

First-Passage Percolation on Ladder-Like Graphs With Inhomogeneous Exponential Times

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Abstract. *We determine the asymptotic speed of the first-passage percolation process on some ladder-like graphs (or width-2 stretches) when the times associated with different edges are independent and exponentially distributed but not necessarily all with the same mean. The method uses a particular Markov chain associated with the first-passage percolation process and properties of its stationary distribution.*

Key words : First-Passage Percolation, Time Constant, Asymptotic Speed.

AMS Subject Classifications : 60K35, 33C10

1. Introduction

Consider a graph G with vertex set V and (undirected) edge set $E \subset V \times V$. If there is an edge $e = \langle v, v' \rangle$ joining vertices v and v' , they are said to be adjacent. A path $\pi(v, v')$ between v and v' is an alternating sequence of vertexes and edges $(v_0, e_1, v_1, \dots, e_n, v_n)$ such that $e_i = \langle v_{i-1}, v_i \rangle$ for $i = 1, \dots, n$, $v_0 = v$ and $v_n = v'$.

We think of each edge e as being associated with a (typically non-negative) random time ξ_e . Formally we define $\xi = \{\xi_e, e \in E\}$ to be a stochastic process indexed by the edges of the graph, and define the time $T\pi(v, v')$ of a path $\pi(v, v')$ to be

$$T\pi(v, v') = \sum_{e \in \pi(v, v')} \xi_e \quad \text{and} \quad T(v, v') = \inf_{\pi(v, v')} T\pi(v, v')$$

as the shortest time of any path between v and v' . $T(v, v')$ is called the first passage time from v to v' and is the subject of investigation in first passage percolation, see e.g. [6].

We may also consider first passage times between sets of vertices. If V_1 and V_2 are such sets, then

$$T(v, v') = \inf_{v \in V_1, v' \in V_2} T\pi(v, v')$$

is the first passage time from V_1 to V_2 .

Usually, all first passage times under consideration are from some fixed and given set V_0 to a varying set V' , in which case we denote by $T(V')$ the first passage time from V_0 to V' , or $T(v')$ if V' is the singleton set $\{v'\}$, and refer to this simply as the first passage time of V' . We can think of this as a model for a contagious disease. At time zero a subset V_0 of the vertices are infected and subsequently the disease spreads to adjacent vertices. The time for the infection to be submitted from v to v' along the edge $\langle v, v' \rangle$ is $\xi_{\langle v, v' \rangle}$.

A typical question, and indeed the one we shall be concerned with, is how *fast* does the infection spread? This of course requires some metric on the set of vertices. We shall look at cases of *ladder-like* graphs with “height” as the measure of distance and independent edge times, each having an exponential distribution.

2. First Passage Percolation on Ladder-Like Graphs

A ladder-like graph, or width-2 stretch, is a graph whose vertex set is $V = \mathbb{N} \times \{0, 1\}$, \mathbb{N} denoting $\{0, 1, 2, \dots\}$, with the ordinary Euclidean metric, considered when each vertex is placed in the (real) plane \mathbb{R}^2 . Our interest is on a particular class \mathcal{L} of such graphs where vertices *may* be adjacent if they are no more than Euclidean distance $\sqrt{2}$ away from each other and, in addition, the graph is translational invariant, see Figure 1. By translational invariant we mean that we can superimpose the graph on itself by shifting it one unit length to the right in Figure 1.

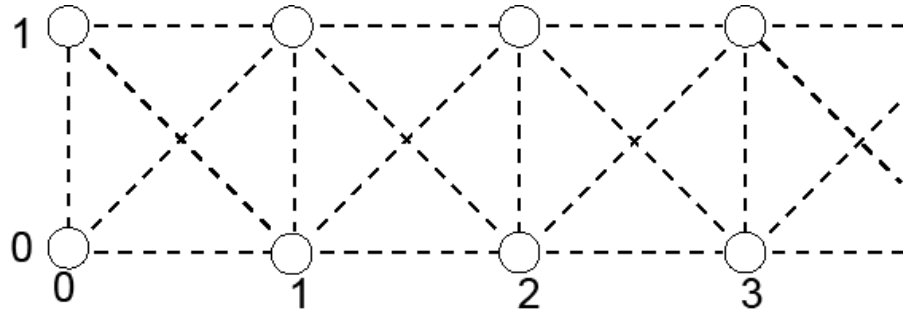


Figure 1: Part of the vertices of a ladder-like graph. The possible edges of a graph in \mathcal{L} are dashed. Note that it must also be translational invariant.

A vertex (x, y) in such a graph is said to be at height $x \in \mathbb{N}$ and level $y \in \{0, 1\}$. We will consider $V_0 = \{(0, 0), (0, 1)\}$ to be infected at time zero, and we consider first passage times of the height n , i.e. the set $V_n = \{(n, 0), (n, 1)\}$.

It should be noted that the Euclidean metric only serves as to define the class of graphs we are interested in. Once this is done, we only consider “height” as a measure of length in the graph. In this sense, the “length” of the (possible) diagonal edges are not longer than that of the (possible) horizontal edges.

Typically the focus is on the a.s. limit

$$\sigma = \sigma(G, \xi) = \lim_{n \rightarrow \infty} \frac{T(V_n)}{n},$$

called the time constant. Now, the first passage times on these graphs are subadditive and the almost sure convergence of $T(V_n)/n$ relies on the subadditive ergodic theorem. See Proposition 4.1 of [9] for the details. In our calculations it is rather the asymptotic speed \mathcal{V} of percolation that appears naturally, but this is just the inverse of the time constant, $\mathcal{V} = 1/\sigma$.

The investigation of the rate, or speed, of percolation on ladder-like graphs began with [2], which consider *the ladder*, i.e. where all vertices at distance 1 are connected by an edge. That paper gives a method of calculating σ when ξ_e 's are independent with the same discrete distribution, as well as a method for getting arbitrarily good bounds for the same quantity when the distribution is continuous (and well behaved).

A few years later [9] and [4] independently studied the case of having independent and identical exponential times. In [9], six graphs in \mathcal{L} are considered and explicit expressions for σ are found for three of them, using recursive distributional equations. It should be noted that these six graphs are in fact “all” graphs in the class \mathcal{L} as far as σ is concerned, in the sense that any other graph is either trivial or it has the same σ as one of these six.

Another method for finding σ is employed in [4], which studies the ladder. There a Markov chain is found and the speed of percolation can be calculated “at stationarity”, a method that will be repeated below in a more general context. The method also allows the study of another aspect of the percolation process called the *residual times*, defined in [4].

Now, we fix some notation that will be used throughout this paper. Let

$$N_t = \sup_{i \in \{0,1\}} \{x : T[(x, i)] \leq t\} \quad (1)$$

denote – in the interpretation of the model of a contagious disease – the height of the infection at time t . And

$$M_t = \sup \{x : \max \{T[(x, 0)], T[(x, 1)]\} \leq t\} \quad (2)$$

denotes the largest height at which both levels have an infected vertex.

Now define the *front process* F_t as

$$F_t = N_t - M_t. \quad (3)$$

In some situations F_t is a Markov chain on \mathbb{N} . As we shall see, the problem of determining the percolation speed \mathcal{V} is more or less equivalent to finding the stationary distribution of this Markov chain.

We will denote by $\varrho_n \sim \mathfrak{G}_n$ the property that

$$\lim_{n \rightarrow \infty} \frac{\varrho_n}{\mathfrak{G}_n} = 1,$$

for real sequences ϱ_n and \mathfrak{G}_n . Let z^n and $z^{\bar{n}}$ denote falling and rising factorials, respectively, which may be expressed as

$$z^n = z(z-1)\cdots(z-n+1) = \frac{\Gamma(z+1)}{\Gamma(z+1-n)},$$

and

$$z^n = z(z+1)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}.$$

3. The Ladder With Heterogeneous Exponential Times

In this section we generalize the result of [4] to a ladder with heterogeneous exponential times. Let each horizontal edge be associated with an exponential time with intensity $\lambda_h > 0$ (i.e. an exponential with expectation $1/\lambda_h$) and each vertical edge be associated with an exponential time with intensity $\lambda_v > 0$. Random variables associated to different edges are independent. Notice, since we may equally well measure intensity in units of λ_h , there is no loss of generality in saying that $\lambda_h = 1$. Let us do so and denote λ_v simply by λ . Note that [4] is concerned with the case $\lambda = 1$.

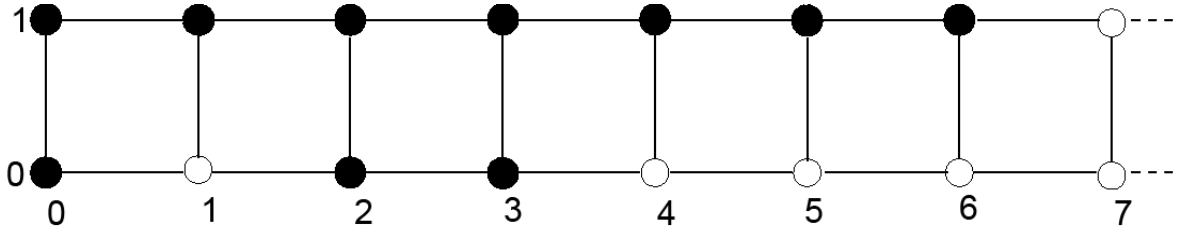


Figure 2: Infected nodes at time t marked as black. Here $M_t = 3$, $N_t = 6$ and $F_t = 3$. Not shown in the picture is that each vertical edge has intensity λ and each horizontal edge has intensity 1.

Now, F_t , defined by (1), (2) and (3), is a continuous time Markov chain on \mathbb{N} . This fact becomes clear as we write down the intensity matrix Q of the process. Denote by $t' = \inf\{t > t_0 : F_t \neq F_{t_0}\}$ the first time, after t_0 , that the F -process changes its value.

Consider first the case of $F_{t_0} = 0$, which means that M_{t_0} and N_{t_0} both equal, say, n . Then both $(n, 0)$ and $(n, 1)$ are infected, and there is no infected node at any greater height. The only states that can be infected at t' are $(n+1, 0)$ or $(n+1, 1)$, both resulting in $F_{t'} = 1$, and both as a result of the infection spreading along a horizontal edge. Informally, we will say that these edges *lead to* the state $F_{t'} = 1$, which is rather inaccurate since edges connect different vertices and not different states of the front process, but a convenient terminology. So, thus far we have established that the intensity from state 0 to state 1, is 2, and the intensity away from state 0 is -2.

If $F_{t_0} = 3$, as in Figure 2, there are 3 vertical edges leading to states 2, 1 and 0 respectively. There are 2 horizontal edges leading to states 2 and 4, respectively. Thus, the intensities to 0 and 1 are both λ , to 2 it is $1 + \lambda$ and to 4 the intensity is 1.

In a similar manner, one derives the following form (4) of the intensity matrix. This Markov chain is irreducible, so if we can find a stationary distribution $\Pi = (\pi_0, \pi_1, \dots)$, on \mathbb{N} , such that $\Pi Q = 0$, then this is the unique stationary distribution of F_t , and as $t \rightarrow \infty$, F_t will converge to it.

$$Q = \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & & \\ 1 + \lambda & -2 - \lambda & 1 & 0 & 0 & & \\ \lambda & 1 + \lambda & -2 - 2\lambda & 1 & 0 & \dots & \\ \lambda & \lambda & 1 + \lambda & -2 - 3\lambda & 1 & & \\ \lambda & \lambda & \lambda & 1 + \lambda & -2 - 4\lambda & & \\ \lambda & \lambda & \lambda & \lambda & 1 + \lambda & & \\ & & \vdots & & & \ddots & \end{pmatrix}. \quad (4)$$

Now, our main question is: what is the asymptotic speed of percolation? When $F_{t_0} = 0$ the intensity by which N_t increases is 2, since there are two horizontal edges from the state $F_{t_0} = 0$ to the state $N_{t'} = N_{t_0} + 1$ (and no other edge that makes any difference to neither of these processes). When $F_{t_0} > 0$, there is one horizontal edge leading to $N_{t'} = N_{t_0} + 1$ (and no other edge that makes any difference to the N_t -process). Hence, at stationarity we get the speed of percolation as

$$V = 2\pi_0 + 1(1 - \pi_0) = 1 + \pi_0. \quad (5)$$

Next, we proceed to find Π through the equation $\Pi Q = 0$. One way of doing this is to express each π_n , $n > 0$, in terms of π_0 ; $\pi_n = a_n\pi_0 - b_n$.

Now, the first entry of $0 = \Pi Q$ is

$$0 = -2\pi_0 + \pi_1 + \lambda \sum_{j \geq 1} \pi_j.$$

As $\sum_{j \geq 1} \pi_j = 1$, which implies $\sum_{j \geq 1} \pi_j = 1 - \pi_0$, we get

$$0 = -(2 + \lambda)\pi_0 + \lambda + \pi_1 \quad \text{yielding} \quad \pi_1 = (2 + \lambda)\pi_0 - \lambda. \quad (6)$$

Repeating this procedure for the equations corresponding to columns 2 and 3 of Q , yields, after isolating π_2 and π_3 ,

$$\pi_2 = (2\lambda^2 + 7\lambda + 2)\pi_0 - (2\lambda^2 + 3\lambda), \quad (7)$$

and

$$\pi_3 = (6\lambda^3 + 26\lambda^2 + 22\lambda + 2)\pi_0 - (6\lambda^3 + 14\lambda^2 + 6\lambda). \quad (8)$$

Next, taking the difference of equations corresponding to columns k and $k + 1$ for any $k \geq 2$ yields

$$0 = \pi_{k-1} - (\lambda k + 3)\pi_k + (\lambda(k + 2) + 3)\pi_{k+1} - \pi_{k+2},$$

which we shall rewrite as

$$\pi_n = (\lambda n + 3)\pi_{n-1} - (\lambda(n - 2) + 3)\pi_{n-2} + \pi_{n-3}, \quad n \geq 4. \quad (9)$$

We formally define a_n and b_n by the relation $\pi_n = a_n\pi_0 - b_n$, for $n \geq 1$. Thus, for $n = 1, 2, 3$ they are defined by equations (6), (7) and (8), respectively, and for $n \geq 4$ can be found iteratively through (9), with a_n or b_n in place of π_n .

As necessarily $\pi_n \rightarrow 0$ as $n \rightarrow \infty$, we can find π_0 as

$$\pi_0 = \lim_{n \rightarrow \infty} \frac{b_n}{a_n}.$$

This far we copied the procedure of [4], which proceeded to prove the link to Bessel functions, along the lines of how this fact was originally discovered. Here, we start instead with the Bessel functions.

3.1. The Bessel functions

The Bessel functions of the first and second kind, $J_\nu(z)$ and $Y_\nu(z)$, as described in e.g. [7], both satisfy the recursion

$$C_{\nu+1}(z) + C_{\nu-1}(z) = \frac{2\nu}{z} \cdot C_\nu(z). \quad (10)$$

Define, for any (real numbers) n, A and $B \neq 0$,

$$\hat{J}_n = \hat{J}_n(A, B) = J_{n+A/B}(2/B) \quad \text{and} \quad \hat{Y}_n = \hat{Y}_n(A, B) = Y_{n+A/B}(2/B).$$

Then \hat{J}_n and \hat{Y}_n both satisfy

$$\hat{C}_{n+1} + \hat{C}_{n-1} = (A + Bn)\hat{C}_n. \quad (11)$$

Define the function

$$\Upsilon_n = \Upsilon(n, m, A, B) = \pi[\hat{J}_n \hat{Y}_m - \hat{J}_m \hat{Y}_n]. \quad (12)$$

This function inherits recursion (11) (in parameter n)

$$\Upsilon_{n+1} + \Upsilon_{n-1} = (A + Bn)\Upsilon_n. \quad (13)$$

Next, define

$$\Delta_n = \Delta(n, m) = \Delta(n, m, A, B) = \Upsilon_n - \Upsilon_{n-1}.$$

When m and/or A and B are clear from the context, or unimportant, we write Δ_n or $\Delta(n, m)$ instead of the lengthy $\Delta(n, m, A, B)$. As the next claim makes clear, this function is designed with (9) in mind.

Claim 3.1. Δ_k satisfies,

$$\Delta_n = [1 + A + B(n-1)]\Delta_{n-1} - [1 + A + B(n-3)]\Delta_{n-2} + \Delta_{n-3}. \quad (14)$$

Proof. By using the definition $\Delta_k = \Upsilon_k - \Upsilon_{k-1}$ on the right hand side of (14), rearranging terms so that relation (13) can be applied (as indicated in (15)), we see that the right hand side of (14) equals

$$\begin{aligned} & \underbrace{[A + B(n-1)]\Upsilon_{n-1}}_{\Upsilon_n + \Upsilon_{n-2}} - 2 \underbrace{[A + B(n-2)]\Upsilon_{n-2}}_{-2\Upsilon_{n-1} - 2\Upsilon_{n-3}} + \\ & + \underbrace{[A + B(n-3)]\Upsilon_{n-3}}_{+\Upsilon_{n-2} + \Upsilon_{n-4}} + \Upsilon_{n-1} - 2\Upsilon_{n-2} + 2\Upsilon_{n-3} - \Upsilon_{n-4} \end{aligned} \quad (15)$$

$$= \Upsilon_n - \Upsilon_{n-1}, \quad (16)$$

which is the same as the left-hand side of (14). ■

Claim 3.2.

$$\Upsilon(n, n, A, B) = 0 \quad (17)$$

and

$$\Upsilon(n + 1, n, A, B) = B. \quad (18)$$

Proof. The first assertion follows trivially from the definition (12).

For the second assertion, we need two important relations for the Bessel functions. The first is that

$$zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z), \quad \forall \nu. \quad (19)$$

and the same holds with Y_ν replacing J_ν , see 3.2(4), p. 45 and 3.56(4), p. 66 of [7], resp. The second is a fact relating to the Wronskian, namely that

$$\frac{2}{\pi z} = J_\nu(z)Y'_\nu(z) - J'_\nu(z)Y_\nu(z), \quad \forall \nu, z \neq 0, \quad (20)$$

see 3.63(1), p. 76 of [7].

Relation (19) when inserted in (20) gives

$$\frac{2}{z} = \pi[J_{\nu+1}(z)Y_\nu(z) - J_\nu(z)Y_{\nu+1}(z)]$$

and $z = 2/B$ gives the desired result. ■

Remark 3.1. It should be noted that the Υ -function does not solve a recursion of the form (14) with *arbitrary* initial conditions. Indeed, by relation (13) we get

$$\Delta_{n+1} - \Delta_n = (A + Bn - 2)\Upsilon(n, m, A, B).$$

Hence, with A and B considered fixed, two points of the sequence of Δ 's (such that $A + Bn - 2 \neq 0$) is enough to determine m - which may not be unique, but in general not arbitrary - which in turn determines the entire sequence.

From the preceding claim, and relation (13), we are able to state the claim that follows.

Claim 3.3.

$$\Delta(m, m, A, B) = B,$$

$$\Delta(m + 1, m, A, B) = B, \quad \text{and}$$

$$\Delta(m + 2, m, A, B) = [A + B(m + 1) - 1]B.$$

Next, we need some asymptotic properties of Υ_n .

Claim 3.4. As $n \rightarrow \infty$,

$$\Upsilon(n, m, A, B) \sim \hat{J}_m \Gamma(n + A/B) B^{n+A/B}. \quad (21)$$

Proof. Reference [1] lists the following asymptotic relations, for fixed x and n tending to

infinity,

$$J_n(x) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ex}{2n}\right)^n \quad \text{and} \quad Y_n(x) \sim -\sqrt{\frac{2}{\pi n}} \left(\frac{2n}{ex}\right)^n,$$

hence, by the well-known Stirling formula $\Gamma(n+1) \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$ we conclude

$$\hat{J}_n \sim \frac{1}{\Gamma(n+A/B+1)B^{n+A/B}}$$

and

$$\hat{Y}_n \sim -\frac{1}{\pi} \Gamma(n+A/B) B^{n+A/B} \tag{22}$$

from which the claim follows. ■

Now, we have the tools to continue with the problem of finding the speed of the percolation process.

3.2. Finding the stationary distribution

If we set $A = 2 + \lambda$ and $B = \lambda$ it is clear that relation (14) fits (9). One way of trying to describe a_n and b_n is through some linear combination of Δ 's. By inspection of Claim 3.3, we choose to work with $\Delta(n, 1)$ and $\Delta(n, 2)$. Then one finds that

$$a_n = \frac{2\lambda^2 + 8\lambda + 5}{\lambda} \cdot \Delta(n, 1) - \frac{\lambda + 3}{\lambda} \cdot \Delta(n, 2),$$

$$b_n = \frac{2\lambda^2 + 4\lambda + 1}{\lambda} \cdot \Delta(n, 1) - \frac{\lambda + 1}{\lambda} \cdot \Delta(n, 2),$$

is true for $n = 1, 2, 3$ which makes it true for every n .

Next, with the aid of Claim 3.4, we calculate

$$\pi_0 = \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \frac{(2\lambda^2 + 4\lambda + 1)J_{2+2/\lambda}(2/\lambda) - (\lambda + 1)J_{3+2/\lambda}(2/\lambda)}{(2\lambda^2 + 8\lambda + 5)J_{2+2/\lambda}(2/\lambda) - (\lambda + 3)J_{3+2/\lambda}(2/\lambda)}, \tag{23}$$

and thereafter, a long, but straightforward, calculation* yields, for $n \geq 1$,

$$\pi_n = c(\hat{J}_{n-1} - \hat{J}_n), \quad \text{with } c = \frac{2}{(2\lambda^2 + 8\lambda + 5)\hat{J}_1 - (\lambda + 3)\hat{J}_2},$$

after which it can be verified that Π is indeed a distribution since,

$$\sum_{k=0}^n \pi_k = 1 - c\hat{J}_n \rightarrow 1,$$

as $\hat{J}_n \rightarrow 0$ by (22).

The speed \mathcal{V} of percolation in this model as given by (5) is $1 + \pi_0$, with π_0 as in (23), also depicted in Figure 3 with λ ranging from close to zero* to 20.

*An analogous calculation may be found in [4].

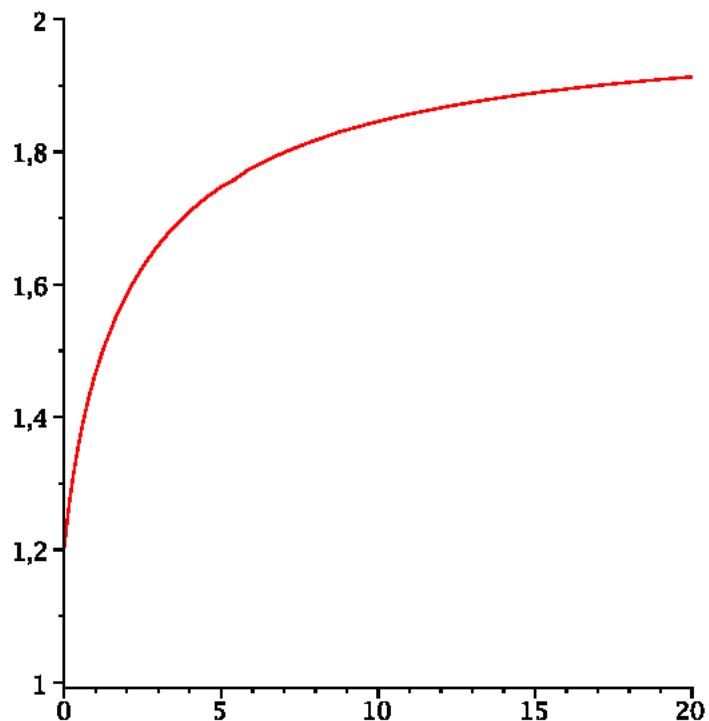


Figure 3: The speed of percolation on the ladder as a function of (exponential) intensity λ on the vertical edges (as measured in units of the intensity of the horizontal edges), plotted by Maple. note that the speed at $\lambda = 0$ is 1.

Although not defined, the speed as a function of $\lambda = 0$ is 1, since $\lambda = 0$ must be thought of as not having any vertical edges at all (or equivalent that the times associated with these are infinite) and as such the time to reach V_n is the time it takes until the first of two independent Poisson processes reaches n . Although this is always a bit faster than the time for a single such process, the effect wears off as n tends to infinity.

Similarly, the speed at $\lambda = \infty$ is 2, if we interpret $\lambda = \infty$ as having zero time associated with vertical edges. Then infection between adjacent vertices on different levels is immediate, and the time it takes to move one step up the ladder is the minimum of two exponential variables each having one unit of intensity, i.e. an exponential random time having intensity 2.

*It appears to be difficult to evaluate the speed for very small values of λ , as $B = 1/\lambda$ becomes very large. In Figure 3 it looks as if the speed at $\lambda = 0$ is ≈ 1.2 . This is, however not the case.

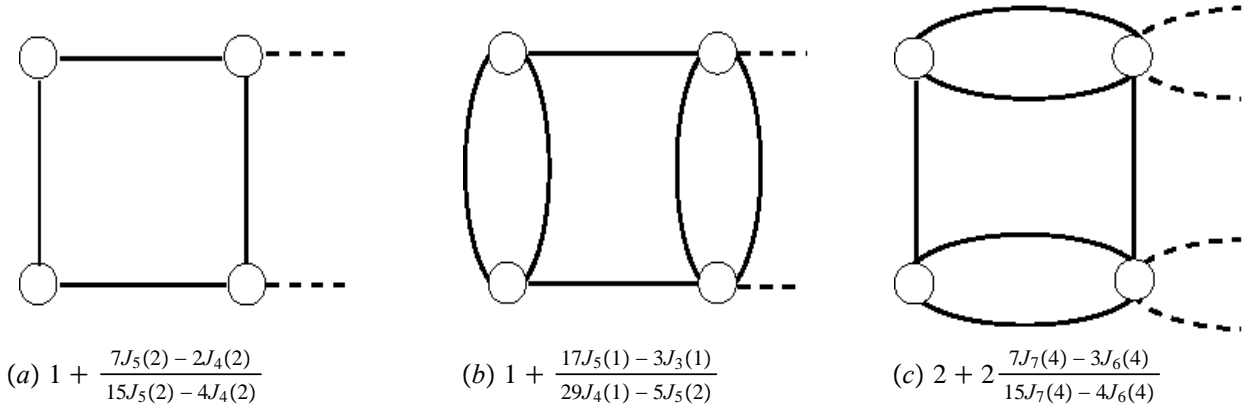


Figure 4: The speed of percolation when all edges are associated with random variable having intensity 1. Numerical values to two digits are, from left to right, 1.47, 1.59 and 2.74.

Furthermore, having an intensity of 2 associated with an edge is equivalent to having two (independent) edges associated with intensity 1, since the minimum time of two intensity 1 exponential random variables is an intensity 2 exponential random variable. Some examples of the speed of percolation on different ladders are shown in Figure 4. Notice that in the right most graph, we calculate the speed as two times that of having $\lambda = 1/2$.

4. Adding Diagonals to the Ladder

Next, we see what happens when diagonals are added to the ladder. For this model we cannot find a solution through Bessel function, as was the case with the ladder. Neither can we consider arbitrary intensities associated to the diagonals, these must be of the same intensity as the horizontal, else the front process becomes intractable.

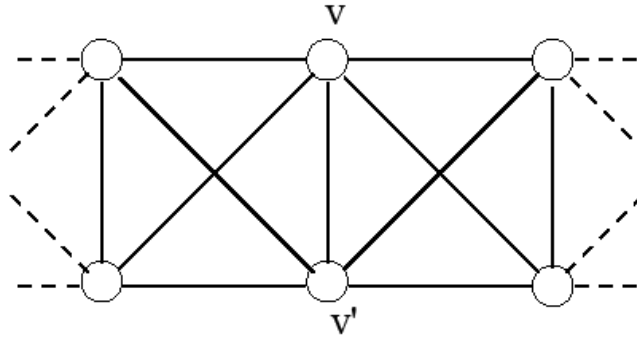


Figure 5: Intensities associated with vertical edges have intensity λ , as measured in the unit of intensity that is associated with both the horizontal and diagonal edges. Note that the graph is the same if interchange v and v' (without breaking any edges).

Part of the graph we are now considering is depicted in Figure 5. In this figure we see two nodes v and v' . If we interchange the positions of these two nodes, without breaking any edges, the graph is effectively unaltered. The vertical edges are still the same (they are now “upside down”, but this is irrelevant as edges are undirected). Four horizontal edges have become diagonal and vice versa, but as these are all associated with the same intensities, nothing has

essentially changed. This is, of course, the reason we need to have the same intensities on horizontal and diagonal edges.

Consider the front process, defined by (1),(2) and (3), at time t . If we disregard all nodes below height M_t , there are 2^k different sets of infected nodes (at t) that could yield $\{F_t = k\}$. From any such state the intensity to move to any other state that yields $\{F'_t = k'\}$ is the same, and as such the front process will be a Markov chain on \mathbb{N} .

In Figure 6 we illustrate this when the front process is found to be 2 at t .

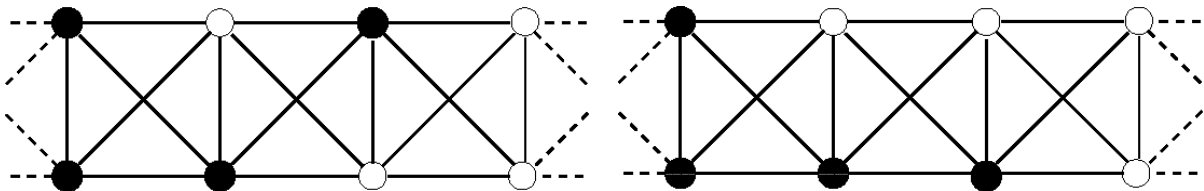


Figure 6: Two of the four possible sets of infected nodes at t (disregarding nodes below height M_t) that would yield $F_t = 2$. The infected nodes are marked in black. The remaining two sets are found by mirroring the cases above through a horizontal line.

Now, it is straightforward to compute the intensity matrix Q of the front process, e.g. when in state 2, as in Figure 6, then intensities to state 0 is $1 + \lambda$, to state 1 is $3 + \lambda$ and to state 3 is 2, and by similar considerations we get

$$Q = \begin{pmatrix} -4 & 4 & 0 & 0 & 0 & & \\ 2 + \lambda & -4 - \lambda & 2 & 0 & 0 & & \\ 1 + \lambda & 3 + \lambda & -6 - 2\lambda & 2 & 0 & \dots & \\ 1 + \lambda & 2 + \lambda & 3 + \lambda & -8 - 3\lambda & 2 & & \\ 1 + \lambda & 2 + \lambda & 2 + \lambda & 3 + \lambda & -10 - 4\lambda & & \\ 1 + \lambda & 2 + \lambda & 2 + \lambda & 2 + \lambda & 3 + \lambda & & \\ & & \vdots & & & \ddots & \end{pmatrix}. \quad (24)$$

We proceed to find the stationary distribution $\Pi = (\pi_0, \pi_1, \dots)$. Analogously to the computation of (6), (7), (8) and (9) we get

$$\pi_n = [(2 + \lambda)n + 3]\pi_{n-1} - [(2 + \lambda)(n - 2) + 4]\pi_{n-2} + 2\pi_{n-3}, \quad n \geq 4, \quad (25)$$

and if we let $\pi_n = c_n\pi_0 - d_n$, for $n \geq 1$, we have

$$(c_1, c_2, c_3) = (5 + \lambda, 28 + 17\lambda + 2\lambda^2, 226 + 226\lambda + 68\lambda^2 + 6\lambda^3),$$

and

$$(d_1, d_2, d_3) = (1 + \lambda, 8 + 9\lambda + 2\lambda^2, 66 + 98\lambda + 44\lambda^2 + 6\lambda^3),$$

where c_n, d_n for $n \geq 4$ can be determined via relation (25) with c_n or d_n in place of π_n . As in the previous section, we aim to determine π_0 via

$$\lim_{n \rightarrow \infty} \frac{d_n}{c_n}. \quad (26)$$

Once, this is done, the speed of percolation \mathcal{V} is

$$\mathcal{V} = 4\pi_0 + 2(1 - \pi_0) = 2(1 + \pi_0).$$

4.1. Use of generating functions

Next, we employ the recursive techniques of [3] to get an explicit expression for c_n and d_n . For ease of comparison, we will adopt the notation of that paper. [3] considers recursively defined sequences a_n of the form

$$a_n = \sum_{i=1}^K [\alpha_i(n-i) + \tilde{\beta}_i] a_{n-i}, \quad n \geq K, \quad (27)$$

where $\alpha_1, \dots, \alpha_K$ and $\tilde{\beta}_1, \dots, \tilde{\beta}_K$ are given numbers, and the initial part of the sequence a_0, \dots, a_{K-1} is fixed. Also, the numbers have been normalized as to give $\alpha_1 = 1$. We may work in this setting if we, for $n \geq 0$ set

$$\hat{a}_n = c_{n+1}(2+\lambda)^{-n} \quad \text{and} \quad a_n^* = d_{n+1}(2+\lambda)^{-n}.$$

Our main interest is $\lim_n d_n/c_n$ which will equal $\lim_n a_n^*/\hat{a}_n$. So, we aim to find a formula for a_n given by (27) where $K = 3$ and $\alpha_1 = 1$, $\alpha_3 = 0$ and α_2 typically is negative. Set $\alpha = -\alpha_2$. Then the parameters

$$\alpha = \frac{1}{2+\lambda},$$

$$\tilde{\beta}_1 = 2 + \frac{3}{2+\lambda} = \frac{7+2\lambda}{2+\lambda},$$

$$\tilde{\beta}_2 = -\frac{1}{2+\lambda} - \frac{4}{(2+\lambda)^2} = -\frac{6+\lambda}{(2+\lambda)^2}$$

and

$$\tilde{\beta}_3 = \frac{2}{(2+\lambda)^3},$$

are the same for the sequences \hat{a}_n and a_n^* , which then “only” differ in initial values

$$(\hat{a}_1, \hat{a}_2, \hat{a}_3) = \left(5 + \lambda, \frac{28+17\lambda+2\lambda^2}{2+\lambda}, \frac{226+226\lambda+68\lambda^2+6\lambda^3}{(2+\lambda)^2} \right),$$

and

$$(a_1^*, a_2^*, a_3^*) = \left(1 + \lambda, \frac{8+9\lambda+2\lambda^2}{2+\lambda}, \frac{66+98\lambda+44\lambda^2+6\lambda^3}{(2+\lambda)^2} \right).$$

Following [3] we define the generating function

$$A(z) = \sum_{k=0}^{\infty} a_k z^k$$

for the sequence a_n . Next, relation (27), with $K = 3$ and parameters as above, yields

$$A(z)q(z) = A^2 p(z) + r(z), \quad (28)$$

with

$$p(z) = \sum_{i=0}^3 \alpha_i z^{i-1} = 1 - \alpha z,$$

$$q(z) = 1 - \sum_{i=0}^3 \tilde{\beta}_i z^i = 1 - \tilde{\beta}_1 z - \tilde{\beta}_2 z^2 - \tilde{\beta}_3 z^3,$$

and

$$r(z) = \sum_{i=0}^2 R_i z^i, \tag{29}$$

with $R_0 = a_0$, $R_1 = a_1 - \tilde{\beta}_1 a_0$, and $R_2 = a_2 - (1 + \tilde{\beta}_1) a_1 - \tilde{\beta}_2 a_2$.

Let $I \subset \mathbb{R}$ be such that $0 \in I$ and $p(z) \neq 0$ on I . Theorem 3.2 (i) and (iv) of [3] reveals that there exists a solution A to (28) in I and any such solution satisfies the asymptotic expansion $A(z) = \sum_{n=0}^N a_n z^n + \mathcal{O}(z^{N+1})$, as $z \rightarrow 0$.

Now, of the three polynomials p , q and r it is only the latter that differs between \hat{a}_n and a_n^* . So let us denote these specific polynomials $\hat{r}(z) = \sum \hat{R}_i z^{i-1}$ and $r^*(z) = \sum R_i^* z^{i-1}$, respectively. The coefficients simplify to

$$\hat{R}_0 = 5 + \lambda,$$

$$\hat{R}_1 = \frac{28+17\lambda+2\lambda^2}{2+\lambda} - \frac{7+2\lambda}{2+\lambda} (5 + \lambda) = -\frac{7}{2+\lambda} = -7\alpha,$$

$$\begin{aligned} \hat{R}_2 &= \frac{226+226\lambda+68\lambda^2+6\lambda^3}{(2+\lambda)^2} - \left(1 + \frac{226+226\lambda+68\lambda^2+6\lambda^3}{(2+\lambda)^2} \right) \\ &\quad + \frac{6+\lambda}{(2+\lambda)^2} (5 + \lambda) = \frac{4}{(2+\lambda)^2} = 4\alpha^2, \end{aligned}$$

$$R_0^* = 1 + \lambda,$$

$$R_1^* = \frac{8+9\lambda+2\lambda^2}{2+\lambda} - \frac{7+2\lambda}{2+\lambda} (1 + \lambda) = \frac{1}{2+\lambda} = \alpha,$$

$$\begin{aligned} R_2^* &= \frac{66+98\lambda+44\lambda^2+6\lambda^3}{(2+\lambda)^2} - \left(1 + \frac{66+98\lambda+44\lambda^2+6\lambda^3}{(2+\lambda)^2} \right) \\ &\quad + \frac{6+\lambda}{(2+\lambda)^2} (1 + \lambda) = 0. \end{aligned} \tag{30}$$

Still using the notation of [3], we implicitly define γ and the rational function $g(z)$ via

$$\frac{q(z)}{z^2 p(z)} = z^{-2} + \gamma z^{-1} + g(z),$$

so that

$$\gamma = \alpha - \tilde{\beta}_1 = -\frac{6+2\lambda}{2+\lambda} \quad \text{and} \quad g(z) = \frac{\alpha(\alpha - \tilde{\beta}_1) - \tilde{\beta}_2 - \tilde{\beta}_3 z}{1 - \alpha z}.$$

We fix an antiderivative G to g as

$$G(z) = Cz + D \ln(1 - \alpha z),$$

where

$$C = \frac{\tilde{\beta}_3}{\alpha} = 2\alpha^2 \quad \text{and} \quad D = \frac{\tilde{\beta}_3}{\alpha^2} + \frac{\tilde{\beta}_2}{\alpha} + \tilde{\beta}_1 - \alpha = \alpha(2 + \lambda) = 1.$$

Now, we can write down what in [3] is called *the principal solution* to (28), which we consider for $z < 0$,

$$\begin{aligned} A_0(z) &= \int_0^\infty (1-zt)^\gamma \exp\left\{-t - G\left(\frac{z}{1-zt}\right) + G(z)\right\} \frac{r\left(\frac{z}{1-zt}\right)}{p\left(\frac{z}{1-zt}\right)} dt \\ &= \int_0^\infty \left[(1-zt)^\gamma \frac{(1-\alpha z)^D (1-zt)^D}{(1-(\alpha+t)z)^D} \exp\left\{-\frac{Ctz^2}{1-zt}\right\} e^{-t} \right. \\ &\quad \left. \times \frac{1-zt}{1-(\alpha+t)z} \sum_{i=0}^2 R_i\left(\frac{z}{1-zt}\right)^i \right], \end{aligned} \quad (31)$$

where by three Taylor expansions, as $z \nearrow 0$, and the relation $(-x)^n = (-1)^n x^n$ we get

$$\begin{aligned} A_0(z) &= \sum_{i=0}^2 R_i \int_0^\infty \left[(1-zt)^{\gamma+D+1-i} \sum_{j=0}^N \frac{(-\alpha)^j D^j z^j}{j!} \right. \\ &\quad \left. \times \sum_{k=0}^N \frac{(-Ctz^2)^k}{k!(1-zt)^k} \sum_{m=0}^N \frac{(-\alpha-t)^m (-D-1)^m z^m}{m!} e^{-tz^i} \right] dt + \mathcal{O}(z^{N+1}) \\ &= \sum_{i=0}^2 R_i \sum_{j=0}^N \frac{(-\alpha)^j D^j}{j!} \sum_{k=0}^N \frac{(-C)^k}{k!} \sum_{m=0}^N \frac{(D+1)^m}{m!} \\ &\quad \cdot \underbrace{\int_0^\infty (1-zt)^{\gamma+D+1-i-k} t^k (\alpha+t)^m e^{-t} dt}_{=I} \cdot z^{i+j+2k+m}. \end{aligned} \quad (32)$$

The integral I in (32), is, by Taylor expansion, as $z \nearrow 0$,

$$\begin{aligned} I &= \int_0^\infty \sum_{l=0}^N \frac{(-t)^l (\gamma+D+1-i-k)^l z^l}{l!} t^k (\alpha+t)^m e^{-t} dt + \mathcal{O}(z^{N+1}) \\ &= \sum_{l=0}^N \frac{(k+i-\gamma-D-1)^l}{l!} z^l \int_0^\infty t^{k+l} \sum_{n=0}^m \binom{m}{n} t^n \alpha^{m-n} e^{-t} dt + \mathcal{O}(z^{N+1}) \\ &= \sum_{l=0}^N \frac{(k+i-\gamma-D-1)^l}{l!} z^l \sum_{n=0}^m \binom{m}{n} \alpha^{m-n} (k+l+n)! + \mathcal{O}(z^{N+1}). \end{aligned} \quad (33)$$

So, from (32) and (33), we get the following formula

$$\begin{aligned} A_0(z) &= \sum_{i=0}^2 R_i \sum_{j=0}^N \frac{(-\alpha)^j D^j}{j!} \sum_{k=0}^N \frac{(-C)^k}{k!} \sum_{l=0}^N \frac{(k+i-\gamma-D-1)^l}{l!} \\ &\quad \cdot \sum_{m=0}^N \frac{(D+1)^m}{m!} \sum_{n=0}^m \binom{m}{n} \alpha^{m-n} (k+l+n)! z^{i+j+2k+l+m} + \mathcal{O}(z^{N+1}), \end{aligned} \quad (34)$$

which simplifies somewhat since $D = 1$ in our present application. Using this, and letting B_0 and B_1 be the nonzero summands of $\sum_{j=0}^{\infty} (-\alpha)^j D^j / j!$, i.e. $B_0 = 1$ and $B_1 = -\alpha$, and by defining

$$F_i(k, l, m) = \frac{(-C)^k}{k!} \frac{(k+i-\gamma-2)^l}{l!} \frac{2^{\bar{m}}}{m!} \sum_{n=0}^m \binom{m}{n} \alpha^{m-n} (k+l+n)!, \quad (35)$$

we can write

$$A_0(z) = \sum_{i=0}^2 R_i \sum_{j=0}^N B_j \sum_{k,l,m \geq 0} F_i(k, l, m) z^{i+j+2k+l+m}.$$

Further, defining $B_j = 0$ if $j \notin \{0, 1\}$ and $R_i = 0$ if $i \notin \{0, 1, 2\}$ makes

$$A_0(z) = \sum_{L=0}^3 \sum_{k,l,m \geq 0} \sum_{i+j=L} B_j R_i F_i(k, l, m) z^{L+2k+l+m}. \quad (36)$$

Since $A_0(z)$ can be written in the form (34) this allows us, by Theorem 3.2 of [3], to identify the coefficients of $A_0(z)$ to those of $A(z)$, so from (36),

$$a_N = \sum_{L=0}^3 \sum_{2k+l+m=N-L} \sum_{i+j=L} B_j R_i F_i(k, l, m), \quad (37)$$

if we follow the convention that an empty sum is zero.

Now, to handle a_N , consider

$$\begin{aligned} \hat{F}_i(M) &= \sum_{2k+l+m=M} F_i(k, l, m) \\ &= \sum_{2k+l+m=M} \frac{(-C)^k}{k!} \frac{(k+i-\gamma-2)^l}{l!} (m+1) \sum_{n=0}^m \binom{m}{n} \alpha^{m-n} (k+l+n)!. \end{aligned} \quad (38)$$

Claim 4.1.

$$L_i := \lim_{M \rightarrow \infty} \frac{\hat{F}_i(M)}{M! M^{\hat{\gamma}+1+i}} = \sum_{k=0}^{\infty} \frac{(-C)^k}{k! \Gamma(k+1+\hat{\gamma}+i)} \int_0^1 (1-x)^{\hat{\gamma}+k+i} x e^{\alpha x} dx, \quad (39)$$

where $\hat{\gamma} = -\gamma - 3 = -\lambda/(2 + \lambda) \in (-1, 0]$.

Proof. The details are given in Appendix A. ■

We need to find a useful expression for the integral that appears in (39).

Claim 4.2. Define, for $n \in \mathbb{N}$,

$$I(n) = \int_0^1 (1-x)^{\hat{\gamma}+n} x e^{\alpha x} dx \quad \text{and} \quad J(n) = \int_0^1 (1-x)^{\hat{\gamma}+n} e^{\alpha x} dx. \quad (40)$$

Then

$$I(n) = \frac{\Gamma(\hat{\gamma}+n+1)}{\alpha^n} \left[\frac{I(0) - \frac{n}{\alpha} J(0)}{\Gamma(\hat{\gamma}+1)} + \frac{1}{\alpha^2} \sum_{m=0}^n \frac{(n+1-m)\alpha^m}{\Gamma(\hat{\gamma}+1+m)} \right].$$

Proof. The details are given in Appendix B. (Note that the result is true for any $\alpha \neq 0$ but that it is essential that $\hat{\gamma} > -1$.) ■

4.2. Evaluating the limit

Now we return to examination of the limits L_i given in (39). By (39), (40) and (61) we have, for $i \in \{0, 1, 2\}$,

$$\begin{aligned} L_i &= \sum_{k=0}^{\infty} \frac{(-C)^k}{k! \Gamma(\hat{\gamma}+k+1+i)} I(k+i) \quad L_i = \sum_{k=0}^{\infty} \frac{(-C)^k}{k! \Gamma(\hat{\gamma}+k+1+i)} I(k+i) \\ &= \frac{I(0) - \frac{i}{\alpha} J(0)}{\alpha^i \Gamma(\hat{\gamma}+1)} S_3 - \frac{J(0)}{\alpha^{i+1} \Gamma(\hat{\gamma}+1)} S_4 + \frac{1}{\alpha^{i+2}} S(i), \end{aligned} \quad (41)$$

where

$$\begin{aligned} S_3 &= \sum_{k=0}^{\infty} \frac{(-C/\alpha)^k}{k!} = e^{-C/\alpha} = e^{-2\alpha}, \\ S_4 &= \sum_{k=0}^{\infty} \frac{k(-C/\alpha)^k}{k!} = -\frac{C}{\alpha} e^{-C/\alpha} = -2\alpha e^{-2\alpha}, \end{aligned}$$

and

$$S(i) = \sum_{k=1}^{\infty} \sum_{m=1}^{k+i} f(k, m, i),$$

where

$$f(k, m, i) = \frac{(-1)^k (2\alpha)^k \alpha^m (k+i+1-m)}{k! \Gamma(\hat{\gamma}+1+m)}. \quad (42)$$

This yields, by rearranging (41) and using $C/\alpha^2 = 2$,

$$L_i = \frac{1}{\alpha^i} \left(\frac{e^{-2\alpha} [I(0) + 2J(0)]}{\Gamma(\hat{\gamma}+1)} - i \cdot \frac{e^{-2\alpha} J(0)}{\alpha \Gamma(\hat{\gamma}+1)} + \frac{1}{\alpha^2} S(i) \right). \quad (43)$$

Recall that $\alpha \in (0, 1/2]$ and that $\hat{\gamma} > -1$ so that f defined in (42) satisfies $|f(k, l, m)| < |k+i+1-m|/k!m!$ when $k \geq 0$ and $m \geq 1$, and so the sum $S(i)$ is absolutely convergent and we are thus free to interchange the order of summation

$$\begin{aligned}
S(0) &= \sum_{k=0}^{\infty} \sum_{m=1}^k f(k, m, 0) = \sum_{k=1}^{\infty} \sum_{m=1}^k f(k, m, 0) = \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} f(k, m, 0) \\
&= \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} f(j+m, m, 0) = \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} f(j+m, m, 0) \\
&= \sum_{j=0}^{\infty} (-2\alpha)^j (j+1) \sum_{m=1}^{\infty} \frac{(-1)^m (2\alpha^2)^m}{\Gamma(m+1+\hat{\gamma})\Gamma(m+1+j)} = \mathfrak{I}(0),
\end{aligned} \tag{44}$$

if we define

$$\mathfrak{I}(k) = \sum_{j=0}^{\infty} (-2\alpha)^j (j+1+k) \sum_{m=1}^{\infty} \frac{(-1)^m (2\alpha^2)^m}{\Gamma(m+1+\hat{\gamma})\Gamma(m+1+j)}. \tag{45}$$

Similarly to how we handled $S(0)$ we get

$$\begin{aligned}
S(1) &= \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} f(j+m-1, m, 1) \\
&= \sum_{j=0}^{\infty} (-2\alpha)^{j-1} (j+1) \sum_{m=1}^{\infty} \frac{(-1)^m (2\alpha^2)^m}{\Gamma(m+1+\hat{\gamma})\Gamma(m+1+j-1)} \\
&= -\frac{1}{2\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m (2\alpha^2)^m}{(m-1)!\Gamma(m+1+\hat{\gamma})} + \mathfrak{I}(1) = \frac{2^{(1-\hat{\gamma})/2}}{2\alpha^{\hat{\gamma}}} J_{1+\hat{\gamma}}(2\sqrt{2}\alpha) + \mathfrak{I}(1),
\end{aligned} \tag{46}$$

where J denotes the Bessel function of the first kind, as well as

$$\begin{aligned}
S(2) &= \sum_{k=0}^{\infty} \sum_{m=1}^{k+2} f(k, m, 2) = \sum_{k=0}^{\infty} \sum_{m=1}^{k+2} f(k, m, 2) + \sum_{k=0}^{\infty} f(k, k+2, 2) \\
&= \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} f(j+m-1, m, 2) + \alpha^2 \sum_{k=0}^{\infty} \frac{(-1)^k (2\alpha^2)^k}{k!\Gamma(k+3+\hat{\gamma})} \\
&= -\frac{2}{2\alpha} \sum_{m=1}^{\infty} \frac{(-2\alpha)^k}{(m-1)!\Gamma(m+1+\hat{\gamma})} + \mathfrak{I}(2) + \frac{2^{-\hat{\gamma}/2}}{2\alpha^{\hat{\gamma}}} J_{2+\hat{\gamma}}(2\sqrt{2}\alpha) \\
&= 2 \frac{2^{(1-\hat{\gamma})/2}}{2\alpha^{\hat{\gamma}}} J_{1+\hat{\gamma}}(2\sqrt{2}\alpha) + \frac{2^{-\hat{\gamma}/2}}{2\alpha^{\hat{\gamma}}} J_{2+\hat{\gamma}}(2\sqrt{2}\alpha) + \mathfrak{I}(2).
\end{aligned} \tag{47}$$

4.3. Evaluating the limiting fraction

Now, we go back to the question of determining the limit in (26). First, the formula for a_N in (37) equals, by the result in (39),

$$\begin{aligned}
a_N &\sim B_0 R_0 L_0 N^{\hat{\gamma}+1} N! + B_1 R_0 L_0 (N-1)^{\hat{\gamma}+1} (N-1)! \\
&\quad + B_0 R_1 L_1 (N-1)^{\hat{\gamma}+2} (N-1)! + B_1 R_1 L_1 (N-2)^{\hat{\gamma}+2} (N-2)! \\
&\quad + B_0 R_2 L_2 (N-2)^{\hat{\gamma}+3} (N-2)! + B_1 R_2 L_2 (N-3)^{\hat{\gamma}+3} (N-3)! \\
&\sim N^{\hat{\gamma}+1} N! B_0 (R_0 L_0 + R_1 L_1 + R_2 L_2).
\end{aligned}$$

Recall, from preceding equation (35), that $B_0 = 1$ and the R_i 's we want to use, which we have denoted \hat{R}_i and R_i^* for \hat{a}_N and a_N^* , respectively, are to be found in (30).

Note also, from (40), that

$$\begin{aligned}
I(0) + J(0) &= \int_0^1 (1-x)^{-\lambda/(2+\lambda)} (x+1) e^{x/(2+\lambda)} dx \\
&= [-(2+\lambda)(1-x)^{1-\lambda/(2+\lambda)} e^{x/(2+\lambda)}]_{x=0}^{x=1} = 2 + \lambda = 1/\alpha,
\end{aligned}$$

and a direct property of $\mathfrak{J}(x)$, as defined in (45), is

$$x\mathfrak{J}(0) + y\mathfrak{J}(1) + z\mathfrak{J}(2) = (x+y+z)\mathfrak{J}\left(\frac{y+2z}{x+y+z}\right). \quad (48)$$

Then, from (30), (43), (44), (46), (47) and (45), recalling also that $\alpha = 1/(2+\lambda)$, we get

$$\begin{aligned}
\frac{\hat{a}_N}{N^{\hat{\gamma}+1} N!} &\sim \sum_{i=0}^2 \hat{R}_i L_i = (5+\lambda)L_0 - 7\alpha L_1 + 4\alpha^2 L_2 \\
&= \frac{e^{-2\alpha}[I(0)+J(0)]}{\alpha\Gamma(\hat{\gamma}+1)} + \frac{1}{\alpha^2} [(5+\nu)S(0) - 7S(1) + 4S(2)] \\
&= \frac{1}{\alpha^2} \left[\frac{e^{-2\alpha}}{\Gamma(\hat{\gamma}+1)} + \frac{2^{(1-\hat{\gamma})/2}}{2\alpha^{\hat{\gamma}}} J_{1+\hat{\gamma}}(2\sqrt{2}\alpha) + 4 \frac{2^{-\hat{\gamma}/2}}{2\alpha^{\hat{\gamma}}} J_{2+\hat{\gamma}}(2\sqrt{2}\alpha) + \frac{1}{\alpha} \mathfrak{J}(\alpha) \right],
\end{aligned}$$

where we in the last equality used the result of (48). Similarly we get

$$\begin{aligned}
\frac{a_N^*}{N^{\hat{\gamma}+1} N!} &\sim \sum_{i=0}^2 R_i^* L_i = (1+\lambda)L_0 + \alpha L_1 + 0L_2 \\
&= \frac{1}{\alpha^2} \left[\frac{e^{-2\alpha}}{\Gamma(\hat{\gamma}+1)} + \frac{2^{(1-\hat{\gamma})/2}}{2\alpha^{\hat{\gamma}}} J_{1+\hat{\gamma}}(2\sqrt{2}\alpha) + \frac{1}{\alpha} \mathfrak{J}(\alpha) \right].
\end{aligned}$$

Next, we note that

$$\begin{aligned}
\mathfrak{J}(\alpha) &= \sum_{j=0}^{\infty} (-2\alpha)^j (j+1+\alpha) \sum_{m=1}^{\infty} \frac{(-1)^m (2\alpha^2)^m}{\Gamma(m+1+\hat{\gamma})\Gamma(m+1+j)} \\
&= \underbrace{\sum_{j=0}^{\infty} (-2\alpha)^j (j+1+\alpha) \sum_{m=0}^{\infty} \frac{(-1)^m (2\alpha^2)^m}{\Gamma(m+1+\hat{\gamma})\Gamma(m+1+j)}}_{=: \mathfrak{J}_\lambda} - \frac{(1-\alpha)e^{-2\alpha}}{\Gamma(\hat{\gamma}+1)}, \quad (49)
\end{aligned}$$

where we think of the sum denoted \mathfrak{J}_λ as a function of the parameter λ .

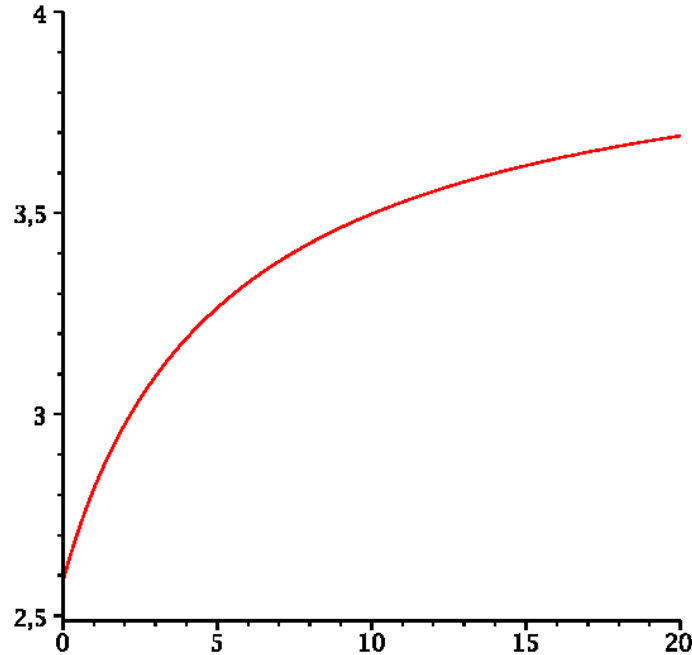


Figure 7: The speed of percolation on the ladder with diagonals (exponential) intensity λ on the vertical edges (as measured in units of the intensity of the horizontal and diagonal edges), plotted by Maple.

Now, we have an expression for the limiting fraction

$$\pi_0 = \lim_{n \rightarrow \infty} \frac{d_n}{c_n} = \frac{\frac{(2\alpha-1)e^{-2\alpha}}{\Gamma(\hat{\gamma}+1)} + \frac{1}{2}(\sqrt{2}\alpha)^{1-\hat{\gamma}} J_{1+\hat{\gamma}} + \mathfrak{I}_\lambda}{\frac{(2\alpha-1)e^{-2\alpha}}{\Gamma(\hat{\gamma}+1)} + \frac{1}{2}(\sqrt{2}\alpha)^{1-\hat{\gamma}} [J_{1-\hat{\gamma}} + 2\sqrt{2}J_{2-\hat{\gamma}}] + \mathfrak{I}_\lambda}, \quad (50)$$

in which we have suppressed the argument $2\sqrt{2}\alpha$ from the Bessel function, i.e. in (50) we should interpret J_ν as $J_\nu(2\sqrt{2}\alpha)$. The speed

$$\mathcal{V} = 2 + 2\pi_0$$

is plotted in Figure 7 for λ ranging from 0 to 20. At zero the speed is $\frac{2\sqrt{2}J_1(\sqrt{2})}{3J_1(\sqrt{2})/\sqrt{2}-J_0(\sqrt{2})} \approx 2.58$ - this exact expression is derived in Section 4.4 below - and as λ increases to infinity, the speed must reach 4.

In Figure 8 we have given numerical values when λ is 0, 1 and 2, respectively.

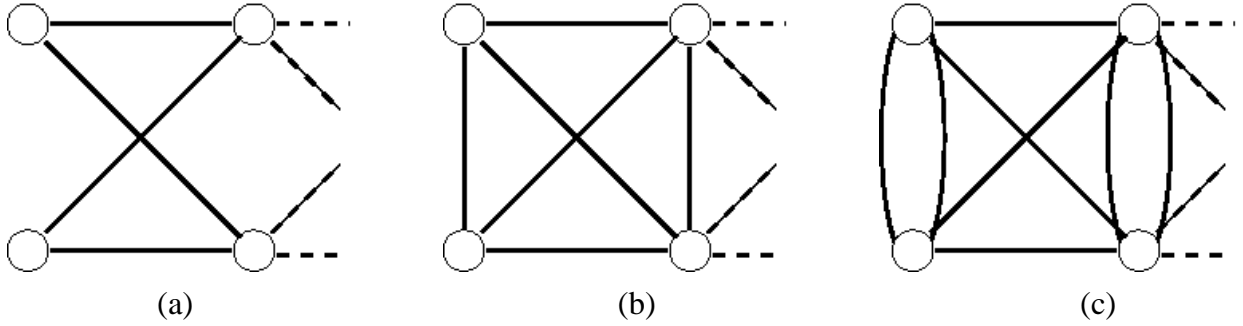


Figure 8: The speed of percolation when all edges are associated with random variables having intensity 1. Numerical values to two digits are, from left to right, 2.58, 2.81 and 2.97.

4.4. The case $\lambda = 0$

This special case has already been investigated in [5] by other methods. In this section we show that our method gives the same result, since this is far from obvious. Also, the case of $\lambda = 0$ appears to be the only case that allows for great simplification of the expression (50). We note that for $\lambda = 0$, since this implies $\alpha = 1/2$ and $\hat{\gamma} = 0$, we get from the definition of \mathfrak{I}_λ in (49),

$$\begin{aligned}
 \mathfrak{I}_0 &= \sum_{j=0}^{\infty} (-1)^j \left(j + \frac{3}{2}\right) \sum_{m=0}^{\infty} \frac{(-1)^m (1/2)^m}{m!(m+j)!} \\
 &= \sum_{j=0}^{\infty} (-1)^j \left(j + \frac{3}{2}\right) (\sqrt{2})^j J_j(\sqrt{2}) \\
 &= \frac{3}{2} \sum_{j=0}^{\infty} (-1)^j (\sqrt{2})^j J_j(\sqrt{2}) + \sum_{j=0}^{\infty} (-1)^j (\sqrt{2})^{j-1} \sqrt{2} j J_j(\sqrt{2}), \tag{51}
 \end{aligned}$$

and since $\sqrt{2} j J_j(\sqrt{2}) = J_{j-1}(\sqrt{2}) + J_{j+1}(\sqrt{2})$, then by relation (10), after cancellation of terms, we get

$$\mathfrak{I}_0 = \frac{1}{2} [J_0(\sqrt{2}) - \sqrt{2} J_1(\sqrt{2})].$$

This, together with the fact that $J_2(\sqrt{2})$ that now appears in the denominator of (50), as a consequence of relation (10), can be written as $\sqrt{2} J_1(\sqrt{2}) - J_0(\sqrt{2})$, simplifies (50) to

$$\frac{J_0(\sqrt{2}) - J_1(\sqrt{2})/\sqrt{2}}{-J_0(\sqrt{2}) + 3J_1(\sqrt{2})/\sqrt{2}},$$

so that

$$\mathcal{V} = 2 + 2\pi_0 = \frac{2\sqrt{2}J_1(\sqrt{2})}{3J_1(\sqrt{2})/\sqrt{2} - J_0(\sqrt{2})},$$

and thus

$$\sigma = \frac{1}{\mathcal{V}} = \frac{3}{4} - \frac{J_0(\sqrt{2})}{2\sqrt{2}J_1(\sqrt{2})},$$

which is what is calculated in [5].

5. Other Graphs in \mathcal{L}

The class \mathcal{L} of graphs, defined in Section 2 and Figure 1, contains (at least) 3 more graphs of interest; these are shown in Figure 9. We will not deal with these graphs here but make some remarks. We believe that the front process together with the results of [3], used in Section 4, can be used to calculate the speed of percolation on the graph depicted in Figure 9(c), in principle with arbitrary intensities associated with the vertical, diagonal and the two different horizontal edges. There would be added complexity as the state space of the front process would be all the integers, not only the non-negative ones. The speed when the intensities are the same is $\frac{2 \tan 1 - 1}{2 \tan 1 - 2} \approx 1.90$, from the exact expression of the rate of percolation calculated in [5].

The problem when trying to apply the front process to the other graphs is that the Markov property is lost.

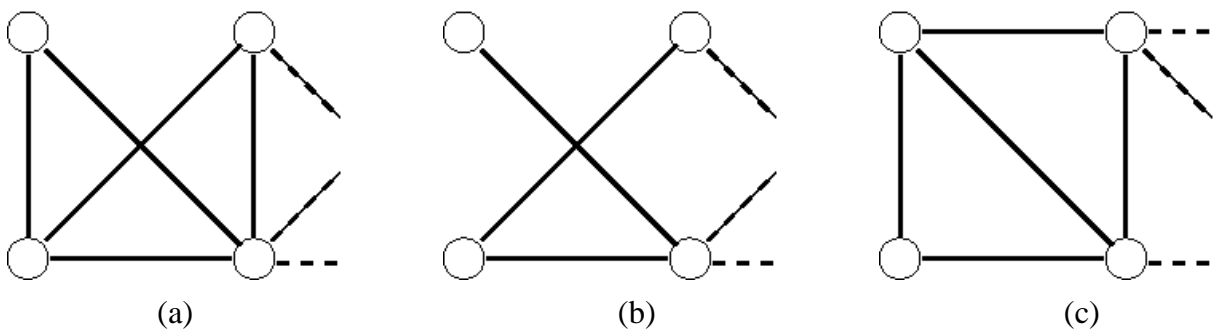


Figure 9: Three more graphs in \mathcal{L}

6. Appendices

6.1. Appendix A (Proof of Claim 4.1)

Step 1. Let α and K be nonnegative real numbers, α is considered fixed, and define

$$\mathfrak{A}_m := \mathfrak{A}_m(K) = \sum_{j=1}^m \binom{m}{j} \alpha^j (m + K - j)! \tag{52}$$

Then we can write \mathfrak{A}_m as

$$\mathfrak{A}_m = (m + K)! \sum_{j=0}^m \frac{\alpha^j}{j!} \prod_{i=0}^{j-1} \frac{m-i}{m+K-i}.$$

We are interested in the limit of $\mathfrak{A}'_m = \mathfrak{A}_m / (m + K)!$ as m tends to infinity. Now, as $(m - i) / (m + K - i)$ is decreasing in i ,

$$\prod_{i=0}^{j-1} \frac{m-i}{m+K-i} \leq \left(\frac{m}{m+K} \right)^j,$$

and as a consequence, we get

$$\mathfrak{A}_m^{l \alpha m / (m+K)}. \quad (53)$$

Next, as long as $j < m$, since $\ln[(m-i)/(m+K-i)]$ is a negative and decreasing function when $0 \leq i < m$, we can calculate,

$$\begin{aligned} \prod_{i=0}^{j-1} \frac{m-i}{m+K-i} &= \exp \left\{ \sum_{i=0}^{j-1} \ln \left[\frac{m-i}{m+K-i} \right] \right\} \\ &\geq \exp \left\{ \int_0^j \ln \left[\frac{m-x}{m+K-x} \right] dx \right\} = \frac{m^m}{(m+K)^{m+K}} \frac{(m+K-j)^{m+K-j}}{(m-j)^{m-i}} \\ &= \left(\frac{m}{m+K} \right)^j \underbrace{\frac{\left(1 - \frac{j}{m+K}\right)^{m+K}}{\left(1 - \frac{j}{m}\right)^m}}_{\geq 1} \left(\frac{1 - \frac{j}{m}}{1 - \frac{j}{m+K}} \right)^j. \end{aligned} \quad (54)$$

Since

$$\frac{1 - \frac{j}{m}}{1 - \frac{j}{m+K}} = 1 - \frac{jK}{m(m+K-i)},$$

the last factor in (54) is, by Taylor expansion around 1,

$$1 - \frac{j^2 K}{m(m+K-j)} + \mathcal{O} \left(\left[\frac{K}{m(m+K-j)} \right]^2 \right),$$

and hence \mathfrak{A}_m^l is bounded below by

$$\sum_{j=1}^{m-1} \frac{[m \alpha / (m+K)]^j}{j!} (1 + \mathcal{O}(1/m)) = e^{\alpha m / (m+K)} (1 + \mathcal{O}(1/m)). \quad (55)$$

In conclusion; from the upper and lower bounds, (53) and (55) respectively,

$$\mathfrak{A}_m = (m+K)! \exp \left\{ \frac{\alpha m}{m+K} \right\} (1 + \mathcal{O}(1/m)). \quad (56)$$

Step 2. Let us examine, for $l \geq 0$ and $K > -1$, the fraction $\Gamma(l+K+1)/\Gamma(l+1)$ for large l . By Stirling's formula

$$\begin{aligned} \frac{\Gamma(l+K+1)}{\Gamma(l+1)} &= \frac{\sqrt{2\pi} (l+K)^{l+K+1/2} e^{-l-K}}{\sqrt{2\pi} l^{l+1/2} e^{-l}} (1 + \mathcal{O}([l+K]^{-1})) \\ &= (l+K)^{l+K} e^{-K} \left(1 + \frac{K}{l}\right)^{l+1/2} (1 + \mathcal{O}([l+K]^{-1})) \\ &= (l+K)^{l+K} (1 + \mathcal{O}(1/l)). \end{aligned} \quad (57)$$

Step 3. Define the function

$$\mathfrak{B}_i(k) = \sum_{l+m=M-2k} \frac{\Gamma(k+i+\hat{\gamma}+l+1)}{\Gamma(l+1)} (m+1) \mathfrak{A}_m(k+l), \quad k \leq M/2,$$

where $\mathfrak{A}_m(k+l)$ is defined in (52). With $\mathfrak{B}_i(k)$ defined in this way we can write $\hat{F}_i(M)$ from (38) as

$$\hat{F}_i(M) = \sum_{k \leq M/2} \frac{(-C)^k}{k! \Gamma(k+i+\hat{\gamma}+1)} \mathfrak{B}_i(k).$$

Now, examine $\mathfrak{B}_i(k)$ scaled by $M^{k+\hat{\gamma}+i+1}(M-k)!$,

$$\frac{\mathfrak{B}_i(k)}{M^{k+\hat{\gamma}+i+1}(M-k)!} = \sum_{l+m=M-2k} \underbrace{\frac{\Gamma(k+i+\hat{\gamma}+l+1)}{\Gamma(l+1)M^{k+\hat{\gamma}+i}}}_a \cdot \underbrace{\frac{m+1}{M}}_b \cdot \underbrace{\frac{\mathfrak{A}_m(k+l)}{(M-k)!}}_c.$$

From (57) and using $l+m = M-2k$ we get

$$a = \left(\frac{\hat{\gamma}+i-k}{M} + 1 - \frac{m}{M} \right)^{k+i+\hat{\gamma}} (1 + \mathcal{O}(1/l)) = \left(1 - \frac{m}{M} \right)^{k+i+\hat{\gamma}} (1 + \mathcal{O}(1/l)).$$

Obviously $b = m/M + 1/M$, and from (56) we get

$$c = \exp\left\{ \frac{\alpha m}{M-k} \right\} (1 + \mathcal{O}(1/m)) = \exp\left\{ \frac{\alpha m}{M} \right\} (1 + \mathcal{O}(1/m)),$$

so that, as $M \rightarrow \infty$,

$$\begin{aligned} \frac{\mathfrak{B}_i(k)}{M^{k+\hat{\gamma}+i+1}(M-k)!} &\sim \sum_{m=0}^M \left(1 - \frac{m}{M} \right)^{k+i+\hat{\gamma}} \frac{m}{M} \exp\left\{ \alpha \frac{m}{M} \right\} \\ &\sim \int_0^1 (1-x)^{k+i+\hat{\gamma}} x e^{\alpha x} dx. \end{aligned}$$

And hence, since also $M^{k+\hat{\gamma}+i+1}(M-k)! \sim M^{\hat{\gamma}+i+1}M!$, we have demonstrated the result in (39). ■

6.2. Appendix B (Proof of Claim 4.2)

Recall that we have defined, for $n \in \mathbb{N}$,

$$I(n) = \int_0^1 (1-x)^{\hat{\gamma}+n} x e^{\alpha x} dx \quad \text{and} \quad J(n) = \int_0^1 (1-x)^{\hat{\gamma}+n} e^{\alpha x} dx, \quad (58)$$

where $\hat{\gamma} \in (-1, 0]$ and $\alpha \in (0, 1/2]$, although in this section it is only essential that $\hat{\gamma} > -1$.

Then $I(0) < \infty$ and $J(0) < \infty$, since $\hat{\gamma} > -1$. For $n \geq 1$, partial integration reveals the recursive relationship

$$\begin{aligned} I(n) &= \left[(1-x)^{\hat{\gamma}+n} x \frac{1}{\alpha} e^{\alpha x} \right]_{x=0}^{x=1} \\ &\quad - \frac{1}{\alpha} \int_0^1 [-(\hat{\gamma}+n)(1-x)^{\hat{\gamma}+n-1} x + (1-x)^{\hat{\gamma}+n}] e^{\alpha x} dx \\ &= \frac{1}{\alpha} I(n-1) - \frac{1}{\alpha} J(n), \end{aligned}$$

from which follows by iteration that

$$I(n) = \frac{(\hat{\gamma}+n)^n}{\alpha^n} I(0) - \frac{1}{\alpha} \sum_{j=1}^n \frac{(\hat{\gamma}+n)^{n-j}}{\alpha^{n-j}} J(j). \quad (59)$$

Note that the above formula holds for any $n \geq 0$ if we interpret sums of the form \sum_n^m to be zero if $m < n$.

Next, we turn our attention to $J(n)$. For $n \geq 1$ we get, again by partial integration,

$$\begin{aligned} J(n) &= \left[(1-x)^{\hat{\gamma}+n} x \frac{1}{\alpha} e^{\alpha x} \right]_{x=0}^{x=1} + \frac{\hat{\gamma}+n}{\alpha} \int_0^1 [(1-x)^{\hat{\gamma}+n-1}] e^{\alpha x} dx \\ &= \frac{\hat{\gamma}+n}{\alpha} J(n-1) - \frac{1}{\alpha}, \end{aligned}$$

from which follows by iteration that

$$J(n) = \frac{(\hat{\gamma}+n)^n}{\alpha^n} J(0) - \frac{1}{\alpha} \sum_{k=1}^{n-1} \frac{(\hat{\gamma}+n)^k}{\alpha^k}. \quad (60)$$

Employment of (60) in (59) leads to

$$I(n) = \frac{(\hat{\gamma}+n)^n}{\alpha^n} I(0) - \frac{J(0)}{\alpha^{n+1}} S_1 + \frac{1}{\alpha^2} S_2,$$

where

$$S_1 = \sum_{j=1}^n (\hat{\gamma}+n)^{n-j} (\hat{\gamma}+j)^j = \frac{n\Gamma(\hat{\gamma}+n+1)}{\Gamma(\hat{\gamma}+1)},$$

and

$$S_2 = \sum_{j=1}^n \sum_{k=0}^{j-1} \frac{(\hat{\gamma}+n)^{n-j} (\hat{\gamma}+j)^k}{\alpha^{k-j}}.$$

The above simplification of S_1 is easily seen to be true by the relation $x^n = \Gamma(x+1)/\Gamma(x+1-m)$. The same relation can be applied to S_2 to give

$$\begin{aligned} S_2 &= \frac{\Gamma(\hat{\gamma}+n+1)}{\alpha^n} \sum_{j=1}^n \sum_{k=0}^{j-1} \frac{\alpha^{j-k}}{\Gamma(\hat{\gamma}+1+j-k)} \\ &= \frac{\Gamma(\hat{\gamma}+n+1)}{\alpha^n} \sum_{m=1}^n \frac{(n+1-m)\alpha^m}{\Gamma(\hat{\gamma}+1+m)}, \end{aligned}$$

where the last equality follows from noting that k and j only appear in the summation as $m = j - k$, and that the values of m , i.e. $1, 2, \dots, n$, appear $n, n-1, \dots, 1$ times, respectively. So, we have the sought formula

$$\begin{aligned}
I(n) &= \frac{\Gamma(\hat{\gamma}+n+1)}{\alpha^n \Gamma(\hat{\gamma}+1)} \left[I(0) - \frac{n}{\alpha} J(0) \right] \\
&+ \frac{\Gamma(\hat{\gamma}+n+1)}{\alpha^{n+2}} \sum_{m=1}^n \frac{(n+1-m)\alpha^m}{\Gamma(\hat{\gamma}+1+m)} \\
&= \frac{\Gamma(\hat{\gamma}+n+1)}{\alpha^n} \left[\frac{I(0) - \frac{n}{\alpha} J(0)}{\Gamma(\hat{\gamma}+1)} + \frac{1}{\alpha^2} \sum_{m=1}^n \frac{(n+1-m)\alpha^m}{\Gamma(\hat{\gamma}+1+m)} \right].
\end{aligned} \tag{61}$$

■

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