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Local Time of Weight Fractional Brownian Motion

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Abstract. We study the existence, the joint continuity and the Hölder regularity of the local time of weight fractional Brownian motion $\xi = \{\xi_t,\ t \geq 0\}$.

Key words : Local Time, Joint Continuity, Hölder Regularity, Weight Fractional Brownian Motion.

AMS Subject Classifications : 60H10, 93E20

1. Introduction

The purpose of this paper is to investigate local times and some related sample paths for Weight fractional Brownian motion (in short WfBm). The WfBm with parameters *a* and *b* is a centered Gaussian process $\xi = \{\xi_t,\ t \geq 0\}$ with a covariance function

$$\mathbb{E}(\xi_s \xi_t) = \int^{t \wedge s} u^a[(t-u)^b + (s-u)^b]du,\ \ s,\ t \geq 0, \tag{1}$$

where $a > -1$, $-1 < b \leq 1$, $|b| \leq 1 + a$. If $a = 0$, ξ is the usual fBm with a Hurst parameter $(b+1)/2$ (up to multiplicative constant). This WfBm was introduced by Bojdecki et al in [15].

We will establish new related results by using an approach based on the concept of local nondeterminism (LND), introduced by Berman [7], to unify and extend his earlier works to local times of stationary Gaussian processes*.

The joint continuity as well as Hölder conditions, in both space and time set of variables of the local time of a locally nondeterministic (LND) Gaussian process and fields, have been studied by Berman [4] and [7], Pitt [14], Kôno [12], Geman and Horowitz [10], and later by Csörgo, Lin and Shao [8] and by Xiao [16]. Recently, Boufoussi, Dozzi, and Guerbaz [2], then Guerbaz [11] and Mendy [13] have studied the local time of multifractional Brownian motion

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(mBm) and the local time of the filtered white noise and the local time of sub-fractional Brownian motion. The multifractional Brownian motion extend the fBm in the sense that its Hurst parameter is no more constant, but a Hölder function of time.

The paper is organized as follows. Section 2 contains a brief review on the local times of Gaussian processes and Berman's concept of local nondeterminism. We also give some properties of the WfBm. In section 3 we prove the existence of a square integrable version of the local time, the joint continuity and Hölder regularity in time and in space. There we will consider $C, C_1, \ldots$ to denote unspecified positive finite constants which may not necessary be the same at each occurrence.

2. Preliminaries

Let us recall some aspects of local times and refer to the paper of Geman and Horowitz [10] for an insightful survey of local times. Assume $X = \{X(t), t \geq 0\}$ to be a real valued separable random process with Borel sample functions. For any Borel set B of the real line, the occupation measure of X is defined viz

$$\mu(A,B) = \lambda\{s \in A : X(s) \in B\} \;\; \forall \; A \in \mathcal{B}(\mathbb{R}^+),$$

where λ is the Lebesgue measure on $\mathbb{R}^+$. If $\mu(A,.)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, we say that X has local times on A and define its local time, $L(A,.)$, as the Radon-Nikodym derivative of $\mu(A,.)$. Here x is the so-called space variable, and A is the time variable. The existence of jointly continuous local time reveals information on the fluctuation of the sample paths of the process itself ([1], Chapter8). There are several approaches for proving the joint continuity of the local times. One of them is the Fourier analytic method developed by Berman to extend his early works on local times of stationary Gaussian processes. The main tool used in Berman's approach of [7] is the local nondeterminism. Here we shall give only a brief review of the concept of local nondeterminism, while more information on the subject can be found in [7].

Let J be an open interval on the t axis, and assume that $\{X(t), t \geq 0\}$ is a zero mean Gaussian process without singularities in any interval of length δ, for some $\delta > 0$, and without fixed zeros; i.e. there exists $\delta > 0$ such that

$$(\mathcal{P}) \begin{cases} \mathbb{E}(X(t) - X(s))^2 > 0, & 0 < |t-s| < \delta \\ \mathbb{E}(X(t))^2 > 0, & t \in J \end{cases}.$$

To introduce the concept of local nondeterminism, Berman defines the relative conditioning error,

$$V_m = \frac{Var\{X(t_m) - X(t_{m-1})/X(t_1), \ldots, X(t_{m-1})\}}{Var\{X(t_m) - X(t_{m-1})\}}, \tag{2}$$

where, for $m \geq 2$, $t_1, \ldots, t_m$ are arbitrary points in J, ordered according to their indices, i.e. $t_1 < t_2 < \ldots < t_m$. We then say that the process X is locally nondeterministic (LND) on J if for every $m \geq 2$,

$$\liminf_{c\searrow 0^+,\, 0<t_m-t_1<c} V_m > 0. \tag{3}$$

This condition means that a small increment of the process is not almost relatively predictable, on the basis of a finite number of observations, from the immediate past. Berman has proved, for Gaussian processes, that the local nondeterminism can be characterized as follows.

Proposition 2.1.[7] *X is LND if and only if for every integer $m \geq 2$, there exists positive constants C and δ (both may depend on m) such that*

$$Var\left(\sum_{j=1}^{m} u_j[X(t_j) - X(t_{j-1})]\right) \geq C\sum_{j=1}^{m} u_j^2\, Var[X(t_j) - X(t_{j-1})], \tag{4}$$

for all ordered points $t_1 < t_2 < \ldots < t_m$ in J, with $t_m - t_1 < \delta$, $t_0 = 0$ and $(u_1, u_2, \ldots, u_m) \in R^m$.

The proof of this proposition is based on Lemmata 2. 1 and 8. 1 of [7].

3. Local Times

The purpose of this section is to present sufficient conditions for existence of the local times of WfBm. Furthermore, using the LND approach and under some conditions on the parameters *a* and *b*, we show that the local times have a jointly continuous version.

3. 1. **Square integrability**

Theorem 3.1. *Assume $a \geq 0$ and $-1 < b \leq 1$. On each time-interval $[\alpha, \beta] \subset (0, \infty)$, the weight fractional Brownian motion ξ has almost surely a local time $L([\alpha, \beta], x)$, continuous in t for a.e. $x \in \mathbb{R}$ and such that*

$$\int_{\mathbb{R}} L([\alpha, \beta], x)^2 dx < \infty.$$

Proof. By using the following Lemma 3.2 together with theorem 3.1 of [3]. ■

So we need the following lemmata for the proof of Theorem 3.1 and for the rest of the paper. The following first lemma is due to Bojdecki et al ([15], theorem 2.4.)

Lemma 3.1. *Assume $a \geq 0$, $-1 < b \leq 1$. Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{-b-1}\, \mathbb{E}(\xi_{t+\varepsilon} - \xi_t)^2 = \frac{2}{b+1} t^a.$$

Proof. From theorem 2.4 of Bojdecki et al [15], we have

$$\begin{aligned}\varepsilon^{-b-1}\mathbb{E}(\xi_{t+\varepsilon}-\xi_t)^2 &= 2\varepsilon^{-b-1}\int_t^{t+\varepsilon} u^a(t+\varepsilon-u)^b du\\ &= 2\int_0^1 (t+\varepsilon y)^a(1-y)^b dy\\ &\to 2t^a\int_0^1(1-y)^b dy = \frac{2}{b+1}t^a,\ as\ \varepsilon\to 0,\end{aligned}$$

where the required result appears at the right hand side. ■

The second lemma relates to the integrability of the characteristic function of the increment $\xi_t-\xi_s$, for $0<\alpha\le s<t\le\beta<\infty$; s and t are sufficiently close.

Lemma 3.2. *Let $a\ge 0$, $-1<b\le 1$. Then there exists positive numbers $(\rho,H)\in(0,\infty)\times(0,1)$ and a positive function $\psi\in L^1(\mathbb{R})$ such that for all $\lambda\in\mathbb{R}$, t, $s\in[\alpha,\beta]$, $0<|t-s|<\rho$ there holds*

$$\left|\mathbb{E}\exp\left(i\lambda\frac{\xi_t-\xi_s}{|t-s|^H}\right)\right|\le\psi(\lambda).$$

Proof. Let us consider $H=(b+1)/2$. Since ξ is Gaussian centered, we have

$$\mathbb{E}\exp\left(i\lambda\frac{\xi_{t+\varepsilon}-\xi_t}{|\varepsilon|^H}\right)=\exp\left(-\frac{\lambda^2}{2}\,\mathbb{E}\left[\left(\frac{\xi_{t+\varepsilon}-\xi_t}{|\varepsilon|^H}\right)^2\right]\right). \tag{5}$$

Now in view of Lemma 3.1 there exists ε_0 such that for every ε satisfying $|\varepsilon|<\varepsilon_0$ and for every t we have

$$\mathbb{E}\left[\left(\frac{\xi_{t+\varepsilon}-\xi_t}{|\varepsilon|^H}\right)^2\right]\ge\frac{c}{2}, \tag{6}$$

where $c=\frac{2}{b+1}\alpha^a$. Thus, combining (5) and (6) we get for every λ, t and s satisfying $|t-s|<\varepsilon_0$,

$$\left|\mathbb{E}\exp\left(i\lambda\frac{\xi_{t+\varepsilon}-\xi_t}{|\varepsilon|^H}\right)\right|\le\exp\left(-\frac{\lambda^2c}{4}\right).$$

Then we choose $\psi(\lambda)=\exp\left(-\frac{\lambda^2c}{4}\right)$ to conclude this proof. ■

3.2. **LND property of weight fractional Brownian motion**

First, we give the following lemma which will be used in the sequel.

Lemma 3.3. *Let $a\ge 0$, $-1<b\le 1$ and WfBm ξ is locally self-similar, then for every $t\in[\alpha,\beta]\subset(0,\infty)$ the following convergence in distribution holds,*

$$\lim_{\varepsilon\to 0}\left(\frac{\xi(t+\varepsilon u)-\xi(t)}{\varepsilon^{(b+1)/2}}\right)_{u\geq 0} = \left(\frac{\sqrt{2}\,t^{a/2}}{\sqrt{b+1}}B^{(b+1)/2}(u)\right)_{u\geq 0},$$

where the convergence is in the sense of the finite dimensional distributions, and $B^{(b+1)/2}$ *is a fractional Brownian motion with a Hurst parameter* $0 < (b+1)/2 < 1$.

Proof. Let us start by proving the convergence of the finite dimensional distribution. Because ξ is Gaussian, it suffices to illustrate the convergence of the second-order moments. By theorem 2.4 of Bojdecki et al [14], we have for every $0 \leq u \leq v$,

$$\frac{1}{\varepsilon^{b+1}}\mathbb{E}[(\xi(t+\varepsilon u)-\xi(t))(\xi(t+\varepsilon v)-\xi(t))]$$
$$= \frac{1}{\varepsilon^{b+1}}\int_t^{t+\varepsilon u} x^a[(t+\varepsilon u-x)^b+(t+\varepsilon v-x)^b]dx = \int_0^u(\varepsilon x+t)^a[(u-x)^b+(v-x)^b]dx$$
$$\to t^a\int_0^u[(u-x)^b+(v-x)^b]dx = \frac{t^a}{b+1}(u^{b+1}+v^{b+1}-(v-u)^{b+1}).$$

Now it remains to prove the tightness in the space of continuous functions endowed by the uniform norm. We also consider $T > 0$ such that t, $t+\varepsilon u$ and $t+\varepsilon v \in [0,T]$ and $u \leq v$ for every $\varepsilon > 0$. Here also by theorem 2.4 of Bojdecki et al [15], we have

$$\mathbb{E}\left[\left(\frac{\xi(t+\varepsilon u)-\xi(t)}{\varepsilon^{(b+1)/2}}-\frac{\xi(t+\varepsilon u)-\xi(t)}{\varepsilon^{(b+1)/2}}\right)^2\right]$$
$$= \frac{1}{\varepsilon^{b+1}}\mathbb{E}\left[(\xi(t+\varepsilon v)-\xi(t+\varepsilon u))^2\right] = 2\int_{t+\varepsilon u}^{t+\varepsilon v}x^a(t+\varepsilon v-x)^b dx$$
$$\leq 2T^a\int_u^v(v-x)^b dx = \frac{2T^a}{b+1}(v-u)^{b+1}.$$

Here the proof ends. ■

Second, in order to study the joint continuity of the local time of the WfBm, we prove now the LND property of WfBm.

Theorem 3.2. *Assume* $a \geq 0$ *and* $-1 < b \leq 1$. *Then for every* $\alpha > 0$, *and* $\beta > \alpha$, *the WfBm is LND on* $[\alpha,\beta]$.

Proof. Let us note that the WfBm ξ is a zero mean Gaussian process, and that

$$\mathbb{E}(\xi(t))^2 = 2\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}t^{a+b+1} > 0,$$

for every $t \in [\varepsilon,T]$.

Moreover, by Lemma 3.1, there exists $\delta > 0$ and $C > 0$ such that

$$\mathbb{E}(\xi(t)-\xi(s))^2 \geq C|t-s|^{b+1},$$

for all $|t-s| < \delta$. Therefore the condition $(\mathcal{P})$ of LND holds.

It remains to show that ξ satisfies (3). In this respect, using theorem 2.4 of [15], leads to

$$Var(\xi(t)-\xi(s)) \leq C(T)|t-s|^{b+1}, \ \forall s,t \in [\varepsilon,T]. \quad (7)$$

Then, for $m \geq 2$ and all points $t < t_1 < \ldots < t_m < t+\delta$, we have

$$Var(\xi(t_m)-\xi(t_{m-1})) \leq C(T)\delta^{b+1}.$$

Therefore,

$$\lim_{\delta\to 0}\frac{Var(\xi(t_m)/\xi(t_1),\ldots,\xi(t_m-1))}{Var(\xi(t_m)-\xi(t_{m-1}))} \geq \lim_{\delta\to 0}\frac{Var(\xi(t)/\xi(t_1),\ldots,\xi(t_{m-1}))}{C(T)\delta^{b+1}}. \tag{8}$$

Moreover, if we add $\xi(t)$ to the conditional set we obtain

$$\frac{Var(\xi(t_m)/\xi(t_1),...,\xi(t_{m-1}))}{C(T)\,\delta^{b+1}} \geq \frac{Var(\xi(t_m)/\xi(t),\xi(t_1),...,\xi(t_{m-1}))}{C(T)\,\delta^{b+1}}$$

$$= Var\left(\frac{\xi(t_m)-\xi(t)}{\delta^{b+1}}/\xi(t),\xi(t_1)-\xi(t),\ldots,\xi(t_{m-1})-\xi(t)\right)$$

$$= Var\left(\frac{\xi(t_m)-\xi(t)}{\delta^{b+1}}/\xi(t),\frac{\xi(t_1)-\xi(t)}{\delta^{b+1}},\ldots,\frac{\xi(t_{m-1})-\xi(t)}{\delta^{b+1}}\right),$$

where the last equality follows from the fact that

$$\sigma(\xi(t),\xi(t_1)-\xi(t),\ldots,\xi(t_{m-1})-\xi(t)) = \sigma\left(\xi(t),\frac{\xi(t_1)-\xi(t)}{\delta^{b+1}},\ldots,\frac{\xi(t_{m-1})-\xi(t)}{\delta^{b+1}}\right).$$

Let $t_i - t = \delta u_i$ with $0 < u_i < t,\ i = 1,\ldots,m$. Therefore, the fraction in (9) becomes

$$lim_{\delta\to 0}\ Var\left(\frac{\xi(t+\delta u_m)-\xi(t)}{\delta^{b+1}}/\xi(t),\frac{\xi(t+\delta u_1)-\xi(t)}{\delta^{b+1}},\ldots,\frac{\xi(t+\delta u_{m-1})-\xi(t)}{\delta^{b+1}}\right)$$

$$= lim_{\delta\to 0}\ Var(\eta_{t,\delta}(u_m)/\xi(t),\eta_{t,\delta}(u_1),\ldots,\eta_{t,\delta}(u_{m-1})),$$

when using the notation

$$\eta_{t,\delta}(u_m) = \frac{\xi(t+\delta u_{m-1})-\xi(t)}{\delta^{b+1}}.$$

Furthermore,

$$Var(\eta_{t,\delta}(u_m)/\xi(t),\eta_{t,\delta}(u_1),\ldots,\eta_{t,\delta}(u_{m-1})) = \frac{detCov(\xi(t),\eta_{t,\delta}(u_1),...,\eta_{t,\delta}(u_m))}{detCov(\xi(t),\eta_{t,\delta}(u_1),...,\eta_{t,\delta}(u_{m-1}))}.$$

Now, since the WfBm is locally asymptotically self similar (Lemma 3.3), $\eta_{t,\delta}$ converges weakly to the fractional Brownian motion $\bar{B}^{(b+1)/2}$ with parameter $(b+1)/2$. Consequently, the fraction above converges to

$$\frac{detCov(\xi(t),\bar{B}^{(b+1)/2}(u_1),...,\bar{B}^{(b+1)/2}(u_m))}{detCov(\xi(t),\bar{B}^{(b+1)/2}(u_1),...,\bar{B}^{(b+1)/2}(u_{m-1}))}.$$

Therefore

$$lim_{\delta\to 0}\ Var(\eta_{t,\delta}(u_m)/\xi(t),\eta_{t,\delta}(u_1),\ldots,\eta_{t,\delta}(u_{m-1}))$$

$$Var(\bar{B}^{(b+1)/2}(u_m)/\bar{B}^{(b+1)/2}(t),\bar{B}^{(b+1)/2}(u_1)\ldots,\bar{B}^{(b+1)/2(u_{m-1})})$$

$$\geq C(T)[(u_m-u_{m-1})\wedge(t-u_m)]^{(b+1)/2},$$

where the last inequality follows from lemma 7.1 of [14], and the last term is strictly positive since $0 < u_1 <\ldots< u_m < t$. ■

3.2. **Joint continuity and Hölder regularity**

Let $T > 0$, $H = (b+1)/2$ and $\mathcal{H}([0,T])$ be the family of intervals $I \subset [0,T]$ of length at most δ (the constant appearing in Theorem 3.2). In this paragraph we shall apply some results of Berman on an LND process to prove the joint continuity of local times of the WfBm. Our main result then follows.

Theorem 3.3. *Assume $a \geq 0$ and $-1 < b \leq 1$. Then the WfBm ξ has, almost surely, a jointly continuous local time $\{L(t,x),\ t \in [0,T],\ x \in \mathbb{R}\}$. It moreover satisfies, for any compact $U \subset \mathbb{R}$,*
(i)

$$\sup_{x\in U} \frac{L(t+h,x) - L(t,x)}{|h|^{\lambda}} < +\infty \ a.s., \tag{9}$$

where $\lambda < 1 - H$ and $|h| < \eta$, η being a small random variable almost surely positive and finite,
(ii) for any $I \in H([0,T])$,

$$\sup_{x,y\in U, x\neq y} \frac{L(I,x) - L(I,y)}{|x-y|^{\gamma}} < +\infty \ a.s., \tag{10}$$

where $\gamma < 1 \wedge \frac{1-H}{2H}$.

The proof of the previous theorem relies on the following upper bounds for the moments of the local times.

Lemma 3.4. *Assume $a \geq 0, -1 < b \leq 1$ and let δ be the constant appearing in Theorem 3.2. Then for any even integer $m \geq 2$ there exists a positive and finite constant C_m such that, for any $t \in [0,+\infty)$, any $h \in (0,\delta)$, any x, $y \in \mathbb{R}$ and any $\alpha < 1 \wedge \frac{1-H}{2H}$, there hold*

$$\mathbb{E}[L(t+h,x) - L(t,x)]^m \leq C_m \frac{h^{m(1-H)}}{\Gamma(1+m(1-H))}, \tag{11}$$

$$\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^m \leq C_m |y-x|^{m\alpha} \frac{h^{m(1-H(1+\alpha))}}{\Gamma(1+m(1-H(1+\alpha)))}. \tag{12}$$

Proof. We will proof only (12), the proof of (11) is similar. It follows from equation (25.7) in Geman and Horowitz [10] (see also Boufoussi et al [2]) that for any $x, y \in \mathbb{R}$, t, $t+h \in [0,+\infty)$ and for every even integer $m \geq 2$,

$$\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^m$$
$$= (2\pi)^{-m} \int_{[t,\,t+h]^m} \int_{\mathbb{R}^m} \prod_{j=1}^{m} [e^{-iyu_j} - e^{-ixu_j}] \times \mathbb{E}\left(e^{i\sum_{j=1}^m u_j \xi(s_j)}\right) \prod_{j=1}^{m} du_j \prod_{j=1}^{m} ds_j.$$

Using the elementary inequality $|1 - e^{i\theta}| \leq 2^{1-\alpha}\, |\theta|^{\alpha}$ for all $0 < \alpha < 1$ and any $\theta \in \mathbb{R}$, we obtain

$$\mathbb{E}[L(t+h,y)-L(t,y)-L(t+h,x)+L(t,x)]^m \leq (2^\alpha\pi)^{-m}m!|y-x|^{m\alpha}$$
$$\times\int_{t<t_1<\ldots<t_m<t+h}\int_{\mathbb{R}^m}\prod_{j=1}^m|u_j|^\alpha\mathbb{E}\Big[\exp\Big(i\sum_{j=1}^m u_j\xi(t_j)\Big)\Big]\prod_{j=1}^m du_j\prod_{j=1}^m dt_j, \tag{13}$$

where in order to apply the LND property of ξ, we replaced the integration over the domain $[t, t+h]$ by the subset $t<t_1<\ldots<t_m<t+h$. We can deal now with the inner multiple integral over the u's. Change the variable of integration by means of the transformation

$u_j = v_j - v_{j+1},\ j = 1,2,\ldots,m-1;\ u_m = v_m.$

Then the linear combination in the exponent in (13) is transformed according to

$$\sum_{j=1}^m u_j\xi(t_j) = \sum_{j=1}^m v_j(\xi(t_j)-\xi(t_{j-1})),$$

where $t_0 = 0$. Since ξ is a Gaussian process, the characteristic function in (13) has the form

$$\exp\left(-\frac{1}{2}Var\left[\sum_{j=1}^m v_j(\xi(t_j)-\xi(t_{j-1}))\right]\right). \tag{14}$$

And since $|x-y|^\alpha \leq |x|^\alpha+|y|^\alpha$ for all $0<\alpha<1$, it follows that

$$\prod_{j=1}^m|u_j|^\alpha = \prod_{j=1}^{m-1}|v_j-v_{j+1}|^\alpha|v_m|^\alpha \leq \prod_{j=1}^{m-1}(|v_j|^\alpha+|v_{j+1}|^\alpha)|v_m|^\alpha. \tag{15}$$

Moreover, the last product is at most equal to a finite sum of 2^{m-1} terms of the form $\prod_{j=1}^m|x_j|^{\alpha\varepsilon_j}$, where $\varepsilon_j = 0,1$ or 2 and $\sum_{j=1}^m\varepsilon_j = m$.

Let us write for simplicity $\sigma_j^2 = \mathbb{E}(\xi(t_j)-\xi(t_{j-1}))^2$. Combining the result of Proposition 2.1, (14) and (15), we observe that the integral in (13) is dominated by the sum over all possible $(\varepsilon_1,\ldots,\varepsilon_m)\in\{0,1,2\}^m$ of the following

$$\int_{t<t_1<\ldots<t_m<t+h}\int_{\mathbb{R}^m}\prod_{j=1}^m|v_j|^{\alpha\varepsilon_j}\exp\left(-\frac{C_m}{2}\sum_{j=1}^m v_j^2\sigma_j^2\right)\prod_{j=1}^m dt_j dv_j,$$

where C_m is the constant given in Proposition 2.1. The change of variable $x_j = v_j\sigma_j$ converts the last integral to

$$\int_{t<t_1<\ldots<t_m<t+h}\prod_{j=1}^m\sigma_j^{-1-\alpha\varepsilon_j}dt_1\ldots dt_m\int_{\mathbb{R}^m}\prod_{j=1}^m|x_j|^{\alpha\varepsilon_j}\exp\left(-\frac{C_m}{2}\sum_{j=1}^m x_j^2\right)\prod_{j=1}^m dx_j.$$

We then denote

$$J(m,\xi) = \int_{\mathbb{R}^m}\prod_{j=1}^m|x_j|^{\alpha\varepsilon_j}\exp\left(-\frac{C_m}{2}\sum_{j=1}^m x_j^2\right)\prod_{j=1}^m dx_j,$$

to arrive at

$$\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^m \leq$$
$$C_m\, J(m,\alpha)|y-x|^{m\alpha} \int_{t<t_1<\ldots<t_m<t+h} \prod_{j=1}^{m} \sigma_j^{-1-\alpha\varepsilon_j} dt_1 \ldots dt_m. \tag{16}$$

According to Lemma 3.1, for h sufficiently small, namely $0 < h < \inf(\delta, 1)$, we have

$$\mathbb{E}(\xi(t_i) - \xi(t_j))^2 \geq C|t_i - t_j|^{2H}, \ \forall\ t_i, t_j \in [t, t+h]. \tag{17}$$

It follows that the integral on the right hand side of (16) is bounded, up to a constant, by

$$\int_{t<t_1<\ldots<t_m<t+h} \prod_{j=1}^{m} (t_j - t_{j-1})^{-H(1+\alpha\varepsilon_j)} dt_1 \ldots dt_m. \tag{18}$$

Since, $(t_j - t_{j-1}) < 1$, for all $j \in \{2, \ldots, m\}$, we have
$(t_j - t_{j-1})^{-H(1+\alpha\varepsilon_j)} < (t_j - t_{j-1})^{-H(1+2\alpha)}\ \forall \varepsilon_j \in \{0, 1, 2\}$,
and by hypothesis $\alpha < \frac{1}{2H} - \frac{1}{2}$, the integral in (18) is finite. Moreover, an elementary calculation(cf. Ehm [9]), for all $m \geq 1$, $h > 0$ and $b_j < 1$, yields

$$\int_{t<s_1<\ldots<s_m<t+h} \prod_{j=1}^{m} (s_j - s_{j-1})^{-b_j} ds_1 \ldots ds_m = h^{m - \sum_{j=1}^{m} b_j} \frac{\prod_{j=1}^{m} \Gamma(1-b_j)}{\Gamma(1+h-\sum_{j=1}^{m} b_j)},$$

where $s_0 = t$. Clearly (18) is dominated by

$$C_m \frac{h^{m(1-H(1+\alpha))}}{\Gamma(1+m(1-H(1+\alpha)))},$$

where $\sum_{j=}^{m} \varepsilon_j = m$. Consequently

$$\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^m \leq C_m \frac{|y-x|^{m\alpha} h^{m(1-H(1+\alpha))}}{\Gamma(1+m(1-H(1+\alpha)))}, \tag{19}$$

and by that the proof ends. ■

3.3. Proof of theorem 3.3

Since $L(0,x) = 0$ for all $x \in \mathbb{R}$, hence if we replace t and $t+h$ by 0 and t respectively in (12), we obtain

$$\mathbb{E}[L(t,y) - L(t,x)]^m \leq \tilde{C}_m|y-x|^{m\alpha}. \tag{20}$$

The joint continuity of the local time follows straightforwardly from (11), (12) and (20) and the classical parameter Kolmogorov's theorem (theorem 5.1 of Berman [5]). The Hölder condition (i) of theorem 3.1 follows from (12) and the one parameter Kolmogorov's theorem (see also the proof of theorem 2 by Pitt in [14]).

Now we move on to the proof of (ii). According to theorem 3.1 by Berman in [6], the inequalities (11), (12) and (20) imply that (ii) holds for any $\lambda < 1 - H(1+\alpha)$, for all $0 < \alpha < 1 \wedge \frac{1-H}{2H}$. Finally, by letting α tends to zero, we obtain the desired result. ■

As a classical consequence, we have the following result on the Hausdorff dimension of the level set. Here we refer to Adler [1] for definitions and results for the fractional Brownian motion.

Proposition 3.1. *With probability one, for any interval* $I \subset [0, T]$, *there holds*

$$dim\{t \in I/\xi = x\} = 1 - H, \tag{21}$$

for all x *such that* $L(t, x) > 0$.

Proof. According to (7) and Kolmogorov's theorem, the WfBm is β-Hölder for every $\beta < H$. Moreover, the WfBm has a jointly continuous local time, then theorem 8.7.3 in Adler [1] completes the proof of the upper bound, i.e. $dim\{t \in I/\xi = x\} \leq 1 - H$, a.s. Now by (i) of Theorem 3.3, the jointly continuous local time of the WfBm is uniform Hölder of any order smaller than $1 - H$. Then theorem 8.7.4 of Adler [1] implies that $dim\{t \in I/\xi = x\} \geq 1 - H$, a.s. for all x such that $L(t, x) > 0$. This completes the proof. ■

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