrival rate, service rate and catastrophe rate are not independent of one another. Specifically, consider a RRBDP with catastrophe process whose rates satisfy the conditions.

$$\lambda_n + \mu_n + \nu_n = 1 \tag{32}$$

$$\lambda_{n-1}\mu_n = q^n, \qquad n = 1, 2, 3, \ldots. \tag{33}$$

with $\mu_0 = 0 = \nu_0$ and $\lambda_0 = 1$.

With this assumption, we have the following result.

Theorem 4.1. *For every $t \geq 0$ and $x \in [0, r_i t)$, the stationary probabilities of the Fluid Model driven by RRBDP with catastrophe are*

$$f_0(x) = ae^{-\frac{1}{r_0}x} + a \sum_{k=1}^{\infty} \left(\frac{q}{r_0}\right)^k \int_0^x \frac{y^k e^{-\frac{1}{r_0}y}}{k!} \phi_1^{*k}(x-y)dy \tag{34}$$

and

$$f_j(x) = q\phi_1(x) * q^2\phi_2(x) * q^3\phi_3(x) * \ldots * q^j\phi_j(x) * F_0(x), \ \ j = 1,2,\ldots. \tag{35}$$

Proof. From (23), using (32) and (33), we have

$$f_0(s) = \frac{r_0 a}{r_0 s + 1 -}\,\frac{q}{r_1 s + 1 -}\,\frac{q^2}{r_2 s + 1 - \ldots} \tag{36}$$

which is the Rogers-Romanujan continued fraction (1) when $s = 0$ and $a = 1/r_0$, hence justifying the fact that the birth and death process under consideration is related to the Rogers-Ramanujan continued fraction (1).
From (24), using (32) and (33), we have

$$\frac{f_j(s)}{f_{j-1}(s)} = \frac{-q^j}{r_j s + 1 -}\,\frac{q^{j+1}}{r_{j+1} s + 1 -}\,\frac{q^{j+2}}{r_{j+2} s + 1 - \ldots}. \tag{37}$$

Make use then of (12) with (32) and (33) to obtain

$$\hat{\phi}_j(s) = \frac{-q^j}{r_j s + 1 -}\,\frac{q^{j+1}}{r_{j+1} s + 1 -}\,\frac{q^{j+2}}{r_{j+2} s + 1 - \ldots}. \tag{38}$$

Moreover, from (37) we have

$$\frac{f_j(s)}{f_{j-1}(s)} = \hat{\phi}_j(s), \ \ j = 1,2,3,\ldots. \tag{39}$$

Using (39) in equation (36) finally yields

$$f_0(s) = \frac{a}{s + \frac{1}{r_0} + \hat{\phi}_1(s)} = a\sum_{k=0}^{\infty}\left(\frac{1}{r_0}\right)^k \frac{(\hat{\phi}_1(s))^k}{(s + \frac{1}{r_0})^{k+1}} \tag{40}$$

and

$$f_j(s) = \left(\prod_{k=1}^{j}(\hat{\phi}_k(s)\right) f_0(s), \ \ j = 1,2,3,\ldots. \tag{41}$$

Inversion of (40)-(41) using (30) and (31) completes this proof. ■

5. Special Rogers-Ramanujan Birth-Death Processes with Catastrophes

We use the Pochhammer symbol for our discussion to let

$$(a;q)_n = \sum_{k=0}^{n-1} (1-aq^k),\ (a;q)_0 = 1,$$

and

$$(a;q)_\infty = \sum_{k=0}^{\infty}(1-aq^k),\ |q|< 1.$$

In terms of this notation, we represent the Rogers-Ramanujan Identity as

$$R(q) = \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^2;q^5)_\infty}.$$

Here we exhibit some examples of continued fractions of Ramanujan and Jacobi which correspond to birth-death processes with catastrophe fluid queueing models of positive rates $\lambda_n,\ \mu_n,$ and ν_n. So we define $\lambda_0 = 1,\ \mu_0 = 0$ and $\nu_0 = 0$.

a) Let $\{\lambda_n\}$,$\{\mu_n\}$ be sequences of real numbers of BDP-1 satisfying

$$b_n = \lambda_n + \mu_n + \nu_n = 1 + q^n,\ \ n = 0,1,2,3,\dots,$$

$$a_n = \lambda_{n-1}\mu_n$$

$$= q^{2n-1},\ \ n = 1,2,3,\dots.$$

The corresponding continued fraction for the model BDP-1 is

$$f_1(q) = \frac{1}{1-}\,\frac{q}{1+q-}\,\frac{q^2}{1+q^2-}\cdots \tag{42}$$

$$= \frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}. \tag{43}$$

This continued fraction is Entry 10 in Ramanujan's second Notebook [3, p.20].

b) Let $\{\lambda_n\}$,$\{\mu_n\}$ be sequences of real numbers of BDP-2 satisfying

$$b_n = \lambda_n + \mu_n + \nu_n = 1 + q^{2n},\ \ n = 0,1,2,3,\dots,$$

$$a_n = \lambda_{n-1}\mu_n,\ n = 0,1,2,3,\dots,$$

$$= q^{2n-1},\ \ n \geq 1.$$

The corresponding continued fraction for the model BDP-2 is

$$f_2(q) = \frac{1}{1-}\,\frac{q}{1+q^2-}\,\frac{q^3}{1+q^4-}\cdots \tag{44}$$

$$= \frac{(q^3;q^4)_\infty}{(q;q^4)_\infty}. \tag{45}$$

This continued fraction is Entry 11 in Ramanujan's second Notebook[3, p.22].

c) Let $\{\lambda_n\}$,$\{\mu_n\}$ be sequences of real numbers of the model BDP-3 satisfying

$b_n = \lambda_n + \mu_n + \nu_n = 1 + q^{2n+1}, \; n = 0, 1, 2, 3, \ldots,$
$a_n = \lambda_{n-1}\mu_n, \; n = 0, 1, 2, 3, \ldots,$
$\quad = q^{2n+1}, \; n = 1, 2, 3, \ldots.$

The corresponding continued fraction for the model BDP-3 is

$$f_3(q) = \frac{1}{1-}\,\frac{q^3}{1+q^3-}\,\frac{q^5}{1+q^5-}\ldots. \tag{46}$$

Note the relation between $f_3(q)$ and the classical Jacobi theta function (see Folsom [10]) is

$$\vartheta(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2qf_3(q) \tag{47}$$

d) Let $\{\lambda_n\}$,$\{\mu_n\}$ be sequences of real numbers of BDP-4 satisfying
$b_n = \lambda_n + \mu_n + \nu_n = 1 + q^{2n}, \; n = 0, 1, 2, 3, \ldots,$
$a_n = \lambda_{n-1}\mu_n, \; n = 0, 1, 2, 3, \ldots,$
$\quad = q^{2n}, \; n = 1, 2, 3, \ldots.$

The corresponding continued fraction birth-death for the model BDP-4

$$f_4(q) = \frac{1}{1-}\,\frac{q^2}{1+q^2-}\,\frac{q^4}{1+q^4-}\ldots \tag{48}$$

Note the relation between $f_4(q)$ and the classical Jacobi theta function (see Folsom [10]) is

$$\vartheta_2(q) = \sum_{n=-\infty}^{\infty} q^{((n+1)/2)^2} = 2q^{1/4}f_4(q)\,. \tag{49}$$

6. Fluid Model Driven by an M/M/1 Queue With a Catastrophe

In this section, we consider the fluid model discussed in Section 2 with the background process as an $M/M/1$ queue with independent rates λ(arrival), μ(service), and ν(catastrophe). Also, let r_i be r for $i = 1, 2, 3, \ldots.$

Theorem 6.1. *The Laplace transforms of the steady state probability distribution functions of a fluid model driven by M/M/1 queueing system with catastrophe are*

$$f_0(s) = \frac{a + \nu/r_0 s}{s + \frac{\lambda+\nu}{r_0}} + \left(a + \frac{\nu}{r_0 s}\right)\sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \frac{[p - \sqrt{p^2-\alpha^2}\,]^k}{\left(s + \frac{\lambda+\nu}{r_0}\right)^{k+1}} \tag{50}$$

$$\hat{F}_j(s) = \left(a + \frac{\nu}{r_0 s}\right)\left(\frac{r}{2\mu}\right)^j \sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \frac{[p - \sqrt{p^2 - \alpha^2}\,]^{j+k}}{\left(s + \frac{\lambda+\nu}{r_0}\right)^{k+1}}, j = 1, 2, 3, \ldots. \tag{51}$$

Proof. Since $\pi_i = \rho^i - \nu\rho^{i-1}$, with $\rho \equiv (\lambda+\nu)/\mu$, it follows from (2) that

$$\rho < \frac{r_0 - r\nu}{r_0 - r}. \tag{52}$$

Substitute $\lambda_j = \lambda, \mu_j = \mu$ and $\nu_j = \nu$ in (23) to get

$$f_0(s) = \cfrac{a + \frac{\nu}{r_0 s}}{s + \frac{\lambda+\nu}{r_0} -} \cfrac{\frac{\lambda\mu}{r_0 r}}{s + \frac{\lambda+\mu+\nu}{r} -} \cfrac{\frac{\lambda\mu}{r^2}}{s + \frac{\lambda+\mu+\nu}{r} - \ldots}$$

which can be written as

$$f_0(s) = \frac{a + \nu/r_0 s}{s + \frac{\lambda+\nu}{r_0} - \frac{\lambda\mu}{r_0 r} f(s)}, \tag{53}$$

where

$$f(s) = \cfrac{1}{s + \frac{\lambda+\mu+\nu}{r} -} \cfrac{\lambda\mu/r^2}{s + \frac{\lambda+\mu+\nu}{r} - \ldots} = \frac{1}{s + \frac{\lambda+\mu+\nu}{r} - \frac{\lambda\mu}{r^2} f(s)}, \tag{54}$$

which results in a quadratic relation for $f(s)$, whose negative root is

$$f(s) = \frac{[p - \sqrt{p^2 - \alpha^2}\,]}{\alpha^2/2}, \tag{55}$$

where $p = s + \frac{\lambda+\mu+\nu}{r}, \alpha = 2\sqrt{\lambda\mu}/r$. By substituting (55) in (54), pulling out the term $s + \frac{\lambda+\mu+\nu}{r_0}$ from the denominator and expanding binomially, we get the first result.

Furthermore, substitution of $\lambda_j = \lambda, \mu_j = \mu$ and $\nu_j = \nu$ in (23) and (24), leads to

$$f_1(s) = \frac{-\lambda\mu}{r_0 r} f_0(s) f(s), \tag{56}$$

$$f_j(s) = \frac{-\lambda\mu}{r^2} f_{j-1}(s) f(s), \ \ j = 2, 3, \ldots \tag{57}$$

which can be put together as

$$f_j(s) = \frac{r}{r_0}\left(-\frac{\lambda\mu}{r^2}\right)^j (f(s))^j f_0(s), \ \ j = 1, 2, 3, \ldots. \tag{58}$$

Since by (11) we have

$$\hat{F}_j(s) = \left(\frac{\lambda}{r}\right)^j (f(s))^j f_0(s), \ j = 0,1,2,\ldots, \tag{59}$$

then substitution of (55) and (53) for $f(s)$ and $f_0(s)$ respectively, allows equation (59) to be written as the second result of this theorem. ■

Remark 6.1. When there is no catastrophe, i.e. $\nu \to 0,$ we have the stationary solution as

$$\hat{F}_o(s) = \frac{a}{s+\frac{\lambda}{r_0}} + a\sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \frac{[p-\sqrt{p^2-\alpha^2}\,]^k}{\left(s+\frac{\lambda}{r_0}\right)^{k+1}} \tag{60}$$

$$\hat{F}_j(s) = a\left(\frac{r}{2\mu}\right)^j \sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \frac{[p-\sqrt{p^2-\alpha^2}\,]^{j+k}}{\left(s+\frac{\lambda}{r_0}\right)^{k+1}}, \ j = 1,2,3,\ldots, \tag{61}$$

which are same as in equations (15) and (19) of Parthasarathy et al [16].

By inverting the results of Theorem 4.1, we arrive at the stationary probabilities as follows.

Theorem 6.2. *For every $t \geq 0$ and $x \in [0, rt),$ the stationary probabilities of the fluid model driven M/M/1 queue with a catastrophe having independent rates λ, μ and ν are*

$$F_0(x) = ae^{\frac{-(\lambda+\nu)x}{r_0}} + \frac{\nu(1+e^{-\frac{\lambda+\nu}{r_0}})}{\lambda+\nu}$$

$$+ ae^{-\frac{(\lambda+\mu+\nu)x}{r}} \sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \frac{\alpha^k}{\Gamma k}\int_0^x \frac{y^k I_k(\alpha(x-y))e^{\frac{\lambda+\mu+\nu}{r}-\frac{\lambda+\nu}{r_0}y}}{x-y}\mathrm{d}y$$

$$+ \frac{\nu}{r_0}\sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{(\lambda+\mu+\nu)(x-y)}{r}} \frac{\alpha^k I_k(\alpha(x-y))}{\Gamma k(x-y)} h_k(y)\mathrm{d}y. \tag{62}$$

For $j = 1,2,3,\ldots$ and $x \geq 0$

$$F_j(x) = a\left(\frac{r}{2\mu}\right)^j e^{-\frac{(\lambda+\mu+\nu)x}{r}} \sum_{k=0}^{\infty}\left(\frac{r}{2r_0}\right)^k \frac{\alpha^{j+k}(j+k)}{\Gamma k+1}$$

$$\int_0^x \frac{y^k I_{j+k}(\alpha(x-y))e^{\frac{\lambda+\mu+\nu}{r}-\frac{\lambda+\nu}{r_0}y}}{x-y}\mathrm{d}y$$

$$+ \frac{\nu}{r_0}\sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{(\lambda+\mu+\nu)(x-y)}{r}} \frac{\alpha^k I_k(\alpha(x-y))}{\Gamma k+1(x-y)} h_k(y)\mathrm{d}y. \tag{63}$$

Proof. To invert $\hat{F}_j(s)$ for $j = 1,2,3,\ldots$ w. r. t. $s,$ consider

$$f_0(s) = \frac{a+\nu/r_0 s}{s+\dfrac{\lambda+\nu}{r_0}} + \left(a+\frac{\nu}{r_0 s}\right)\sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \frac{[p-\sqrt{p^2-\alpha^2}\,]^k}{\left(s+\dfrac{\lambda+\nu}{r_0}\right)^{k+1}}.$$

Obviously

$$\mathcal{L}_s^{-1}((\theta-\sqrt{\theta^2-\alpha^2}\,)^k) = \mathcal{L}_s^{-1}((s+\frac{\lambda+\nu+\mu}{r}) - \sqrt{(s+(\lambda+\nu+\mu)/\,r)^2-\alpha^2})^k$$

$$= e^{-\dfrac{(\lambda+\nu+\mu)x}{r}}\mathcal{L}_s^{-1}((s-\sqrt{s^2-\alpha^2}\,))^k \qquad (64)$$

$$= e^{-\dfrac{(\lambda+\nu+\mu)x}{r}}\frac{\alpha^k k I_k(\alpha x)}{x}, \qquad (65)$$

$$\mathcal{L}_s^{-1}\left(\frac{1}{s+\dfrac{\lambda+\nu}{r_0}}\right)^k = \frac{x^k e^{-\dfrac{(\lambda+\nu)x}{r_0}}}{k!}, \qquad (66)$$

$$\mathcal{L}_s^{-1}\left(\frac{F(s)}{s}\right) = \int_0^x f(u)\mathrm{d}u, \qquad (67)$$

and

$$\mathcal{L}_s^{-1}\left(\frac{1}{s(s+\dfrac{\lambda+\nu}{r_0})^{k+1}}\right) = \int_0^x \frac{u^k e^{-\dfrac{(\lambda+\nu)u}{r_0}}}{k!}\mathrm{d}u = h_k(x), \qquad (68)$$

say.
Using (64), (66), (67) and (68), on inverting (50) w. r. t. the s variable, we get the first result of Theorem 6.2 for $x \geq 0$.

Now, to obtain other probabilities $F_j(x), j = 1,2,3,\ldots,$ we consider

$$\hat{F}_j(s) = \left(a+\frac{\nu}{r_0 s}\right)\left(\frac{r}{2\mu}\right)^j \sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \frac{[p-\sqrt{p^2-\alpha^2}\,]^k}{\left(s+\dfrac{\lambda+\nu}{r_0}\right)^{k+1}}, \text{for } j = 1,2,3,\ldots.$$

By similar arguments as before, inverting $\hat{F}_j(s)$ w.r.t. s yields the second result of this theorem. ■

Remark 6.2. When there is no catastrophe, i.e. $\nu \to 0,$ we have the stationary solution as

$$F_0(x) = ae^{\dfrac{-\lambda x}{r_0}} + ae^{-\dfrac{(\lambda+\mu)x}{r}}\sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k$$

$$\times \frac{\alpha^k}{\Gamma k}\int_0^x \frac{y^k I_k(\alpha(x-y))e^{(\dfrac{\lambda+\mu}{r}-\dfrac{\lambda}{r_0})y}}{x-y}\mathrm{d}y. \qquad (69)$$

For $j = 1,2,3,\ldots$ and $x \geq 0,$ we have

$$F_j(x) = a\left(\frac{r}{2\mu}\right)^j e^{-\frac{(\lambda+\mu)x}{r}} \sum_{k=0}^{\infty}\left(\frac{r}{2r_0}\right)^k \frac{\alpha^{j+k}(j+k)}{\Gamma k+1}$$

$$\int_0^x \frac{y^k I_{j+k}(\alpha(x-y))e^{(\frac{\lambda+\mu}{r}-\frac{\lambda}{r_0})y}}{x-y}\,\mathrm{d}y \tag{70}$$

which are the same as in (15) and (19) of Parthasarathy et al [16].

7. Buffer Content Probability

In this section we obtain the Laplace transform of the probability distribution of the buffer content of the fluid queue with a catastrophe.

Theorem 7.1. *Let $F(x) \equiv Pr(C \leq x) = \sum_{j=0}^{\infty} F_j(x)$. be the probability distribution of the buffer content of a fluid queue driven by an M/M/1 queue with catastrophe having independent rates $\lambda, \mu,$ and ν. Then the Laplace Transform $\hat{F}(s)$ of $F(x)$ is*

$$\hat{F}(s) = \left(\frac{ar_0\, s+\nu}{2s(rs+\nu)}\right)$$

$$\times\left(\frac{r\, s^2(r_0-r)+\lambda s(r_0-r)+\nu s(r_0-r)-\mu s(r_0+r)+\eta s(r_0-r)-2\nu\mu}{r_0\, s^2(r_0-r)+\lambda s(r_0-r)+\nu s(r_0-r)-s\mu\, r_0-\nu\mu}\right). \tag{71}$$

Proof. Using (59), the Laplace transform $\hat{F}(s)$ of $F(x)$ is

$$\hat{F}(s) = \frac{f_0(s)}{1-(\lambda/r)f(s)}. \tag{72}$$

According to (53) and (55), the above relation can be written as

$$\hat{F}(s) = \frac{4r_0(a+\nu/r_0 s)\mu}{[(2r_0-r)s+\lambda-\mu+\nu+\eta][\eta-(rs+\lambda+\nu-\mu)]},$$

where $\eta = \sqrt{(rs+\lambda+\nu+\mu)^2-4\lambda\mu}$. By multiplying and dividing the above expression by the term $(rs+\lambda+\nu-\mu)+\eta,$ we obtain,

$$\hat{F}(s) = \left(\frac{r_0(a+\nu/r_0 s)(rs+\lambda+\nu-\mu)+\eta}{rs[(2r_0-r)s+\lambda-\mu+\nu+\eta]}\right). \tag{73}$$

Further Multiplication and division of the above relation by $(2r_0-r)s+\lambda-\mu+\nu)-\eta$ leads, after some manipulations, to the required result. ■

Remark 7.1. As $\nu \to 0$ in (71), the above equation reduces to

$$\hat{F}(s) = \left(\frac{a\, r_0}{2\, rs}\right)\left(\frac{r\, s(r_0 - r) + \lambda(r_0 - r) - \mu(r_0 + r) + \eta(r_0 - r)}{r_0\, s(r_0 - r) + \lambda(r_0 - r) - \mu r_0}\right), \tag{74}$$

which coincides with (22) of Parthasarathy et al [16].

Remark 7.2. When $r_0 = -1$ and $r = 1,$ equation (71) becomes

$$\hat{F}(s) = \left(\frac{a\, s - \nu}{s + \nu}\right)\left(\frac{s(s + \lambda + \nu) + \mu\nu + s\sqrt{(rs + \lambda + \nu + \mu)^2 - 4\lambda\mu}}{s(s(2s - 2\lambda - 2\nu + \mu) - \mu\nu)}\right). \tag{75}$$

The Laplace-Stieltjes transform $\hat{F}(s)$ of $F(x)$ is therefore given by

$$\hat{F}(s) = \left(\frac{a\, s - \nu}{s + \nu}\right)\left(\frac{s(s + \lambda + \nu) + \mu\nu + s\sqrt{(rs + \lambda + \nu + \mu)^2 - 4\lambda\mu}}{s(2s - 2\lambda - 2\nu + \mu) - \mu\nu}\right). \tag{76}$$

Obviously, when there is no catastrophe, i.e. $\nu \to 0,$ the last relation becomes

$$\hat{F}(s) = \frac{a[s + \lambda + \sqrt{(s + \lambda + \mu)^2 - 4\lambda\mu}\,]}{2s - 2\lambda + \mu}. \tag{77}$$

Furthermore, under the assumption that $a = (1 - 2\rho),$ the above result happens to coincide with $T^*(s)$ in Adan and Resing [1].

Remark 7.3. As $\nu \to 0$ in equation (73), the resulting relation coincides with equation (21) of Parthasarathy et al [16],

$$\hat{F}(s) = \left(\frac{r_0 a(rs + \lambda - \mu) + \eta}{rs[(2r_0 - r)s + \lambda - \mu + \eta]}\right).$$

8. Conclusion

This paper reveals the amazing connection between fluid queueing models and the legendry Rogers-Ramanujan Continued fraction in a novel way to get elegant results. In this paper, a fluid model driven by birth and death processes (BDPs) with catastrophes is discussed. A system of differential equations satisfied by the fluid model has been set up. The Laplace transform of the stationary-state probability distribution of the fluid queue is obtained through continued fractions and a stationary solution for an M/M/1 fluid queue is obtained. Also birth and death related to the Rogers-Ramanujan continued fraction is discussed as an initial step and some special BDPs with a catastrophe are discussed as particular cases. The pertaining system has moreover been studied, when there are no catastrophes, and hence confirmed the respective results obtained by other researchers.

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