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New Higher Order Derivative-Free Methods for Solving Nonlinear Equations

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Abstract. *In this paper, two new derivative-free methods with* $2k$ *and Fibonacci number order of convergence for solution of nonlinear equations are presented. Methods of different order of convergence are constructed using a suitable parametric function and an arbitrary real parameter. For some values of* k*, we have proved the order of convergence of the new derivative-free methods. Consequently, we have examined the effectiveness of the new iterative methods by approximating the simple root of given nonlinear equations. The implementation of the new derivative-free methods is shown using different numerical examples.*

Key words : Fibonacci Numbers, Derivative-Free Methods, Nonlinear Equations, Order of Convergence, Steffensen's Method, Kung-Traub Conjecture.

AMS Subject Classifications : 65H05

1. Introduction

Multipoint iterative methods for solving nonlinear equations are of great practical importance since they overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency. The new derivative-free methods are applied to find a simple root of the nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D and it is sufficiently smooth in a neighborhood of α. In this paper, two new derivative-free iterative methods of the $2k$ and Fibonacci number order are constructed.

If the derivative of the function f is difficult to compute or is expensive to obtain, then a derivative-free method is required. In this study, the new derivative-free iterative methods are based on a classical Steffensen's method [5], which actually replaces the derivative in the classical Newton's method with suitable approximations based on finite difference,

$$w_n = x_n - f(x_n), \tag{2}$$

$$f'(x_n) = \left(\frac{f(w_n)-f(x_n)}{w_n - x_n}\right) = f[w_n, x_n] \tag{3}$$

Therefore, the Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{4}$$

becomes the Steffensen's method

$$x_{n+1} = x_n - \left(\frac{f^2(x_n)}{f(w_n)-f(x_n)}\right), \tag{5}$$

In fact, it is well known that the Newton's method (4) and the Steffensen's method (5) have the convergence order of two [1].
Furthermore, we shall briefly state the essentials of Fibonacci numbers and how they are generated [4]. The Fibonacci numbers are given by a simply recursion formula

$$F_{k+1} = F_k + F_{k-1}, \tag{6}$$

where $F_1 = 1$, $F_0 = 1$ and $k \in \mathbb{N}$. The first few numbers are : $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots..$ and the associated golden number is given as

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = 2^{-1}(\sqrt{5}+1) = 1.61803\ldots.. \tag{7}$$

In this paper we shall prove and demonstrate that one of the new higher order methods involves generating an order of convergence similar to the Fibonacci numbers given by (6).

The prime motive of this study is to develop a class of very efficient derivative-free methods for solving nonlinear equations. The new methods presented in this paper are derivative-free and have an order of convergence of $2k$ and F_{k+1}. Two new derivative-free iterative methods are constructed in the next section. Then the effectiveness of these two new iterative methods is compared. Consequently, we have found that the new derivative-free methods are efficient and robust.

2. New Derivative-Free Methods and Convergence Analysis

In order to establish the order of convergence of these new derivative-free methods we state the three essential definitions.

Definition 2.1. Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converges towards α. The order of convergence m is given by

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^m} = \zeta \neq 0, \tag{8}$$

where ζ is the asymptotic error constant and $m \in \mathbb{R}^+$.

Definition 2.2. Suppose that x_{n+1}, x_n and x_{n-1} are three successively close iterations to the root α of (1). Then the computational order of convergence [7] may be approximated by

$$\text{COC} \cong \frac{\ln |(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln |(x_n-\alpha)/(x_{n-1}-\alpha)|}, \tag{9}$$

where $n \in \mathbb{N}$

Definition 2.3. Let p be the number of function evaluations required by the new method. The efficiency of the new method is measured by the concept of efficiency index [3, 9] which is defined as

$$q^{1/p}, \tag{10}$$

where q is the order of the method.

2.1. The $2k$ derivative-free methods

In this section we define a derivative-free iterative method with $2k$ order of convergence. The first step of the new formula is the classical Steffensen second-order method [5]. Then we use a particular weight function in the iterative process to produce the $2k$ higher derivative-free method. Therefore, the scheme for generating the iterative method of order $2k$ is given as

$$u_1 = x_n - \left(\frac{f^2(x_n)}{f(w_n)-f(x_n)}\right), \tag{11}$$

$$u_2 = u_1 - \left(1 + \frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_1)}{f[u_1,u_0]}\right), \tag{12}$$

$$u_3 = u_1 - \left(1 + \frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_1)}{f[u_1,u_0]}\right) - \left(1 + \frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_2)}{f[u_1,u_0]}\right), \tag{13}$$

$$\vdots$$

$$u_{n+1} = u_1 - \left(1 + \frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_1)}{f[u_1,u_0]}\right) - \cdots\cdots - \left(1 + \frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_{k-1})}{f[u_1,u_0]}\right), \tag{14}$$

The $2k$ convergence order method formed by the above scheme will require $k+1$ functions and no derivative evaluation per iterative step. Furthermore, we have found that the $2k$ convergence order method given by (14) can be expressed in a simpler form

$$u_{n+1} = u_1 - f^{-1}[u_1, u_0]\left(1 + \frac{f(u_1)}{f(x_n)}\right)\sum_{i=1}^{k-1} f(u_i), \tag{15}$$

where $n, k \in \mathbb{N}$, $u_0 = x_n$, $k > 2$, and provided that the denominator in (15) is not equal to zero. Thus the scheme (15) defines new higher order derivative-free methods with a weight function. To obtain the solution of (1) by this method, we must set a particular initial approximation x_0, ideally close to the simple root.

Remark 2.1. We can use an elementary method to determine the sequence for the order of convergence of (15). Namely by simply applying the ratio test to (15), we have found that the following limit illustrates that (15) has a sequence of order 2,

$$\lim_{k\to\infty} \frac{u_1 - f^{-1}[u_1,u_0]\left(1+\frac{f(u_1)}{f(x_n)}\right)\sum_{i=1}^{k+1} f(u_i)}{u_1 - f^{-1}[u_1,u_0]\left(1+\frac{f(u_1)}{f(x_n)}\right)\sum_{i=1}^{k} f(u_i)} = \left(4c_1^{-2}\, c_2^2 + 3c_1^{-1}\, c_2^2 - c_1^{-1}\, c_3 - c_3\right)e_n^2. \tag{16}$$

Remark 2.2. Using (16) we have found that the asymptotic error constant of (15) may be obtained by

$$AEC(2k) = c_1^{-1}c_2(1+c_1)\left(3c_1^{-2}\, c_2^2 + 2c_1^{-1}\, c_2^2 - c_1^{-1}\, c_3 - c_3\right)e_n^4$$
$$\times \left[\left(4c_1^{-2}\, c_2^2 + 3c_1^{-1}\, c_2^2 - c_1^{-1}\, c_3 - c_3\right)e_n^2\right]^{k-2}, \tag{17}$$

for $k = 2, 3, 4, 5, \ldots..$

The expression (17) is a simple formula for calculating the asymptotic error constant for the $2k$ higher derivative-free method given by (15).
In numerical mathematics it is very useful and essential to know the behavior of an approximate method. Therefore, we shall prove the order of convergence of the new higher order derivative-free method.

Theorem 2.1. *Let $\alpha \in D$ be a simple root of a sufficiently differentiable function $f : D \subset R \to R$ in an open interval D . If x_0 is sufficiently close to α, then the order of convergence of the new derivative-free method defined by (15) is $2k$.*

Proof. Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the error is expressed as

$$e = x - \alpha. \tag{18}$$

Using Taylor's expansion, we have

$$f(x_n) = f(\alpha) + f'(\alpha)\, e_n + 2^{-1}\, f''(\alpha)e_n^2 + 6^{-1}f'''(\alpha)e_n^3 + 24^{-1}f^{IV}(\alpha)e_n^4 + \cdots \tag{19}$$

Taking $f(\alpha) = 0$ and simplifying, expression (19) becomes

$$f(x_n) = c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots \tag{20}$$

where $n \in \mathbb{N}$ and

$$c_k = \frac{f^{(k)}(\alpha)}{k!}, \quad k = 1,2,3,\ldots \tag{21}$$

Expanding the Taylor series of $f(w_n)$ and substituting $f(x_n)$ given by (20), we have

$$f(w_n) = c_1(1+c_1)e_n + \left(3c_1\, c_2 + c_1^2c_2 + c_2\right)e_n^2 + \cdots \tag{22}$$

By substituting (22) and (20) in (11), we get

$$u_1-\alpha = x_n-\alpha-\left(\frac{f^2(x_n)}{f(w_n)-f(x_n)}\right) = \frac{c_2}{c_1}(c_1+1)e_n^2 + \cdots \tag{23}$$

The expansion of $f(u_1)$ about α is given as

$$f(u_1) = c_2(c_1+1)e_n^2 + \frac{1}{c_1}(c_1^3\,c_3-2c_2^2+3c_1^2\,c_3+2c_1\,c_3-c_1^2c_2^2-2c_1\,c_2^2)e_n^3+\cdots \tag{24}$$

Substitution of (24) in (15) leads to

$$\begin{aligned} u_2-\alpha &= u_1-\alpha-\left(1+\frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_1)}{f[u_1,u_0]}\right) \\ &= c_1^{-1}c_2(1+c_1)\left(3c_1^{-2}\,c_2^2+2c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right)e_n^4+\cdots \end{aligned} \tag{25}$$

The expansion of $f(u_2)$ about α is given as

$$f(u_2) = c_2(c_1+1)\left(3c_1^{-2}\,c_2^2+2c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right)e_n^4+\cdots \tag{26}$$

Substituting (26) in (15), we get

$$\begin{aligned} u_3-\alpha &= u_1-\alpha-\left(1+\frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_1)}{f[u_1,u_0]}\right)-\left(1+\frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_2)}{f[u_1,u_0]}\right) \\ &= c_1^{-1}c_2(1+c_1)\left(3c_1^{-2}\,c_2^2+2c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right) \\ &\quad\times\left(4c_1^{-2}\,c_2^2+3c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right)e_n^6+\cdots \end{aligned} \tag{27}$$

The expansion of $f(u_3)$ about α is given as

$$\begin{aligned} f(u_3) &= c_1^{-1}c_2(1+c_1)\left(3c_1^{-2}\,c_2^2+2c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right) \\ &\quad\times\left(4c_1^{-2}\,c_2^2+3c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right)e_n^6+\cdots \end{aligned} \tag{28}$$

Substitution of (28) in (15) yields

$$\begin{aligned} u_4-\alpha &= u_1-\alpha-\left(1+\frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_1)}{f[u_1,u_0]}\right)-\left(1+\frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_2)}{f[u_1,u_0]}\right) \\ &\quad-\left(1+\frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_3)}{f[u_1,u_0]}\right) = c_1^{-1}c_2(1+c_1)\left(3c_1^{-2}\,c_2^2+2c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right) \\ &\quad\times\left(4c_1^{-2}\,c_2^2+3c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right)^2e_n^8+\cdots \end{aligned} \tag{29}$$

The expansion of $f(u_4)$ about α is given as

$$\begin{aligned} f(u_4) &= c_2(1+c_1)\left(3c_1^{-2}\,c_2^2+2c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right) \\ &\quad\times\left(4c_1^{-2}\,c_2^2+3c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right)^2e_n^8+\cdots \end{aligned} \tag{30}$$

Substituting (30) in the expression (15), we get

$$\begin{aligned} u_5-\alpha &= u_1-\alpha-\left(1+\frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_1)}{f[u_1,u_0]}\right) -\cdots \\ &\quad-\left(1+\frac{f(u_1)}{f(x_n)}\right)\left(\frac{f(u_4)}{f[u_1,u_0]}\right) = c_1^{-1}c_2(1+c_1)\left(3c_1^{-2}\,c_2^2+2c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right) \\ &\quad\times\left(4c_1^{-2}\,c_2^2+3c_1^{-1}\,c_2^2-c_1^{-1}\,c_3-c_3\right)^3e_n^{10}+\cdots \end{aligned} \tag{31}$$

The expressions (23),(25),(27),(29) and (31) establish the asymptotic error constant for $2(2)10$

order of convergence, respectively, for the new derivative-free iterative method defined by (15). ■

2.2. The Fibonacci sequence order method

In this section we define an improvement of the previous 2k derivative-free iterative method. with 2k order of convergence. The F_{k+1} derivative-free iterative method also has the first step of the new formula the classical Steffensen second-order method. The improvement in this method is achieved by using the latest value in the divided difference and we do not use a weight function in the iterative process. Consequently, we have found that the improved method, namely F_{k+1} higher order derivative-free method has an order of convergence similar to the Fibonacci sequence. Therefore, the scheme for generating iterative method of order the F_{k+1} is given as

$$u_1 = x_n - \left(\frac{f^2(x_n)}{f(w_n) - f(x_n)} \right), \tag{32}$$

$$u_2 = u_1 - \left(\frac{f(u_1)}{f[u_1, u_0]} \right), \tag{33}$$

$$u_3 = u_1 - \left(\frac{f(u_1)}{f[u_1, u_0]} \right) - \left(\frac{f(u_2)}{f[u_2, u_1]} \right), \tag{34}$$

$$\vdots$$

$$u_{n+1} = u_1 - \left(\frac{f(u_1)}{f[u_1, u_0]} \right) - \cdots\cdots - \left(\frac{f(u_k)}{f[u_k, u_{k-1}]} \right), \tag{35}$$

The F_{k+1} convergence order method formed by the above scheme will require $k+1$ functions and again requires no derivative evaluation per iterative step. Furthermore, we have found that the convergence order method given by (35) can be expressed in the simpler form

$$u_{n+1} = u_1 - \sum_{i=1}^{k} \frac{f(u_i)}{f[u_i, u_{i-1}]}, \tag{36}$$

where $n, k \in \mathbb{N}$, and provided that the denominators in (36) is not equal to zero. Thus the scheme (36) defines new higher order derivative-free methods with no weight function. Here also, to obtain the solution of (1) by the new derivative-free method, we must set a particular initial approximation , ideally close to the simple root.

As before, it is very useful and essential to know the behavior of an approximate method. Therefore, we shall prove the order of convergence of the new derivative-free method.

Theorem 2.2. *Let $\alpha \in D$ be a simple root of a sufficiently differentiable function $f : D \subset R \to R$ in an open interval D . If x_0 is sufficiently close to α, then the order of convergence of the new derivative-free method defined by (36) is F_{k+1}.*

Proof. Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the error is $e = x - \alpha$. Using the appropriate expressions (19)-(23), without loss generality, we repeat the process to prove the order of convergence. As before, the expression of u_1 and $f(u_1)$ are given by (23) and (24) respectively.

Substitute (23) and (24) in the expression (36) to obtain

$$u_2 - \alpha = u_1 - \alpha - \left(\frac{f(u_1)}{f[u_1,u_0]}\right) = c_1^{-2}c_2(1+c_1)e_n^3 + \cdots \tag{37}$$

This proves that the formula given by (36) has the order of convergence three.
The expansion of $f(u_2)$ about α is given as

$$f(u_2) = c_1^{-1}c_2(1+c_1)e_n^3 + \cdots \tag{38}$$

Substitution of (38) in (36) leads to

$$u_3 - \alpha = u_1 - \alpha - \left(\frac{f(u_1)}{f[u_1,u_0]}\right) - \left(\frac{f(u_2)}{f[u_2,u_1]}\right) = \left(\frac{c_2}{c_1}\right)^4 (c_1+1)^2\, e_n^5 + \cdots \tag{39}$$

In this case we find that the formula (36) has the order of convergence five.
For the next case, the expansion $f(u_3)$ about α is

$$f(u_3) = c_1^{-3}c_2^4\,(c_1+1)^2\, e_n^5 + \cdots \tag{40}$$

Substituting (40) in (36) we get

$$\begin{aligned} u_4 - \alpha &= u_1 - \alpha - \left(\frac{f(u_1)}{f[u_1,u_0]}\right) - \left(\frac{f(u_2)}{f[u_2,u_1]}\right) - \left(\frac{f(u_3)}{f[u_3,u_2]}\right) \\ &= \left(\frac{c_2}{c_1}\right)^7 (c_1+1)^3 e_n^8 + \cdots \end{aligned} \tag{41}$$

Here, we find that (36) has the order of convergence eight.
Repeating the process, the expansion of $f(u_4)$ about α is given as

$$f(u_4) = c_1^{-6}c_2^7\,(c_1+1)^3\, e_n^8 + \cdots \tag{42}$$

Substitution of (42) in (36) leads to

$$\begin{aligned} u_5 - \alpha &= u_1 - \alpha - \left(\frac{f(u_1)}{f[u_1,u_0]}\right) - \left(\frac{f(u_2)}{f[u_2,u_1]}\right) - \left(\frac{f(u_3)}{f[u_3,u_2]}\right) - \left(\frac{f(u_4)}{f[u_1,u_0]}\right) \\ &= \left(\frac{c_2}{c_1}\right)^{12} (c_1+1)^5 e_n^{13} + \cdots \end{aligned} \tag{43}$$

Here we find that (36) has the order of convergence thirteen. The process maybe repeated to any desired order.

The expressions (37), (39), (41), and (43) establish the asymptotic error constant for the 3,5,8,13 order of convergence for the new derivative-free iterative method defined by (36).■

Conjecture 2.1. Empirically we have found a simple formula for calculating the asymptotic error constant for the F_{k+1} higher order derivative-free method given by (36). From a simple observation of the above proof, its sequence of lower order of convergence, we conjecture that the asymptotic error constant for the higher order of convergence may be calculated by the following simple formula:

$$AEC\,(F_{k+1}) = (c_1^{-1}c_2)^{F_k}\,(c_1+1)^{F_{k-1}}\, e_n^{F_{k+1}}\,, \tag{44}$$

where F_k is given by (6) and $n,k \in \mathbb{N}$.

3. Application of the New Derivative-Free Iterative Methods

We employ the present derivative-free methods to solve some four nonlinear equations and demonstrate their performance. We shall determine the consistency and stability of results by examining the convergence of these new iterative methods. The findings are generalized by illustrating the effectiveness of these methods for determining the simple root of a nonlinear equation. Consequently, we shall give estimates of the approximate solution produced by the eighth-order methods and list the errors obtained by each of the methods. The numerical computations listed in the following tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in them.

Remark 3.1. The test functions and their exact root are displayed in the following tables together with the difference between the root and the approximation for test functions with initial approximation. In fact, α is calculated by using the same total number of function evaluations (TNFE) for all methods. In the calculations, 12 TNFE are used by each method and the limit of the precision used for our calculations is $|x_n - \alpha| < 10^{-5000}$.

Example 1. In our first example we shall demonstrate the convergence of the new higher order derivative-free iterative methods for the following nonlinear equation

$$f(x) = (x-2)(x^{10}+x+1)\exp[-(x+1)] \quad (45)$$

and the exact value of the simple root of (45) is $\alpha = 2$. Tables 1 and 2 display errors obtained by the $2k$ and F_{k+1} convergence order methods, based on the same initial approximation $x_0 = 3^{-1}$.

Table 1: Errors occurring in the estimates of the root of (45) by the $2k$ method (15).

k	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	COC
1	0.113	0.113	0.113	0.99
2	0.875e-1	0.217e-1	0.165e-2	1.85
3	0.697e-1	0.531e-2	0.166e-5	3.13
4	0.565e-1	0.106e-2	0.102e-11	5.22
5	0.463e-1	0.160e-3	0.257e-22	7.64
6	0.382e-1	0.180e-4	0.639e-38	10.1

Table 2: Errors occurring in the estimates of the root of (45) by the F_{k+1} method (36).

k	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	COC
1	0.113	0.113	0.113	0.99
2	0.358e-2	0.608e-4	0.188e-9	3.11
3	0.127e-2	0.307e-8	0.187e-36	5.02
4	0.183e-4	0.282e-28	0.899e-219	8.00
5	0.927e-7	0.231e-75	0.330e-967	13.0
6	0.676e-11	0.149e-208	0.238e-4359	21.0

We observe that the $2k$ method is converging at a lower order than expected, whereas the F_{k+1} method is converging to the expected order.

Example 2. Our second example shall demonstrate the convergence of the new higher order derivative-free iterative methods for the different type of nonlinear equation

$$f(x) = \exp(-x^2 + x + 2) - \cos(x + 1) + x^3 + 1, \qquad (46)$$

and the exact value of the simple root of (46) is $\alpha = -1$. In Tables 3 and 4 we display the errors obtained by the $2k$ and F_{k+1} convergence order methods, based on the same initial approximation $x_0 = -1.6$. We observe that both methods are converging to the expected order.

Table 3: Errors occurring in the estimates of the root of (46) by the $2k$ method (15).

k	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	COC
1	0.396	0.281	0.158	1.68
2	0.308e-1	0.115e-5	0.523e-23	3.92
3	0.262e-2	0.219e-14	0.764e-87	5.92
4	0.211e-3	0.641e-28	0.459e-224	8.00
5	0.170e-4	0.759e-46	0.238e-459	9.99
6	0.137e-5	0.376e-68	0.688e-819	12.0

Table 4: Errors occurring in the estimates of the root of (46) by the F_{k+1} method (36).

k	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	COC
1	0.396	0.281	0.158	1.68
2	0.401e-1	0.187e-5	0.128e-17	2.81
3	0.388e-3	0.329e-18	0.146e-93	5.00
4	0.234e-5	0.112e-47	0.294e-386	8.00
5	0.152e-9	0.172e-132	0.906e-1731	13.0
6	0.592e-16	0.262e-349	0	21.0

Example 3. Here we take another nonlinear equation.

$$f_6(x) = \sin(x)^2 - x^2 + 1, \qquad (47)$$

and the exact value of the simple root of (47) is $\alpha = 1.40449155\cdots$. As before, Tables 5 and 6 contain displays of the errors obtained by the $2k$ and F_{k+1} convergence order methods, based on the same initial approximation, $x_0 = -1.5$. Here we observe that the new F_{k+1} and the $2k$ convergence order methods are converging to the expected order.

Table 5: Errors occurring in the estimates of the root of (47) by the $2k$ method (15).

k	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	COC
1	0.996	0.981	0.110e-3	1.94
2	0.170e-1	0.173e-6	0.176e-26	4.01
3	0.484e-2	0.640e-13	0.352e-78	6.00
4	0.127e-2	0.934e-22	0.768e-175	8.00
5	0.343e-3	0.787e-33	0.314e-329	10.0
6	0.920e-4	0.330e-46	0.151e-555	12.0

Table 6: Errors occurring in the estimates of the root of (47) by the F_{k+1} method (36).

k	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	COC
1	0.996e-1	0.981e-2	0.110e-3	1.94
2	0.454e-1	0.891e-4	0.643e-12	2.60
3	0.367e-2	0.545e-12	0.400e-61	5.00
4	0.127e-3	0.396e-31	0.359e-251	8.00
5	0.366e-6	0.800e-84	0.210e-1093	13.0
6	0.363e-10	0.104e-219	0	21.0

Example 4. In the last but not least of the examples, we take another different type of nonlinear equations,

$$f(x) = \ln(x^2 + x + 2) - x + 1, \tag{48}$$

and the exact value of the simple root of (48) is $\alpha = 4.152591\cdots$. Tables 7 and 8 display the errors obtained by the $2k$ and F_{k+1} convergence order methods, based on the same initial approximation, $x_0 = 5$. In this particular case, we observe that both methods are converging to the expected order.

Table 7: Errors occurring in the estimates of the root of (48) by the $2k$ method (15).

k	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	COC
1	0.489e-1	0.564e-4	0.764e-10	2.00
2	0.586e-3	0.156e-16	0.778e-71	4.00
3	0.869e-5	0.397e-36	0.361e-224	6.00
4	0.129e-6	0.500e-63	0.250e-514	8.00
5	0.192e-8	0.308e-97	0.342e-985	10.0
6	0.286e-10	0.928e-139	0.127e-1680	12.0

Table 8: Errors occurring in the estimates of the root of (48) by the F_{k+1} method (36).

k	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	COC
1	0.489e-1	0.564e-4	0.764e-10	2.00
2	0.418e-2	0.106e-9	0.172e-32	3.00
3	0.122e-4	0.574e-30	0.130e-156	5.00
4	0.309e-8	0.151e-77	0.488e-632	8.00
5	0.228e-14	0.103e-206	0.332e-2707	13.0
6	0.424e-24	0.385e-539	0	21.0

4. Conclusion

In this study, we have constructed two new higher order derivative-free methods for solving nonlinear equations. Convergence analysis proves that the new methods preserve their order of convergence. From the results in the previous tables and a number of numerical experiments, it can be concluded that the convergence of the new multipoint methods is remarkably fast. After an extensive experimentation we conjecture that the Fibnoacci sequence order F_{k+1} method is superior to the $2k$ method.

There are two major advantages of these new derivative-free methods. Firstly, we do not have to evaluate the derivative of the functions; therefore they are especially efficient where the computational cost of the derivative is expensive, and secondly we have established a new higher order of convergence method which is simple to construct. We have examined the effectiveness of the new derivative-free methods by estimating the accuracy of the simple root in some nonlinear equations. The main purpose of demonstrating the new higher order derivative-free methods for four different types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative methods. Finally, further investigation is needed to improve these new iterative methods so as to define the optimal order of convergence based on the Kung-Traub conjecture [3].

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