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Rates of Convergence to Central Limit Theorems Via Esscher Transformed Berry-Esseen Bounds

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Abstract. *By applying the Esscher transform to the expected value of a differentiable random function, it is shown how classical and new elementary bounds of Chernoff and Hoeffding type for such expected values are quite simply obtained. A further use of the Berry-Esseen theorems leads to a substantial strengthening of these bounds. This general method, seemingly due to Talagrand [20], is illustrated at concrete central limit theorems for the survival function and the stop-loss transform. In particular, simple explicit rates of convergence for the Gamma distribution are obtained. More general asymptotic expansions allow to improve on known speeds of convergence. Details for the important compound Poisson distribution are also reported. Numerical examples demonstrate the obtained effects quantitatively.*

Key words : Esscher Transform, Berry-Esseen Theorems, Inequality of Chernoff, Inequality of Hoeffding, Central Limit Theorem, Rate of Convergence, Asymptotic Expansion, Stop-Loss Transform, Gamma Distribution, Compound Poisson Distribution.

AMS Subject Classifications : 60E15, 60F05, 62E20, 62P05, 65C50

1. Inequalities of Chernoff and Hoeffding Types

Consider a random variable X with a probability distribution $F(x)$. Its survival function is denoted by $\overline{F}(x) = 1 - F(x)$ and $K(\theta) = \ln\{E[e^{\theta x}]\}$ is its cumulant generating function (cgf), where one assumes that this function is so many times differentiable as needed in the displayed formulas. The Esscher transform $F_\theta(x)$ of the distribution $F(x)$ is defined by

$$dF_\theta(x) = \exp\{\theta x - K(\theta)\} \cdot dF(x). \qquad (1)$$

We shall derive bounds on the expected value $E_g := E[g(X)]$, where $g(x)$ is a differentiable function defined on an interval $[a,b]$, $-\infty \le a < b \le \infty$. Through partial integration, one obtains the basic formula

$$E_g = \int_a^b g(t)dF(t) = \int_a^b g(t) \cdot e^{K(\theta)-\theta t} dF_\theta(t)$$
$$= e^{K(\theta)} \cdot \left\{ \begin{array}{c} g(a)e^{-\theta a}\overline{F}_\theta(a) - g(b)e^{-\theta b}\overline{F}_\theta(b) \\ +\int_a^b e^{-\theta t} g'(t)\overline{F}_\theta(t)dt - \theta \cdot \int_a^b e^{-\theta t} g(t)\overline{F}_\theta(t)dt \end{array} \right\}. \quad (2)$$

Main examples, which are of primary importance in Actuarial Science and Finance, include the following ones.

Example 1.1. Survival function. If $g(t) = 1,\ a = x,\ b = \infty,\ g'(t) = 0$ on the interval $[x, \infty)$ one has

$$\overline{F}(x) = e^{K(\theta)-\theta x} \cdot \left\{ \overline{F}_\theta(x) - \theta e^{\theta x} \int_x^\infty e^{-\theta t} \overline{F}_\theta(t)dt \right\}. \quad (3)$$

Example 1.2. Stop-loss transform. If $g(t) = t - x,\ a = x,\ b = \infty,\ g'(t) = 1$ on the interval $[x, \infty)$ one has

$$\pi(x) = E[(X-x)_+] = e^{K(\theta)} \cdot \left\{ \int_x^\infty e^{-\theta t}\overline{F}_\theta(t)dt - \theta \int_x^\infty (t-x)e^{-\theta t}\overline{F}_\theta(t)dt \right\}. \quad (4)$$

Clearly, higher order stop-loss moments can also be considered. For example, the second-order stop-loss moment is obtained by setting $g(t) = (t-x)^2,\ a = x,\ b = \infty$, $g'(t) = 2(t-x) \geq 0$. Now, assume that $g(x)$ is a non-negative increasing function on $[a, b]$. Then the rough inequality $0 \leq \overline{F}_\theta(x) \leq 1$ implies by (2) the rather crude upper bound

$$E_g \leq e^{K(\theta)} \cdot \left\{ g(a)e^{-\theta a} + \int_a^b e^{-\theta t} g'(t)dt \right\}. \quad (5)$$

In application to Examples 1.1 and 1.2 , one obtains the inequalities

$$\overline{F}(x) \leq e^{K(\theta)-\theta x}, \quad (6)$$

$$\pi(x) \leq \tfrac{1}{\theta} \cdot e^{K(\theta)-\theta x}. \quad (7)$$

On one hand, by the Cauchy-Schwarz inequality we have

$$K''(\theta) = \frac{E\left[X^2 e^{\theta X}\right] \cdot E\left[e^{\theta X}\right] - E\left[X e^{\theta X}\right]^2}{E\left[e^{\theta X}\right]^2} \geq 0. \quad (8)$$

It follows that the minimum, with respect to θ, of the function $K(\theta) - \theta x$ is attained at $x = K'(\theta)$, and thus

$$\overline{F}(K'(\theta)) \leq e^{K(\theta) - \theta K'(\theta)}. \quad (9)$$

Similarly, the minimum, with respect to θ, of the function $K(\theta) - \theta x - \ln\{\theta\}$ is attained at $x = K'(\theta) - \frac{1}{\theta}$, with

$$\pi(K'(\theta) - \tfrac{1}{\theta}) \leq \tfrac{e}{\theta} \cdot e^{K(\theta) - \theta K'(\theta)}. \quad (10)$$

Now, assume $X = \sum_{i=1}^n X_i$ is a sum of independent identically distributed random variables, each component with a cgf $C(\theta)$. Then $K(\theta) = n\,C(\theta)$, which implies the inequalities

$$\overline{F}(K'(\theta)) \leq \exp\{-n\theta \cdot (C'(\theta) - \tfrac{1}{\theta}C(\theta))\}, \quad (11)$$ (*inequality of Chernoff* [2]),

$$\pi(K'(\theta) - \tfrac{1}{\theta}) \leq \tfrac{e}{\theta} \cdot \exp\left\{-n\theta \cdot \left(C'(\theta) - \tfrac{1}{\theta}C(\theta)\right)\right\}. \quad (12)$$

On the other hand, it is possible to optimize the right-hand sides in (6) and (7), under the assumption that only incomplete information about the summands is available, say on a given finite range $[a,b]$ and known mean, or on a given one-sided infinite range $(-\infty,b]$ and known values of the mean and variance. By solving the extremal moment problems

$$\overline{F}(x) \leq \inf_{\theta \geq 0} \left\{ e^{-\theta x} \cdot \max_{X_i \in D} E\left[\exp\left\{ \theta\left(\sum_{i=1}^n X_i \right) \right\} \right] \right\}, \tag{13}$$

$$\pi(x) \leq \inf_{\theta \geq 0} \left\{ \tfrac{1}{\theta} e^{-\theta x} \cdot \max_{X_i \in D} E\left[\exp\left\{ \theta\left(\sum_{i=1}^n X_i \right) \right\} \right] \right\}, \tag{14}$$

where D is the domain of variation of the summands, one obtains bounds of the type first derived by Hoeffding [9].

Remark 1.1. Concerning actuarial applications, it is possible to apply such bounds to determine the unknown loading of various pricing principles in Insurance and Finance. For example, Kremer [13] has used the inequality of Chernoff, and a refined version of it, to estimate the loading factor θ of the Esscher pricing principle, which associates to a risk X the actuarial risk premium defined by the functional $P_\theta[X] = K'(\theta)$.

2. Improved Inequalities of Chernoff and Hoeffding Types

Consider the sum $X = \sum_{i=1}^n X_i$ of independent and identically distributed random variables with cgf $K(\theta) = nC(\theta)$. Let P be the probability measure associated to X, and let Q be the corresponding Esscher transformed measure defined by $dQ = e^{\theta X - K(\theta)} \cdot dP$. The fact that the X_i's are again independent when Q is the basic probability to be used. Moments with respect to Q will be indexed with θ.

The basic method has been suggested to us by Talagrand [20]. According to Berry [1] and Esseen [3], it is well-known (e.g. Feller [6], p. 542) that the following confidence bounds for $Q(X \geq x) = \overline{F}_\theta(x)$ hold.

$$\left| \overline{F}_\theta(x) - \overline{\Phi}\left(\frac{x-\mu_\theta}{\sigma_\theta}\right) \right| \leq \epsilon(\theta), \tag{15}$$

where $\overline{\Phi}(x) = 1 - \Phi(x)$, with $\Phi(x)$ the standard normal distribution,

$$\mu_\theta = \int X dQ = K'(\theta), \quad \sigma_\theta^2 = \int X^2 dQ - \mu_\theta^2 = K''(\theta), \quad \epsilon(\theta) = c_0 \frac{\beta(\theta)}{\sigma_\theta^3},$$

$$\beta(\theta) = \sum_{i=1}^n \int |X_i - \mu_{i,\theta}|^3 dQ,$$

and the constants satisfy $c_0 \geq \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.4097$ (Esseen [4]), and $c_0 \leq 0.7655$ (Shiganov [19]) ≤ 0.7882 (van Beek [21],[22]).

Under the assumption that $g(x)$ is a non-negative increasing differentiable function on $[a,b]$, insert (15) into (12) to get confidence bounds for the expected value $E_g = E[g(X)]$. Consequently

$$e^{-K(\theta)}\cdot E_g \le g(a)e^{-\theta a}\cdot\left\{\overline{\Phi}\left(\frac{a-\mu_\theta}{\sigma_\theta}\right)+\varepsilon(\theta)\right\}-g(b)e^{-\theta b}\cdot\left\{\overline{\Phi}\left(\frac{b-\mu_\theta}{\sigma_\theta}\right)-\varepsilon(\theta)\right\}$$
$$+I_1+\varepsilon(\theta)\cdot I_2-\theta\cdot I_3+\varepsilon(\theta)\cdot\theta\cdot I_4, \quad (16)$$

with

$$I_4=\int_a^b e^{-\theta t}g(t)dt,\quad I_2=\int_a^b e^{-\theta t}g'(t)dt=g(b)e^{-\theta b}-g(a)e^{-\theta a}+I_4,$$

$$I_3=\int_a^b e^{-\theta t}g(t)\overline{\Phi}\left(\frac{t-\mu_\theta}{\sigma_\theta}\right)dt,$$

$$I_1=\int_a^b e^{-\theta t}g'(t)\overline{\Phi}\left(\frac{t-\mu_\theta}{\sigma_\theta}\right)dt=g(b)e^{-\theta b}\cdot\overline{\Phi}\left(\frac{t-\mu_\theta}{\sigma_\theta}\right)-g(a)e^{-\theta a}\cdot\overline{\Phi}\left(\frac{a-\mu_\theta}{\sigma_\theta}\right)$$
$$-\theta\cdot I_3+\int_a^b e^{-\theta t}g'(t)\frac{1}{\sigma_\theta}\varphi\left(\frac{t-\mu_\theta}{\sigma_\theta}\right)dt,\quad \varphi(x)=\Phi'(x).$$

It follows that

$$e^{-K(\theta)}\cdot E_g \le I_g+2\varepsilon(\theta)\cdot\left\{g(b)e^{-\theta b}+\theta\int_a^b e^{-\theta t}g(t)dt\right\}, \quad (17)$$

with

$$I_g=\int_a^b e^{-\theta t}g'(t)\frac{1}{\sigma_\theta}\varphi\left(\frac{t-\mu_\theta}{\sigma_\theta}\right)dt,$$

where a change of variables and a standard calculation reveals that

$$I_g=\exp\left\{-\theta\mu_\theta+\tfrac{1}{2}(\theta\sigma_\theta)^2\right\}\cdot\int_{\frac{a-\mu_\theta}{\sigma_\theta}+\theta\sigma_\theta}^{\frac{b-\mu_\theta}{\sigma_\theta}+\theta\sigma_\theta} g(\sigma_\theta u+\mu_\theta-\theta\sigma_\theta^2)\varphi(u)du. \quad (18)$$

A lower bound with $\epsilon(\theta)$ replaced by $-\epsilon(\theta)$ follows in a similar fashion. The special instances of Examples 1.1 and 1.2 yield separate results.

Proposition 2.1. (Esscher transformed Berry-Esseen survival bounds) *Let* $X=\sum_{i=1}^n X_i$ *be a sum of independent and identically distributed random variables with cgf* $K(\theta)=nC(\theta)$. *Then the following confidence bound holds.*

$$\left|\overline{F}(x)-e^{K(\theta)-\theta K'(\theta)+\frac{1}{2}\theta^2K''(\theta)}\overline{\Phi}\left(\frac{x-K'(\theta)+\theta K''(\theta)}{\sqrt{K''(\theta)}}\right)\right|\le 2\cdot\varepsilon(\theta)\cdot e^{K(\theta)-\theta x}. \quad (19)$$

Proof. Set $g(t)=1,\ a=x,\ b=\infty$ in (17), (18) and utilize $\mu_\theta=K'(\theta)$ and $\sigma_\theta^2=K''(\theta)$. ■

Proposition 2.2. (Esscher transformed Berry-Esseen stop-loss bounds) *Let* $X=\sum_{i=1}^n X_i$ *be a sum of independent and identically distributed random variables with cgf* $K(\theta)=nC(\theta)$. *Then the following confidence bound holds*

$$\left|\pi(x)-e^{K(\theta)-\theta K'(\theta)+\frac{1}{2}\theta^2K''(\theta)}\sqrt{K''(\theta)}\,\pi_N\left(\frac{x-K'(\theta)+\theta K''(\theta)}{\sqrt{K''(\theta)}}\right)\right|\le 2\cdot\varepsilon(\theta)\cdot\frac{1}{\theta}e^{K(\theta)-\theta x}. \quad (20)$$

Proof. Set $g(t)=t-x,\ a=x,\ b=\infty$ in (17), (18). ■

Remarks 2.1. (i) Setting simply $x=K'(\theta)$ and $x=K'(\theta)-\frac{1}{\theta}$ in (19) and (20), respectively, the choices which minimize the exponential error components independent of $\epsilon(\theta)$ lead to a strengthening of the inequalities of Chernoff type. This happens when

$$e^{\frac{1}{2}\theta^2 K''(\theta)}\overline{\Phi}\left(\theta\sqrt{K''(\theta)}\right) + 2\epsilon(\theta) < 1 \tag{21}$$

for the inequality (11), respectively

$$\frac{\theta\sqrt{K''(\theta)}}{e}\pi_N\left(\theta\sqrt{K''(\theta)} - \frac{1}{\theta\sqrt{K''(\theta)}}\right) + 2\epsilon(\theta) < 1 \tag{22}$$

for the inequality (12). Another method to improve Chernoff's inequality (11), which is also based on the Berry-Esseen theorem, has been proposed by Salikhov [18] in 1992.
(ii) As suggested by Talagrand [20], the bounds (19), (20) may also be used to strengthen the inequalities of Hoeffding type (6) and (7).
(iii) The method can also be applied in more general situations than the examples suggest. Namely in general statistics $g(X) = T(X_1,\dots,X_n)$, especially U-statistics, which have been introduced by Hoeffding [8].
(iv) Further improvement can be obtained through application of a non-uniform version of the Berry-Esseen theorem, first proved by Nagaev [16], for which the constant has been estimated by Michel [14] and others. However, this can only be achieved at a price of some additional technical details. In fact one has to replace $\epsilon(\theta)$ in (15) by

$$\epsilon(x,\theta) = \frac{\omega(\theta)\cdot\sigma_\theta^3}{\sigma_\theta^3 + |x-\mu_\theta|^3}, \quad \omega(\theta) = \frac{c_1\cdot\beta(\theta)}{\sigma_\theta^3}, \quad c_1 \le c_0 + 8(1+e). \tag{23}$$

When doing the calculations, the error component $2\epsilon(\theta)$ in (19) is replaced by

$$\omega(\theta)\cdot\left\{1+\theta\sqrt{K''(\theta)}\cdot\int_0^\infty \frac{e^{-\theta\sqrt{K''(\theta)}x}}{1+x^3}dx\right\} = \omega(\theta)\cdot\left\{2-3\int_0^\infty\left(\frac{x}{1+x^3}\right)^2 e^{-\theta\sqrt{K''(\theta)}x}dx\right\}, \tag{24}$$

which will improve (19) in case this is strictly less than $2\epsilon(\theta)$. A similar statement can be made for (20). As always, for the sake of simplicity, a more tractable mathematics must be preferred in a first analysis.

3. Estimates of the Esscher Transformed Berry-Esseen Error

To verify the criteria (21), (22) in practical work, it is necessary to estimate the error term $\epsilon(\theta)$, that is $\beta(\theta) = \sum_{i=1}^n \int |X_i - \mu_{i,\theta}|^3 dQ$ in the notation of Section 2. For this, it suffices to determine a possibly sharp upper bound on the absolute mean deviation $|X-\mu|$ of an arbitrary random variable with given range, "say$[A,B]$," $-\infty \le A < B \le \infty$. If $A \ge 0$, then the simple estimate $|X-\mu| \le X+\mu$ leads to the bound $|X-\mu|^3 \le (X-\mu)^3 + 2\mu(X-\mu)^2$, which yields without difficulty the estimate

$$\epsilon(\theta) \le \frac{c_0}{\sqrt{n}}\cdot\left\{\frac{C'''(\theta)+2C'(\theta)\,C''(\theta)}{C''(\theta)^{\frac{3}{2}}}\right\}. \tag{25}$$

However, this simple estimate is not the best one. A better estimate is obtained from the general inequality

$$|X-\mu| = |(X-\mu)_+ - (\mu-X)_+| \le (X-\mu)_+ + (\mu-X)_+ = 2(X-\mu)_+ - (X-\mu), \tag{26}$$

which is easily shown to be better than the inequality $|X-\mu| \le X+\mu$ when $X \ge 0$. In applying (26), it remains to find an estimate for $(X-\mu)_+$. By known mean, variance and range, it suffices to construct a majorizing quadratic polynomial $q(X) = aX^2 + bX + c \ge (X-\mu)_+$. This estimate is best provided when the equality is attained for some finite atomic (here biatomic) random variable. This answer is well-known from the theory of maximal stop-loss transforms (see e.g. the monograph [10]).

Lemma 3.1. Let $X \in D([A,B];\mu,\sigma)$ be a random variable with finite mean μ, standard deviation σ and range $[A,B]$. Then a best quadratic upper bound $q(X) \ge (X-\mu)_+$ is determined as follows:
Case 1 : $A \ge \mu-\sigma \quad (\Rightarrow\ B \ge \mu+\sigma)$

$$q(X) = \frac{(\mu-A)^3 \cdot (X-\mu)^2 + \left\{(\mu-A)^4 + \sigma^4\right\} \cdot (X-\mu) + (\mu-A)\cdot\sigma^4}{\left\{(\mu-A)^2+\sigma^2\right\}^2}. \tag{27}$$

The equality is attained for a biatomic random variable with support
$\left\{A, A + \frac{\sigma^2}{\mu-A}\right\}$.
Case 2 : $A \le \mu-\sigma, \quad B \ge \mu+\sigma$

$$q(X) = \frac{1}{4\sigma^2}(X-\mu)^2 + \frac{1}{2}(X-\mu) + \frac{1}{4}\sigma. \tag{28}$$

The equality is attained for a biatomic random variable with support
$\{\mu-\sigma, \mu+\sigma\}$.
Case 3 : $B \ge \mu+\sigma \quad (\Rightarrow\ A \le \mu-\sigma)$

$$q(X) = \frac{(B-\mu)^3 \cdot (X-\mu)^2 + 2\sigma^2\ (B-\mu)^2\ (X-\mu) + (B-\mu)\cdot\sigma^2}{\left\{(B-\mu)^2+\sigma^2\right\}^2}. \tag{29}$$

The equality is attained for a biatomic random variable with support
$\left\{B - \frac{\sigma^2}{B-\mu}, B\right\}$.

Proof. Consult for example Jansen et al [11]. ■

With the aid of (26), one obtains, from the following inequalities for $|X-\mu|^3$, which are sharp because they are attained at the biatomic random variables of Lemma 3.1, the uniform error estimates . To avoid excessive technical details, we restrict our attention to two main situations.

Example 3.1. Double-sided infinite range $(-\infty,\infty)$. From (28) one obtains the strict inequality

$$|X-\mu|^3 \le \frac{1}{2\sigma}(X-\mu)^4 + \frac{1}{2}\sigma(X-\mu)^2, \tag{30}$$

which yields, after some calculations, the Esscher transformed Berry-Esseen uniform error estimate

$$\epsilon(\theta) \le \frac{c_0}{\sqrt{n}} \cdot \left\{2 + \frac{1}{2}\frac{C^{(4)\prime}(\theta)}{C''(\theta)^2}\right\}. \tag{31}$$

Note that this estimate can always be applied. However, if one of $\mu-\sigma$, and $\mu+\sigma$ does not belong in the given range $[A,B]$, then it is not optimal.

Example 3.2. One-sided infinite range $[0,\infty)$. With Lemma 3.1, two cases must be distinguished. If $\sigma \le \mu$ then one uses (3.6), otherwise (3.3) is used with $A = 0$. For the uniform error estimate, this means that (3.7) applies provided $\sigma_{i,\theta}^2 = C''(\theta) \le \mu_{i,\theta}^2 = C'(\theta)^2$. Otherwise the sharp inequality

$$|X-\mu|^3 \le \frac{2(X-\mu)^4 + (k^2-1)^2\,\mu\,(X-\mu)^3 + 2k^4\mu^2}{(1+k^2)^2\mu}, \quad k = \frac{\sigma}{\mu}, \tag{32}$$

leads after some calculations to the (complex) uniform error estimate

$$\epsilon(\theta) \le \frac{c_0}{\sqrt{n}} \cdot \left\{ \frac{2C'(\theta)^3 C^{(4)}(\theta) + (C''(\theta)-C'(\theta)^2)^2 \cdot C'''(\theta) + 2C'(\theta)C''(\theta)}{(C''(\theta)-C'(\theta)^2)^2 \cdot C''(\theta)^{\frac{3}{2}}} \right\}. \tag{33}$$

Note however that when $C''(\theta) > C'(\theta)^2$, it is simpler to apply the estimates (25) or (31).
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4. Improved Rates of Convergence for Central Limit Theorems

Besides improved classical and new inequalities, the elementary derivation in Section 2 of Esscher transformed Berry-Esseen bounds for expected values provides also an elementary approach to diverse central limit theorems and effective rates of convergence. Let us illustrate with a central limit theorem (CLT) for the stop-loss transform. Clearly, the classical CLT

$$\lim_{n\to\infty} \Pr(X \ge n\mu + z\sigma\sqrt{n}\,) = \overline{\Phi}(z),$$

could also be derived by the same way. However, this would not yield a new proof, because our method is equivalent to the Berry-Esseen theorem, which has been assumed, and obviously implies the usual CLT. This holds of course under a more severe condition than the required necessary and sufficient condition of Lindeberg, proved first by Feller [5] (see also Feller [6], Section XV.6).

Proposition 4.1. (Stop-loss transform CLT) *Let* $X = \sum_{i=1}^n X_i$ *be a sum of independent and identically distributed random variables with cgf* $K(\theta) = nC(\theta)$. *Under the simplifying assumption that all cumulants exist, the following central limit theorem holds.*

$$\lim_{n\to\infty} E\left[\left(\frac{X-n\,\mu}{\sigma\sqrt{n}} - z \right)_+ \right] = \pi_N(z) = \varphi(z) - z\overline{\Phi}(z). \tag{34}$$

Proof. Assume that $C(\theta)$ is expandible in power series viz,

$$C(\theta) = \sum_{j=1}^\infty \frac{\kappa_j}{j!}\theta^j. \tag{35}$$

Consider (20), multiplied by θ, and the error estimate (31). Set $x = K'(\theta) = nC'(\theta)$, $\theta = \frac{z}{\sigma\sqrt{n}}$, and expand all terms in power series using (35). After some elementary calculations, one obtains the following expressions:

$$\theta\pi(x) = z \cdot E\left[\left(\frac{X-n\mu}{\sigma\sqrt{n}} - z \cdot \left\{1 + \sum_{j=1}^{\infty} \frac{\kappa_{j+2}}{\sigma^2 (j+1)!}\left(\frac{z}{\sigma\sqrt{n}}\right)^j\right\}\right)\right], \tag{36}$$

$$K(\theta) - \theta K'(\theta) + \tfrac{1}{2}\theta^2 K''(\theta) = \frac{z^3}{2\sqrt{n}} \cdot \sum_{j=0}^{\infty} \frac{\kappa_{j+3}}{\sigma^2 (j+3) j!}\left(\frac{z}{\sigma\sqrt{n}}\right)^j, \tag{37}$$

$$\theta\sqrt{K''(\theta)} = z \cdot \left(\sum_{j=0}^{\infty} \frac{\kappa_{j+2}}{\sigma^2 j!}\left(\frac{z}{\sigma\sqrt{n}}\right)^j\right)^{\frac{1}{2}}, \tag{38}$$

$$K(\theta) - \theta K'(\theta) = -z^2 \cdot \sum_{j=0}^{\infty} \frac{\kappa_{j+2}}{\sigma^2 (j+1)!}\left(\frac{z}{\sigma\sqrt{n}}\right)^j, \tag{39}$$

$$\epsilon(\theta) \le \frac{c_0}{\sqrt{n}} \cdot \left\{2 + \frac{1}{2}\frac{C^{(4)'}(\theta)}{C''(\theta)^2}\right\} = \frac{c_0}{\sqrt{n}} \cdot \left\{2 + \frac{1}{2}\frac{\sum_{j=0}^{\infty} \frac{\kappa_{j+4}}{j!}\left(\frac{z}{\sigma\sqrt{n}}\right)^j}{\left(\sum_{j=0}^{\infty} \frac{\kappa_{j+2}}{j!}\left(\frac{z}{\sigma\sqrt{n}}\right)^j\right)^2}\right\}. \tag{40}$$

Taking limits as $n \to \infty$, one obtains immediately (34). ■

Remark 4.1. Applying the *non-uniform* version of the Berry-Esseen theorem, another simple proof of (34) can be developed along the lines of the proof of Theorem 3 in Michel (1993).

Though (19) does not lead to a new proof of the classical CLT, under certain circumstances, it may provide improved rates of convergence to the CLT when compared to the uniform Berry-Esseen bound. The following simple example suffices to illustrate this point.

Example 4.1. Rates of convergence for a Gamma distribution. Suppose each X_i has cgf $C(\theta) = \alpha \cdot \ln \left\{\frac{\alpha}{\alpha - \mu\theta}\right\}$, and assume for simplicity that the relative variance is $k^2 = \left(\frac{\sigma}{\mu}\right)^2 = \frac{1}{\alpha} \le 1$. The classical Berry-Esseen theorem implies the upper bound

$$\overline{F}_n^S(z) := \Pr\left(\frac{X - n\mu}{\sigma\sqrt{n}} \ge z\right) \le \overline{\Phi}(z) + \varepsilon_n \ , \quad \varepsilon_n = \frac{c_0}{\sqrt{n}} \cdot \frac{E\left[\,|X_i - \mu|^3\right]}{\sigma^3}. \tag{41}$$

Since $k \le 1$, we proceed as in Example 3.2 to obtain the sharp distribution-free error estimate

$$\varepsilon_n = \frac{c_0}{\sqrt{n}} \cdot \{2 + \tfrac{1}{2}\gamma_2\}, \quad \gamma_2 = \frac{E\left[(X-\mu)^4\right]}{\sigma^4} - 3 \ \ (\textit{Excess kurtosis}). \tag{42}$$

For the Gamma distribution, one has $\gamma_2 = 6k^2$, and thus

$$\overline{F}_n^S(z) \le \overline{\Phi}(z) + \frac{c_0}{\sqrt{n}} \cdot \{2 + 3k^2\}. \tag{43}$$

On the other hand, an elementary evaluation of (19) by setting $z\sigma\sqrt{n} = n(C'(\theta) - \mu)$ and using the estimate (31) - which is justified because $\frac{C''(\theta)}{C'(\theta)^2} = k^2 \le 1$ - yields without difficulty (use a series expansion for the logarithm) the alternate upper bound

Table 1: rates of convergence for an exponential distribution with $k = 1$.

z and n	classical Berry-Esseen bounds		Esscher transformed bounds	
$z = 0$				
$n = 10^3$	0.378964	0.621036	0.257928	0.742072
$n = 10^4$	0.461725	07.53825	0.423450	0.576550
$n = 10^6$	0.496173	0.503827	0.492345	0.507655
$n = 10^{10}$	0.499962	0.500038	0.499923	0.500077
$n = 10^{14}$	0.5	0.5	0.499999	0.500001
$z = 1$				
$n = 10^3$	0.037619	0.279691	0.011953	0.308641
$n = 10^4$	0.120380	0.196930	0.112597	0.205765
$n = 10^6$	0.154828	0.162483	0.154064	0.163353
$n = 10^{10}$	0.158617	0.158694	0.158609	0.158702
$n = 10^{14}$	0.158655	0.158656	0.158655	0.158656
$z = 2$				
$n = 10^4$	-0.015525	0.061025	0.012720	0.033991
$n = 10^6$	0.018923	0.026578	0.021772	0.023850
$n = 10^{10}$	0.022712	0.022788	0.022740	0.022761
$n = 10^{14}$	0.022750	0.022751	0.022750	0.022750
$z = 3$				
$n = 10^6$	$-2.47760 \bullet 10^{-3}$	$5.17740 \bullet 10^{-3}$	$1.27627 \bullet 10^{-3}$	$1.44788 \bullet 10^{-3}$
$n = 10^8$	$9.67148 \bullet 10^{-4}$	$1.73265 \bullet 10^{-3}$	$1.34260 \bullet 10^{-3}$	$1.35963 \bullet 10^{-3}$
$n = 10^{10}$	$1.31162 \bullet 10^{-3}$	$1.38817 \bullet 10^{-3}$	$1.34917 \bullet 10^{-3}$	$1.35087 \bullet 10^{-3}$
$n = 10^{14}$	$1.34952 \bullet 10^{-3}$	$1.35028 \bullet 10^{-3}$	$1.34989 \bullet 10^{-3}$	$1.34991 \bullet 10^{-3}$
$n = 10^{16}$	$1.34986 \bullet 10^{-3}$	$1.34994 \bullet 10^{-3}$	$1.349897 \bullet 10^{-3}$	$1.34990 \bullet 10^{-3}$

$$\overline{F}_n^S(z) \le \exp\left\{ z^2 \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j+2} \left(\frac{k\,z}{\sqrt{n}} \right)^j \right\} \left\{ \overline{\Phi}(z) + \frac{2\,C_0}{\sqrt{n}} \cdot (2 + 3k^2) \cdot \exp\{-\tfrac{1}{2} z^2\} \right\}. \qquad (44)$$

By virtue of the exponential factor $\exp\{-\frac{1}{2}z^2\}$ in (44), this rate of convergence to the CLT improves the classical one (43) provided n and z are sufficiently large. For completeness, the stop-loss transform version of (44) obtained from (20) is for $z \geq 0$,

$$\pi_N^S(z) := E\left[\left(\frac{X-n\mu}{\sigma\sqrt{n}}-z\right)_+\right] \leq \left(1+\frac{kz}{\sqrt{n}}\right)\cdot\exp\left\{z^2\cdot\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j+2}\left(\frac{kz}{\sqrt{n}}\right)^j\right\}$$
$$\cdot\left\{\pi_N(z)+\frac{2C_0}{\sqrt{n}}\cdot(2+3k^2)\cdot\frac{1}{z}\exp\{-\tfrac{1}{2}z^2\}\right\}. \tag{45}$$

Table 1 exhibits a numerical comparison between various bounds. The derivation of similar explicit convergence results for any random variable with known cgf is left to the interested reader. For a wider range of applications, it may be more useful to derive asymptotic rates of convergence, which depend only on the first cumulants.

5. Asymptotic Expansion of the Rates of Convergence

Suppose that the power series expansion holds. We derive general asymptotic bounds of the type (44), (45), where terms are expanded up to the order $O\left(n^{-\frac{3}{2}}\right)$. It will turn out that the main terms of these bounds depend only on the skewness and kurtosis. In general, the remainder error term depends also on the fifth order cumulant. In the special case of non-negative random variables, only the skewness and kurtosis are required, as in Section 6.

To simplify calculations, set $x = K'(\theta)$ in (19) and (20). This "plausible" choice minimizes the exponential factor of the remainder error term. With
$K(\theta) = nC(\theta),\ \mu = C'(0),\ \sigma^2 = C''(0),\ z\sigma\sqrt{n} = K'(\theta) - n\mu = n(C'(\theta)-\mu)$,

$$\overline{F}_n^S(z) := \Pr\left(\frac{X-n\mu}{\sigma\sqrt{n}} \geq z\right) = \overline{F}(x),\ \ \pi_n^S(z) := E\left[\left(\frac{X-n\mu}{\sigma\sqrt{n}}-z\right)_+\right] = \frac{1}{\sigma\sqrt{n}}\pi(x),$$

the restatement of (19), (20), using the estimate (31), reads

$$\left|\overline{F}_n^S(z) - e^{n\left\{C(\theta)-\theta C'(\theta)+\frac{1}{2}\theta^2C''(\theta)\right\}}\overline{\Phi}\left(\theta\sqrt{nC''(\theta)}\right)\right| \leq 2\cdot\varepsilon_n(\theta)\cdot e^{n\{C(\theta)-\theta x\}}, \tag{46}$$

$$\left|\pi_N^S(z) - \frac{\exp\left\{n\left(C(\theta)-\theta C'(\theta)+\frac{1}{2}\theta^2C''(\theta)\right)\right\}}{\theta\sigma\sqrt{n}}\theta\sqrt{nC''(\theta)}\,\pi_N\left(\theta\sqrt{nC''(\theta)}\right)\right|$$
$$\leq 2\cdot\frac{\varepsilon_n(\theta)}{\theta\sigma\sqrt{n}}\cdot e^{n\{C(\theta)-\theta x\}}, \tag{47}$$

$$\epsilon_n(\theta) \leq \frac{C_0}{\sqrt{n}}\cdot\left\{2+\frac{1}{2}\frac{C^{(4)'}(\theta)}{C''(\theta)^2}\right\}. \tag{48}$$

Besides the usual manipulations on power series for sums, products and arbitrary powers, the main computational tool for the determination of asymptotic expansions is reversion of power series (originally due to Lagrange) in the slightly generalized version by Knuth [12], Section 4.7 and Exercise 8. In the following consider four power series $P(z) = \sum_{j=0}^{\infty} p_j z^j$, $Q(z) = \sum_{j=0}^{\infty} q_j z^j$, $R(z) = \sum_{j=0}^{\infty} r_j z^j$, $S(z) = \sum_{j=0}^{\infty} s_j z^j$. First, recall the well-known recursive formula to calculate the coefficients of a power series $P(z) = Q(z)^{\alpha}$, $\alpha \in \mathbb{R}$ (e.g. Gould [7], Pourahmadi [17]). If $q_0 = 1$ then $p_0 = 1$ and

$$np_n = \sum_{j=1}^{n}\{(\alpha+1)j-n\}\cdot q_j\cdot p_{n-j},\quad n = 2,3,\ldots \tag{49}$$

Second, consider the following generalized Lagrange reversion of power series, where the problem calls for solving the equation

$$z = Q(\theta) = \theta + q_2\,\theta^2 + q_3\,\theta^3 + \ldots \tag{50}$$

for θ, and for obtaining the coefficients p_j in the identity

$$R(\theta) = r_1\,\theta + r_2\,\theta^2 + r_3\,\theta^3 + \ldots = p_1\,z + p_2\,z^2 + p_3\,z^3 + \ldots = P(z). \tag{51}$$

The special case $r_1 = 1, r_2 = r_3 = \ldots = 0$, is simply reversion of a power series. The generalized classical Lagrange inversion formula states that

$$p_n = \frac{s_{n-1}}{n}, \quad n = 1,2,3,\ldots \tag{52}$$

where the involved power series satisfies the relation

$$S(z) = R'(z) \cdot \left\{\tfrac{1}{z} Q(z)\right\}^{-n}. \tag{53}$$

This solution can be translated into an effective computational algorithm, which requires $O(N^3)$ multiplications to evaluate the first N coefficients. For relevant details consult Knuth [12]. For our purpose, we will need only the first four coefficients, which are determined as follows:

$$p_1 = r_1,\; p_2 = r_2 - q_2 r_1,$$

$$p_3 = r_3 - 2q_2 r_2 + (2q_2^2 - q_3) r_1,$$

$$p_4 = r_4 - 3q_2 r_3 + (5q_2^2 - 2q_3) r_2 - (5q_2^3 - 5q_2 q_3 + q_4) r_1. \tag{54}$$

This relatively simple computational tool leads to the following main asymptotic expansion for the bounds (46), (47).

Proposition 5.1. (Asymptotic expansion of the Esscher transformed Berry-Esseen survival and stop-loss bounds) *Let* $X = \sum_{i=1}^{n} X_i$ *be a sum of independent and identically distributed random variables with cgf* $K(\theta) = nC(\theta)$, *and let* $\mu = C'(0)$, $\sigma^2 = C''(0)$, $\gamma\sigma^3 = C'''(0)$, $\gamma_2\sigma^4 = C^{(4)}(0)$, $\kappa_5 = C^{(5)}(0)$, *be the mean, variance, skewness, kurtosis and fifth order cumulant of each summand. Then the following asymptotic expansions in the parameter* n *hold.*

$$\left|\overline{F}_n^S(z) - \exp\{A_n(z)\}\overline{\Phi}(B_n(z))\right| \le D_n(z)\exp\{C_n(z)\}, \tag{55}$$

$$\left|\pi_n^S(z) - \frac{B_n(z)}{E_n(z)}\exp\{A_n(z)\}\pi_N(B_n(z))\right| \le \frac{D_n(z)}{E_n(z)}\exp\{C_n(z)\}. \tag{56}$$

Here the individual expansions are:

$$A_n(z) = z^2 \cdot \left\{\tfrac{1}{6}\gamma\frac{z}{\sqrt{n}} + \tfrac{1}{8}(\gamma_2 - 2\gamma^2)\frac{z^2}{n} + O\left(z^3 n^{-\frac{3}{2}}\right)\right\}, \tag{57}$$

$$B_n(z) = z \cdot \left\{1 + \tfrac{1}{24}(2\gamma_2 - 3\gamma^2)\frac{z^2}{n} + O\left(z^3 n^{-\frac{3}{2}}\right)\right\}, \tag{58}$$

$$C_n(z) = -\tfrac{1}{2}z^2 \cdot \left\{1 - \tfrac{1}{3}\gamma\frac{z}{\sqrt{n}} + \tfrac{1}{12}(3\gamma_2 - \gamma^2)\frac{z^2}{n} + O\left(z^3 n^{-\frac{3}{2}}\right)\right\}, \tag{59}$$

$$D_n(z) = \frac{2C_0}{\sqrt{n}} \cdot \left\{2 + \tfrac{1}{2}\gamma_2 + \left(\tfrac{1}{2}\frac{\kappa_5}{\sigma^5} - \gamma\gamma_2\right)\frac{z}{\sqrt{n}} + O\left(z^2 n^{-\frac{3}{2}}\right)\right\}, \tag{60}$$

$$E_n(z) = z \cdot \left\{ 1 - \tfrac{1}{2}\gamma \frac{z}{\sqrt{n}} + \tfrac{1}{6}(3\gamma_2 - \gamma^2)\frac{z^2}{n} + O\left(z^3 n^{-\frac{3}{2}}\right) \right\}. \tag{61}$$

Proof. Using (46) to (48), it suffices to apply generalized reversion of power series to solve

$$\frac{z}{\sigma\sqrt{n}} = Q(\theta) = \frac{C'(\theta) - \mu}{\sigma^2} = \sum_{j=1}^{\infty} \frac{\kappa_{j+1}}{\sigma^2\, j!}\theta^j \ , \tag{62}$$

for θ, and to obtain the coefficients of the first powers of z in the following identities

$$n\{C(\theta) - \theta C'(\theta) + \tfrac{1}{2}\theta^2 C''(\theta)\} = A_n(z),\ \ \theta\sqrt{nC''(\theta)} = B_n(z),$$

$$n\{C(\theta) - \theta C'(\theta)\} = C_n(z),\ \frac{c_0}{\sqrt{n}} \cdot \left\{ 2 + \tfrac{1}{2}\frac{C^{(4)'}(\theta)}{C''(\theta)^2} \right\} = D_n(z),\ \theta\sigma\sqrt{n} = E_n(z). \tag{63}$$

For this one uses five times the formulas (54) identifying left hand sides of (63) with $R(\theta)$ in (51) and right hand sides of (63) with $P(\frac{z}{\sigma\sqrt{n}})$. In this regard, for the convenience of the reader, we provide a summary of the routine calculations. In all of the following steps one has by (62):

$$q_1 = 1,\ q_2 = \tfrac{1}{2}\gamma\sigma,\ \ q_3 = \tfrac{1}{6}\gamma_2\sigma^2,\ \ q_4 = \tfrac{1}{24}\frac{\kappa_5}{\sigma^2}. \tag{64}$$

Step 1: $R(\theta) := C(\theta) - \theta C'(\theta) + \tfrac{1}{2}\theta^2 C''(\theta) = \tfrac{1}{2}\sum_{j=0}^{\infty}\frac{\kappa_{j+3}}{(j+3)\, j!}\theta^{j+3}.$

With $r_1 = r_2 = 0,\ \ r_3 = \tfrac{1}{6}\gamma\sigma^3,\ \ r_4 = \tfrac{1}{8}\gamma_2\sigma^4$, one obtains from (54):

$$p_1 = p_2 = 0,\ p_3 = \tfrac{1}{6}\gamma\sigma^3,\ p_4 = \tfrac{1}{8}(\gamma_2 - 2\gamma^2)\sigma^4. \tag{65}$$

Step 2: $R(\theta) := \theta\sqrt{C''(\theta)} = \sigma\theta \cdot \left\{ 1 + \sum_{j=1}^{\infty}\frac{\kappa_{j+2}}{\sigma^2\, j!}\theta^j \right\}^{\frac{1}{2}}.$

With $r_1 = \sigma,\ \ r_2 = \tfrac{1}{6}\gamma\sigma^2,\ \ r_3 = \tfrac{1}{8}(2\gamma_2 - \gamma^2)\sigma^3$, one obtains from (54):

$$p_1 = p_2 = 0,\ p_3 = \tfrac{1}{6}\gamma\sigma^3,\ p_4 = \tfrac{1}{8}(\gamma_2 - 2\gamma^2)\sigma^4. \tag{66}$$

Step 3: $R(\theta) := C(\theta) - \theta C'(\theta) = -\sum_{j=0}^{\infty}\frac{\kappa_{j+2}}{(j+2)\, j!}\theta^{j+2}.$

With $r_1 = 0,\ \ r_2 = -\tfrac{1}{2}\sigma^2,\ \ r_3 = -\tfrac{1}{6}\gamma\sigma^3,\ \ r_4 = -\tfrac{1}{8}\gamma_2\sigma^4$, one gets from (54):

$$p_1 = 0,\ p_2 = -\tfrac{1}{2}\sigma^2,\ p_3 = \tfrac{1}{6}\gamma\sigma^3,\ p_4 = -\tfrac{1}{24}(3\gamma^2 - 2\gamma_2)\sigma^4. \tag{67}$$

Step 4: $R(\theta) := \dfrac{C^{(4)'}(\theta)}{C''(\theta)^2} - \gamma_2 = \gamma_2 \cdot \left\{ 1 + \sum_{j=1}^{\infty}\frac{\kappa_{j+4}}{\kappa_4}\theta^j \right\} \cdot \left\{ 1 + \sum_{j=1}^{\infty}\frac{\kappa_{j+2}}{\sigma^2\, j!}\theta^j \right\}^{-2} - \gamma_2.$

One requires only that

$$p_1 = r_1 = \left\{ \frac{\kappa_5}{\sigma^5} - 2\gamma\gamma_2 \right\}\sigma. \tag{68}$$

Step 5: $R(\theta) := \theta.$

With $r_1 = 1,\ \ r_2 = r_3 = 0$, one obtains from (54):

$$p_1 = 1,\ \ p_2 = -\tfrac{1}{2}\gamma\sigma,\ \ p_3 = \tfrac{1}{6}(3\gamma^2 - \gamma_2)\sigma^2. \tag{69}$$

The resulting asymptotic expansions follow immediately.

■

6. Compound Poisson Improved Berry-Esseen Results

The asymptotic method of the preceding Section can be applied to derive "modified" normal approximations to some infinitely divisible distributions. We illustrate at the important compound Poisson distribution, of interest in insurance applications among others.

Let $X = \sum_{i=1}^{n} X_i$ be a random variable such that $N, Y_1, Y_2, \ldots, Y_N$ are independent, the Y_i's are non-negative identically distributed random variables with moment generating function $M(\theta) = E[e^{\theta Y_i}]$ and N is Poisson(λ) distributed. Its cgf is $K(\theta) = \lambda\{M(\theta) - 1\}$. Since X is infinitely divisible, it is possible to write $X = \sum_{i=1}^{n} X_i$ for each natural number n, where the X_i's are independent and identically distributed with cgf $C(\theta) = \frac{\lambda}{n}\{M(\theta) - 1\}$. This setting fits the assumptions of Proposition 5.1. However, since X is non-negative, it is more convenient to use the remainder error estimate (25) instead of (31). One sees immediately that this estimate may be replaced by

$$\epsilon(\theta) \leq \lim_{n\to\infty} \frac{c_0}{\sqrt{n}} \cdot \left\{ \frac{\frac{\lambda}{n} M'''(\theta) + 2\left(\frac{\lambda}{n}\right)^2 M'(\theta)\, M''(\theta)}{\left(\frac{\lambda}{n}\right)^{\frac{3}{2}} M''(\theta)^{\frac{3}{2}}} \right\} = \frac{c_0}{\sqrt{\lambda}} \frac{M'''(\theta)}{M''(\theta)^{\frac{3}{2}}} = \frac{c_0\, C'''(\theta)}{C''(\theta)^{\frac{3}{2}}}. \tag{70}$$

Expanding the right-hand side using generalized reversion of power series as in Section 5, one obtains the following asymptotic expansions, which provide (modified) normal approximations to compound Poisson quantities, for which the speed of convergence as the Poisson parameter gets larger are under explicit quantitative control.

Proposition 6.1. (Normal approximation to the compound Poisson survival distribution and stop-loss transform) *Let* $X = \sum_{i=1}^{N} Y_i$ *be a compound Poisson random variable with cgf* $K(\theta) = \lambda\{M(\theta) - 1\}$, *let* $m_j = M^{(j)}(0)$, $j = 1,2,3,4$, *be the first four moments of* Y_i, *and set* $c_3 = m_3 m_2^{-\frac{3}{2}}$, $c_4 = m_4 m_2^{-2}$. *Furthermore let*

$$\overline{F}_\lambda^S(z) := \Pr\left(\frac{X - \lambda m_1}{\sqrt{\lambda m_2}} \geq z\right), \quad \pi_\lambda^S(z) := E\left[\left(\frac{X - \lambda m_1}{\sqrt{\lambda m_2}} - z\right)_+\right]$$

be the standardized compound Poisson survival distribution and stop-loss transform. Then the following asymptotic expansions in λ *hold.*

$$\left|\overline{F}_\lambda^S(z) - \exp\{A_\lambda(z)\}\overline{\Phi}(B_\lambda(z))\right| \leq D_\lambda(z)\exp\{C_\lambda(z)\}, \tag{71}$$

$$\left|\pi_\lambda^S(z) - \frac{B_\lambda(z)}{E_\lambda(z)}\exp\{A_\lambda(z)\}\pi_N(B_\lambda(z))\right| \leq \frac{B_\lambda(z)}{E_\lambda(z)}\exp\{C_\lambda(z)\}, \tag{72}$$

with

$$A_\lambda(z) = z^2 \cdot \left\{\tfrac{1}{6}c_3 \frac{z}{\sqrt{\lambda}} + \tfrac{1}{8}(c_4 - 2c_3^2)\frac{z^2}{\lambda} + O\left(z^3\lambda^{-\frac{3}{2}}\right)\right\}, \tag{73}$$

$$B_\lambda(z) = z \cdot \left\{1 + \tfrac{1}{24}(2c_4 - 3c_3^2)\frac{z^2}{\lambda} + O\left(z^3\lambda^{-\frac{3}{2}}\right)\right\}, \tag{74}$$

$$C_\lambda(z) = -\tfrac{1}{2}z^2 \cdot \left\{1 - \tfrac{1}{3}c_3\frac{z}{\sqrt{\lambda}} + \tfrac{1}{12}(3c_4 - c_3^2)\frac{z^2}{\lambda} + O\left(z^3\lambda^{-\frac{3}{2}}\right)\right\}, \quad (75)$$

$$D_\lambda(z) = \frac{2C_0}{\sqrt{\lambda}} \cdot \left\{c_3 + (c_4 - \tfrac{3}{2}c_3^2)\frac{z}{\sqrt{\lambda}} + O\left(z^2\lambda^{-\frac{3}{2}}\right)\right\}, \quad (76)$$

$$E_\lambda(z) = z \cdot \left\{1 - \tfrac{1}{2}c_3\frac{z}{\sqrt{\lambda}} + \tfrac{1}{6}(3c_4 - c_3^2)\frac{z^2}{\lambda} + O\left(z^3\lambda^{-\frac{3}{2}}\right)\right\}. \quad (77)$$

Proof. As seen in the discussion above, Proposition 5.1 applies (up to the remainder error asymptotic expansion). Since $\gamma = \sqrt{\frac{n}{\lambda}}c_3$, $\gamma_2 = \frac{n}{\lambda}c_4$, the dependence upon n cancels out, and is replaced by the completely similar dependence upon λ. To get the asymptotic expansion of the estimate (70), proceed within the setting of Section 5. A calculation shows that

$$R(\theta) := C'''(\theta)C''(\theta)^{-\frac{3}{2}} - \gamma = \{\gamma_2 - \tfrac{3}{2}\gamma^2\}\sigma\,\theta + \ldots \quad (78)$$

By definition of z in (62) and (54), one obtains immediately

$$C'''(\theta)C''(\theta)^{-\frac{3}{2}} = \gamma + \{\gamma_2 - \tfrac{3}{2}\gamma^2\}\frac{z}{\sqrt{n}}, \quad (79)$$

from which the compound Poisson version (76) follows immediately by inserting the corresponding values of the skewness and kurtosis.

■

Table 2: Rates of convergence for a compound Poisson survival function.

z and λ	Michel bounds (80)		Esscher transformed bounds	
$z = 1$				
$\lambda = 10^4$	0.133	0.184	0.128	0.191
$\lambda = 10^6$	0.156	0.161	0.156	0.162
$\lambda = 10^8$	0.158	0.159	0.158	0.159
$z = 2$				
$\lambda = 10^4$	$-2.47 \cdot 10^{-3}$	0.048	0.017	0.031
$\lambda = 10^6$	0.02	0.025	0.022	0.024
$\lambda = 10^8$	0.022	0.023	0.023	0.023
$z = 3$				
$\lambda = 10^4$	-0.024	0.027	$9.136 \cdot 10^{-4}$	$2.252 \cdot 10^{-3}$
$\lambda = 10^6$	$-1.172 \cdot 10^{-3}$	$3.872 \cdot 10^{-3}$	$1.313 \cdot 10^{-3}$	$1.427 \cdot 10^{-3}$
$\lambda = 10^8$	$1.098 \cdot 10^{-3}$	$1.602 \cdot 10^{-3}$	$1.346 \cdot 10^{-3}$	$1.358 \cdot 10^{-3}$
$\lambda = 10^{10}$	$1.325 \cdot 10^{-3}$	$1.375 \cdot 10^{-3}$	$1.35 \cdot 10^{-3}$	$1.351 \cdot 10^{-3}$

Similarly to the improved rates of convergence to the CLT for the Gamma distribution in Example 4.1, Proposition 6.1 also provides improved rates of convergence for the compound

Poisson distribution. Applying the classical Berry-Esseen theorem and its non-uniform version, the following bounds have been derived in Michel [15], Theorems 1, 2, 3.

$$\left|\overline{F}_{\lambda}^{S}(z)-\overline{\Phi}(z)\right| \leq \frac{\min\left\{c_0\,,\,\frac{c_0+8(1+e)}{1+z^3}\right\}}{\sqrt{\lambda}}c_3, \tag{80}$$

$$|\pi_{\lambda}^{S}(z)-\pi_N(z)| \leq \frac{1}{2}\frac{c_0+8(1+e)}{z^2\sqrt{\lambda}}c_3. \tag{81}$$

As the Poisson parameter increases and z is sufficiently large, the Esscher transformed Berry-Esseen approximations (71), (72) yield improved approximations. This follows because the exponential factor $\exp\{C_{\lambda}(z)\}$ is of the order of magnitude $\exp\{-\frac{1}{2}z^2\}$, and thus can be made arbitrarily smaller than z^{-3} and z^{-2}. Tables 2 and 3 provide a numerical illustration based on the real-life insurance estimates $c_3 = 3.295, c_4 = 19.997$.

Table 3: Rates of convergence for a compound Poisson stop-loss transform.

z and λ	Michel bounds (81)		Esscher transformed bounds	
$z = 1$				
$\lambda = 10^4$	-0.419	0.586	0.054	0.116
$\lambda = 10^6$	0.033	0.134	0.080	0.087
$\lambda = 10^8$	0.078	0.088	0.083	0.084
$z = 2$				
$\lambda = 10^4$	-0.117	0.134	$1.898 \cdot 10^{-3}$	0.016
$\lambda = 10^6$	$-4.075 \cdot 10^{-3}$	0.021	$7.869 \cdot 10^{-3}$	$9.244 \cdot 10^{-3}$
$\lambda = 10^8$	$7.234 \cdot 10^{-3}$	$9.747 \cdot 10^{-3}$	$8.429 \cdot 10^{-3}$	$8.566 \cdot 10^{-3}$
$z = 3$				
$\lambda = 10^6$	$-5.203 \cdot 10^{-3}$	$5.967 \cdot 10^{-4}$	$3.328 \cdot 10^{-4}$	$4.469 \cdot 10^{-4}$
$\lambda = 10^8$	$-1.763 \cdot 10^{-4}$	$9.406 \cdot 10^{-4}$	$3.773 \cdot 10^{-4}$	$3.885 \cdot 10^{-4}$
$\lambda = 10^{10}$	$3.263 \cdot 10^{-4}$	$4.38 \cdot 10^{-4}$	$3.817 \cdot 10^{-4}$	$3.828 \cdot 10^{-4}$
$z = 4$				
$\lambda = 10^6$	$-3.134 \cdot 10^{-3}$	$3.149 \cdot 10^{-3}$	$5.692 \cdot 10^{-6}$	$9.213 \cdot 10^{-6}$
$\lambda = 10^8$	$-3.07 \cdot 10^{-4}$	$3.213 \cdot 10^{-4}$	$7.005 \cdot 10^{-6}$	$7.345 \cdot 10^{-6}$
$\lambda = 10^{10}$	$-2.427 \cdot 10^{-5}$	$3.856 \cdot 10^{-5}$	$7.131 \cdot 10^{-6}$	$7.165 \cdot 10^{-6}$

7. Optimal Esscher Transformed Berry-Esseen Bounds

The asymptotic expansions of the preceding Section, based on the first four moments of the severity distribution of the compound Poisson random sum, have shown the potential for improved rates of convergence to the central limit theorem. It is of interest to ask whether better or even optimal bounds may be achieved. We show that this is possible under the additional knowledge of further moments. Since generalized reversion of power series is quite cumbersome in the optimal situation, we shall dispense us with the technical details and just present a simplified version, which is strong enough to generate useful practical results.

Our starting point is again (19), (20), where (20) has been multiplied with θ. Then the remainder error term for both Esscher transformed Berry-Esseen bounds equals (in the compound Poisson case):

$$\varepsilon(\theta,x) = 2\varepsilon(\theta)e^{K(\theta)-\theta x}, \quad K(\theta) = \lambda\{M(\theta)-1\}, \tag{82}$$

where, with (70), one can choose

$$\epsilon(\theta) = \frac{c_0}{\sqrt{\lambda}}\frac{M'''(\theta)}{M''(\theta)^{\frac{3}{2}}}. \tag{83}$$

The first order derivatives with respect to θ of $E(\theta,x) = \ln\{\varepsilon(\theta,x)\}$ are

$$E'(\theta,x) = K'(\theta) - x + \frac{\varepsilon'(\theta)}{\varepsilon(\theta)}, \quad \frac{\varepsilon'(\theta)}{\varepsilon(\theta)} = \frac{M^{(4)}(\theta)}{M''(\theta)} - \frac{3}{2}\frac{M'''(\theta)}{M''(\theta)}, \tag{84}$$

$$E''(\theta,x) = K'(\theta) + \frac{M^{(5)}(\theta)}{M''(\theta)} - \left(\frac{M^{(4)}(\theta)}{M''(\theta)}\right)^2 - \frac{3}{2}\left\{\frac{M^{(4)}(\theta)}{M''(\theta)} - \left(\frac{M'''(\theta)}{M''(\theta)}\right)^2\right\}. \tag{85}$$

From elementary calculus, it follows that "optimal" confidence bounds (19), (20), in the sense of smallest remainder error term, are obtained by setting $x = K'(\theta) + \frac{\varepsilon'(\theta)}{\varepsilon(\theta)}$, which will yield a minimum provided $E''(\theta,x) > 0$.

Example 7.1. Normal approximation to the Poisson distribution.
If $M(\theta) = e^\theta$, one has $\frac{\varepsilon'(\theta)}{\varepsilon(\theta)} = -\frac{1}{2}$, $E''(\theta,x) = \lambda e^\theta > 0$, and $x = \lambda e^\theta - \frac{1}{2}$ yields optimal Esscher transformed Berry-Esseen bounds.

In general, the optimal bounds will be of the form

$$\left|\overline{F}\left(\lambda M'(\theta) + \frac{\varepsilon'(\theta)}{\varepsilon(\theta)}\right) - \exp\{\lambda(M(\theta)-1-\theta M'(\theta) + \tfrac{1}{2}\theta^2 M''(\theta))\} \cdot \overline{\Phi}\left(\theta\sqrt{\lambda M''(\theta)} + \frac{\varepsilon'(\theta)}{\varepsilon(\theta)\sqrt{\lambda M''(\theta)}}\right)\right| \le 2\varepsilon(\theta)\cdot\exp\left\{\lambda(M(\theta)-1-\theta M'(\theta)) - \theta\frac{\varepsilon'(\theta)}{\varepsilon(\theta)}\right\}, \tag{86}$$

$$\Big|\theta \cdot \pi\left(\lambda M'(\theta) + \frac{\varepsilon'(\theta)}{\varepsilon(\theta)}\right) - \exp\{\lambda(M(\theta) - 1 - \theta M'(\theta) + \tfrac{1}{2}\theta^2 M''(\theta))\}$$
$$\cdot\, \theta\sqrt{\lambda M''(\theta)} \cdot \pi_N\left(\theta\sqrt{\lambda M''(\theta)} + \frac{\varepsilon'(\theta)}{\varepsilon(\theta)\sqrt{\lambda M''(\theta)}}\right)\Big|$$
$$\leq 2\varepsilon(\theta) \cdot \exp\left\{\lambda(M(\theta) - 1 - \theta M'(\theta)) - \theta\frac{\varepsilon'(\theta)}{\varepsilon(\theta)}\right\}, \quad (87)$$

$$\epsilon(\theta) = \frac{c_0}{\sqrt{\lambda}}\frac{M'''(\theta)}{M''(\theta)^{\frac{3}{2}}}, \quad \frac{\varepsilon'(\theta)}{\varepsilon(\theta)} = \frac{M^{(4)}(\theta)}{M''(\theta)} - \frac{3}{2}\frac{M'''(\theta)}{M''(\theta)}. \quad (88)$$

Now consider the moment generating function in the power series expansion form

$$M(\theta) = \textstyle\sum_{j=0}^{\infty} m_j\theta^j, \quad m_0 = 1, \quad (89)$$

with a standard normalization

$$\frac{\lambda M'(\theta) + \frac{\varepsilon'(\theta)}{\varepsilon(\theta)} - \lambda m_1}{\sqrt{\lambda m_2}} = z, \quad (90)$$

and apply the algorithm for generalized reversion of power series, as in Sections 5 and 6, to obtain asymptotic expansions for the optimal normalized bounds. Since the technical details for this are quite tedious, it is preferable to apply a more direct method. Setting $\theta = z/\sqrt{\lambda m_2}$, and expanding all terms in power series using (89), as in the proof of Proposition 4.1, yields practical expansions of the optimal normalized bounds.

Making use of the standardized compound Poisson survival function $\overline{F}_\lambda^S(z)$ and stop-loss transform $\pi_\lambda^S(z)$, one obtains the relationships

$$\begin{aligned} \overline{F}_\lambda^S(\psi(\lambda,z)) &= \overline{F}(\lambda m_1 + \psi(\lambda,z)\sqrt{\lambda m_2}), \\ z \cdot \pi_\lambda^S(\psi(\lambda,z)) &= \theta \cdot \pi(\lambda m_1 + \psi(\lambda,z)\sqrt{\lambda m_2}), \end{aligned} \quad (91)$$

with

$$\psi(\lambda,z) = \sqrt{\frac{\lambda}{m_2}}\{M'(\theta) - m_1\} + \frac{1}{\sqrt{\lambda m_2}}\frac{\varepsilon'(\theta)}{\varepsilon(\theta)}$$
$$= z \cdot \left\{1 + \sum_{j=1}^{\infty}\frac{m_{j+2}}{m_2\,(j+1)!}\left(\frac{z}{\sqrt{\lambda m_2}}\right)^j\right\} + \frac{\varepsilon(\lambda,z)}{\sqrt{\lambda m_2}}, \quad (92)$$

$$\varepsilon(\lambda,z) = \frac{M_4(\lambda,z)}{M_3(\lambda,z)} - \frac{3}{2}\frac{M_3(\lambda,z)}{M_2(\lambda,z))}, \quad (93)$$

with

$$M_i(\lambda,z) = \sum_{j=0}^{\infty}\frac{m_{j+i}}{j!}\left(\frac{z}{\sqrt{\lambda m_2}}\right)^j, \quad i = 2,3,4. \quad (94)$$

Furthermore, one gets without difficulty the expansions

$$A(\lambda,z) = \lambda\{M(\theta)-1-\theta M'(\theta)+\tfrac{1}{2}\theta^2 M''(\theta)\} = \frac{z^3}{2\sqrt{\lambda m_2}} \cdot \sum_{j=0}^{\infty} \frac{m_{j+3}}{m_2\,(j+3)\,j!}\left(\frac{z}{\sqrt{\lambda m_2}}\right)^j , \quad (95)$$

$$B(\lambda,z) = \theta\sqrt{\lambda\, M''(\theta)} + \frac{\varepsilon'(\theta)}{\varepsilon(\theta)\sqrt{\lambda\, M''(\theta)}} = z \cdot \sqrt{\frac{M_2(\lambda,z)}{m_2}} + \frac{\varepsilon(\lambda,z)}{\sqrt{\lambda\, M_2(\lambda,z)}} , \quad (96)$$

$$C(\lambda,z) = \lambda\{M(\theta)-1-\theta M'(\theta)\} - \theta\frac{\varepsilon'(\theta)}{\varepsilon(\theta)} = -z^2 \cdot \sum_{j=0}^{\infty} \frac{m_{j+2}}{m_2(j+2)j!}\left(\frac{z}{\sqrt{\lambda m_2}}\right)^j - \frac{z\cdot\varepsilon(\lambda,z)}{\sqrt{\lambda m_2}}, \quad (97)$$

$$\varepsilon(\lambda,z) = \frac{2C_0}{\sqrt{\lambda}}\frac{M'''(\theta)}{M''(\theta)^{\frac{3}{2}}} = \frac{2C_0}{\sqrt{\lambda}}\frac{M_3(\lambda,z)}{M_2(\lambda,z)^{\frac{3}{2}}} , \quad (98)$$

$$E(\lambda,z) = \frac{\theta\sqrt{\lambda M''(\theta)}}{z} = \sqrt{\frac{M_2(\lambda,z)}{m_2}} . \quad (99)$$

In insurance applications, one is often more interested in confidence bounds for stop-loss rates $\frac{\pi(x)}{\lambda m_1}$. Taking into account (91), one search bounds for the expression

$$\frac{\pi\left(\{1+k(\lambda)\,\psi(\lambda,z)\}\cdot\lambda m_1\right)}{\lambda m_1} = k(\lambda)\cdot\pi_\lambda^S(\psi(\lambda,z)), \quad (100)$$

where

$$k(\lambda) = \sqrt{\frac{1}{\lambda}\frac{m_2}{m_1^2}} , \quad (101)$$

is the coefficient of variation of the compound Poisson distribution.

Table 4: Rates of convergence for a compound Poisson survival function.

$\xi(\lambda,z)$		Michel bounds		Opt. Esscher transf. bounds	
$\xi(10^5,1)$	1.00598	0.14854	0.16449	0.14715	0.16643
$\xi(10^6,1)$	1.00188	0.15546	0.1605	0.15501	0.16113
$\xi(10^8,1)$	1.00019	0.15834	0.15884	0.15829	0.1589
$\xi(10^5,2)$	1.012	0.01347	0.02942	0.01965	0.02385
$\xi(10^6,2)$	1.00376	0.01981	0.02486	0.02176	0.02311
$\xi(10^8,2)$	1.00038	0.02246	0.02296	0.02265	0.02279
$\xi(10^5,3)$	1.01809	$-6.83726\cdot10^{-3}$	$9.11348\cdot10^{-3}$	$1.03427\cdot10^{-3}$	$1.35615\cdot10^{-3}$
$\xi(10^6,3)$	1.00565	$-1.24166\cdot10^{-3}$	$3.80241\cdot10^{-3}$	$1.24537\cdot10^{-3}$	$1.35414\cdot10^{-3}$
$\xi(10^8,3)$	1.00056	$1.09064\cdot10^{-3}$	$1.59504\cdot10^{-3}$	$1.33925\cdot10^{-3}$	$1.35042\cdot10^{-3}$
$\xi(10^5,4)$	1.02424	$-4.55973\cdot10^{-3}$	$4.60298\cdot10^{-3}$	$2.00697\cdot10^{-5}$	$2.8577\cdot10^{-5}$
$\xi(10^6,4)$	1.00755	$-1.48713\cdot10^{-3}$	$1.54345\cdot10^{-3}$	$2.76074\cdot10^{-5}$	$3.07597\cdot10^{-5}$
$\xi(10^8,4)$	1.00075	$-1.23034\cdot10^{-4}$	$1.85645\cdot10^{-4}$	$3.12475\cdot10^{-5}$	$3.15835\cdot10^{-5}$

Summarizing, the optimal Esscher transformed Berry-Esseen bounds for the survival function and the stop-loss rates are derivable as follows.

$$|\overline{F}(\xi(\lambda,z)\cdot\lambda m_1) - \exp\{A(\lambda,z)\}\overline{\Phi}(B(\lambda,z))|$$
$$\le D(\lambda,z)\exp\{C(\lambda,z)\},\ \xi(\lambda,z) = 1 + k(\lambda)\psi(\lambda,z), \tag{102}$$

$$\left|\frac{\pi(\xi(\lambda,z)\cdot\lambda m_1)}{\lambda m_1} - k(\lambda)E(\lambda,z)\exp\{A(\lambda,z)\}\pi_N(B(\lambda,z))\right|$$
$$\le k(\lambda)\frac{D(\lambda,z)}{z}\exp\{C(\lambda,z)\},\ z > 0. \tag{103}$$

Again, these bounds must be compared with the bounds for $\overline{F}_\lambda^S(\psi(\lambda,z))$ and $k(\lambda)\cdot\pi_\lambda^S(\psi(\lambda,z))$ obtained from (80) and (81). The Tables 4 and 5 provide a numerical evaluation based on real-life data. The expansions have been truncated after the second term, hence the first six moments of the severity distributions are required. A scrutiny look shows that the optimal bounds do not differ very much from the simpler general asymptotic bounds obtained in Section 6. Expressed in some unit of money, the required moments are given by

$$m_1 = 1.2\cdot 10^2,\ m_2 = 5.06\cdot 10^4,\ m_3 = 3.75\cdot 10^7,$$
$$m_4 = 5.12\cdot 10^{10},\ m_5 = 1.12\cdot 10^{12},\ m_6 = 3.2\cdot 10^{17}. \tag{104}$$

Table 5: Rates of convergence for compound Poisson stop-loss rates.

$\xi(\lambda,z)$	Michel bounds		Opt. Esscher transf. bounds	
$\xi(10^5,1)$ 1.00598	-4.40078•10^{-4}	1.41122•10^{-3}	4.31847•10^{-4}	5.46139•10^{-4}
$\xi(10^6,1)$ 1.00188	6.16552•10^{-3}	2.49047•10^{-4}	1.49965•10^{-4}	1.61422•10^{-4}
$\xi(10^8,1)$ 1.00019	1.46679•10^{-5}	1.65512•10^{-5}	1.55556•10^{-5}	1.56703•10^{-5}
$\xi(10^5,2)$ 1.012	-1.82724•10^{-4}	2.76881•10^{-4}	4.20441•10^{-5}	5.44825•10^{-5}
$\xi(10^6,2)$ 1.00376	-7.7847•10^{-6}	3.89611•10^{-5}	1.50752•10^{-5}	1.63435•10^{-5}
$\xi(10^8,2)$ 1.00038	1.35296•10^{-6}	1.82369•10^{-6}	1.58315•10^{-6}	1.59594•10^{-6}
$\xi(10^5,3)$ 1.01809	-9.9291•10^{-5}	1.03062•10^{-4}	1.69382•10^{-6}	2.32983•10^{-6}
$\xi(10^6,3)$ 1.00565	-9.68072•10^{-6}	1.10342•10^{-5}	6.56436•10^{-7}	7.24399•10^{-7}
$\xi(10^8,3)$ 1.00056	-3.33427•10^{-8}	1.75809•10^{-7}	7.10253•10^{-8}	7.17235•10^{-8}
$\xi(10^5,4)$ 1.02424	-5.63121•10^{-5}	5.63688•10^{-5}	2.63171•10^{-8}	3.89245•10^{-8}
$\xi(10^6,4)$ 1.00755	-5.79601•10^{-6}	5.81969•10^{-6}	1.16145•10^{-8}	1.30917•10^{-8}
$\xi(10^8,4)$ 1.00075	-5.74826•10^{-8}	6.01289•10^{-8}	1.32084•10^{-9}	1.33658•10^{-9}

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