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On the Degree of Approximation of Functions of the Lipschitz Class by $(E,q)(C,\alpha,\beta)$ Means

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Abstract. *In this paper two theorems are proved on the degree of approximation of functions belonging to the Lipschitz classes of the type* $\mathrm{Lip}\alpha$, $0 < \alpha \leq 1$, *and* $W(L_p,\xi(t))$, $p \geq 1$, *by* $(E,q)(C,\alpha,\beta)$ *means. The first one gives the degree of approximation with respect to the* L_∞ *−norm, and the second one with respect to* L_p *−norms.*

Key words : Lipschitz Classes, Fourier Series, Weighted $W(L_p,\xi(t))$.

AMS Subject Classifications : 42B05, 42B08

1. Introduction and Preliminaries

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with its partial sums s_n. We denote by $C_n^{(\alpha,\beta)}$ the n-th Cesáro means of order (α,β), with $\alpha+\beta > -1$ of the sequence (s_n), i.e. (see [12])

$$C_n^{(\alpha,\beta)} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} s_v,$$

where $A_n^{\alpha+\beta} = O(n^{\alpha+\beta})$, $\alpha+\beta > -1$ and $A_0^{\alpha+\beta} = 1$. The series $\sum_{n=0}^{\infty} u_n$ is said to be summable (C,α,β) to the definite number s if

$$C_n^{(\alpha,\beta)} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} s_v \to s, \quad \text{as} \quad n \to \infty.$$

Furthermore, for $q > 0$ a real number, the Euler means (E,q) of the sequence (s_n) are defined (see for example [12]) to be

$$E_n^q = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v.$$

The series $\sum_{n=0}^{\infty} u_n$ is said to be summable (E,q) to the definite number s if

$$E_n^q = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v \to s, \quad \text{as} \quad n \to \infty.$$

The (E,q) transform of the (C,α,β) transform defines an $(E,q)(C,\alpha,\beta)$ transform and we shall denote it by $(EC)_n^{q,\alpha,\beta}$.
Moreover, if

$$(EC)_n^{q,\alpha,\beta} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} C_k^{(\alpha,\beta)} \to s, \quad \text{as} \quad n \to \infty,$$

then we shall say that the infinite series $\sum_{n=0}^{\infty} u_n$ is $(E,q)(C,\alpha,\beta)$ summable to the definite number s. We note that for $q = 1$, $\alpha = 1$ and $\beta = 0$ the concept of $(E,q)(C,\alpha,\beta)$ summability reduces to the $(E,1)(C,1)$ summability introduced in [10].

Let $f(x)$ be a 2π periodic function and integrable in the sense of Lebesgue. Then, let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \tag{1}$$

be its Fourier series with n –th partial sum $s_n(f;x)$. For a function $f : R \to R$ the equalities

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\}$$

and

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}, \quad p \geq 1,$$

denote the L_∞ –norm and L_p –norm respectively.

The degree of approximation of a function f by a trigonometric polynomial t_n of order n under the norm $\|\cdot\|_\infty$ is defined by Zygmund [13] with

$$\|f - t_n\|_\infty = \sup\{|f(x) - t_n(x)| : x \in R\},$$

and the best approximation $E_n(f)$ of a function $f \in L_p$ is defined by the equality

$$E_n(f) = \min_{t_n} \|f - t_n\|_p.$$

A function $f \in \text{Lip}\alpha$ or $f \in \text{Lip}(\alpha,p)$ if respectively

$$|f(x+t) - f(x)| = O(|t|^\alpha) \quad \text{for} \quad 0 < \alpha \leq 1,$$

or

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(|t|^\alpha) \quad \text{for} \quad 0 < \alpha \leq 1 \quad \text{and} \quad p \geq 1.$$

For a given positive increasing function $\xi(t)$ and an integer $p \geq 1$, $f \in \text{Lip}(\xi(t),p)$ (see [8]) if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(\xi(t))$$

and $f \in W(L_p, \xi(t))$ if

$$\left(\int_0^{2\pi} |[f(x+t) - f(x)] \sin^\gamma x|^p dx\right)^{1/p} = O(\xi(t)), \quad \gamma \geq 0, \quad p \geq 1.$$

We note here in these definitions that for $\beta = 0$ the class $W(L_p, \xi(t))$ reduces to the class $\text{Lip}(\xi(t),p)$ and if $\xi(t) = t^\alpha$ then the class $W(L_p, \xi(t))$ reduces to the class $\text{Lip}(\alpha,p)$, and if $p \to \infty$, then the class $\text{Lip}(\alpha,p)$ reduces to the class $\text{Lip}\alpha$.

A large number of authors have determined the degree of approximation of functions from the above mentioned classes , using Cesáro and generalized Nörlund means (we refer the reader for details to the papers [1]-[10]). Quite recently H. K. Nigam has studied in [10]-[11] the same problem proving two interesting theorems using $(E,1)(C,1)$ means. Here in this paper we shall generalize his theorems using $(E,q)(C,\alpha,\beta)$ means instead of the $(E,1)(C,1)$ means that are particular cases of them. Before doing this we shall use the notation

$$\phi(t) := f(x+t) + f(x-t) - 2f(x)$$

$$K_n^{q;\alpha,\beta}(t) := \frac{1}{\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{1}{A_k^{\alpha+\beta}} \sum_{\nu=0}^{n} \frac{A_{k-\nu}^{\alpha-1} A_\nu^{\beta} \sin(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t}$$

and prove the following two lemmas, that are needed for the proof of the main results.

Lemma 1.1. *The estimate* $|K_n^{q;\alpha,\beta}(t)| = O(n+1)$ *holds true for* $0 \le t \le \frac{1}{n+1}$.

Proof. Since for $0 \le t \le \frac{1}{n+1}$, $\sin nt \le n \sin t$, and by virtue of

$$\sum_{\nu=0}^{n} A_{k-\nu}^{\alpha-1} A_\nu^{\beta} = A_k^{\alpha+\beta} \quad and \quad \sum_{k=0}^{n} \binom{n}{k} q^{n-k} = (1+q)^n, \tag{2}$$

we have

$$\begin{aligned} |K_n^{q;\alpha,\beta}(t)| &= \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{1}{A_k^{\alpha+\beta}} \sum_{\nu=0}^{n} \frac{A_{k-\nu}^{\alpha-1} A_\nu^{\beta} \sin(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t} \right| \\ &\le \frac{1}{\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{A_k^{\alpha+\beta}} \sum_{\nu=0}^{n} A_{k-\nu}^{\alpha-1} A_\nu^{\beta} \left| \frac{(2\nu+1)\sin t}{4\sin\frac{1}{2}t} \right| \\ &\le \frac{1}{\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k}(k+1) \\ &= O(n+1). \end{aligned}$$

Here the proof ends. ■

Lemma 1.2. *The estimate* $|K_n^{q;\alpha,\beta}(t)| = O(1/t)$ *holds true for* $\frac{1}{n+1} \le t \le \pi$.

Proof. Apply the well-known inequality $\sin(\frac{1}{2}t) \ge t/\pi$ for $\frac{1}{n+1} \le t \le \pi$, together with (2), to obtain

$$
\begin{aligned}
|K_n^{q;\alpha,\beta}(t)| &= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{A_k^{\alpha+\beta}}\sum_{v=0}^{n}\frac{A_{k-v}^{\alpha-1}A_v^{\beta}\sin(v+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right| \\
&\leq \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\alpha+\beta}}\sum_{v=0}^{n}A_{k-v}^{\alpha-1}A_v^{\beta}\frac{\pi}{t} \\
&\leq \frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k} \\
&= O\left(\frac{1}{t}\right).
\end{aligned}
$$

■

2. Main Results

Theorem 2.1. *If f is a 2π periodic function that belongs to the* Lipα *class, then its degree of approximation is given by*

$$\|(EC)_n^{q;\alpha,\beta}(f)-f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), \quad 0<\alpha<1.$$

Proof. Since

$$s_n(f;x)-f(x) = \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt,$$

then

$$C_n^{(\alpha,\beta)}(f;x)-f(x) = \frac{1}{2\pi A_n^{\alpha+\beta}}\int_0^\pi \phi(t)\sum_{k=0}^{n}A_{n-k}^{\alpha-1}A_v^{\beta}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt.$$

Now by denoting the $(E,q)(C_n^{\alpha,\beta})$ transform of $s_n(f;x)$ by $(EC)_n^{(q;\alpha,\beta)}(f;x)$ we have

$$
\begin{aligned}
(EC)_n^{(q;\alpha,\beta)}(f;x)-f(x) &= \frac{1}{\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\times \\
&\times \int_0^\pi \frac{\phi(t)}{2\sin\frac{1}{2}t}\sum_{v=0}^{n}\frac{A_{k-v}^{\alpha-1}A_v^{\beta}\sin(v+\frac{1}{2})t}{A_k^{\alpha+\beta}}dt \\
&= \int_0^\pi \phi(t)K_n^{q;\alpha,\beta}(t)dt \\
&= \left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)\phi(t)K_n^{q;\alpha,\beta}(t)dt := S_1+S_2. \qquad (3)
\end{aligned}
$$

Using lemma 1.1 leads to

$$|S_1| \le \int_0^{\frac{1}{n+1}} |\phi(t)||K_n^{q;\alpha,\beta}(t)|dt$$
$$= O(n+1)\int_0^{\frac{1}{n+1}} t^\alpha dt = O\Big(\frac{1}{(n+1)^\alpha}\Big). \tag{4}$$

And lemma 1.2 gives

$$|S_2| \le \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||K_n^{q;\alpha,\beta}(t)|dt$$
$$= \int_{\frac{1}{n+1}}^{\pi} |t^\alpha| O(\tfrac{1}{t})dt = \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1}dt = O\Big(\frac{1}{(n+1)^\alpha}\Big). \tag{5}$$

Insertion of (4) and (5) in (3) completes this proof. ■

Theorem 2.2. *If f is a* 2π *periodic function that belongs to the weighted* $W(L_p,\xi(t))$ *class, then its degree of approximation is given by*

$$\|(EC)_n^{q;\alpha,\beta}(f) - f\|_p = O\Big((n+1)^{\gamma+\frac{1}{p}}\ \xi\Big(\frac{1}{n+1}\Big)\Big), \tag{6}$$

provided that $\xi(t)$ satisfies the following conditions:

$$\Big\{\frac{\xi(t)}{t}\Big\} \quad \text{is a decreasing sequence}, \tag{7}$$

$$\left\{\int_0^{\frac{1}{n+1}} \Big(\frac{t|\phi(t)|}{\xi(t)}\Big)^p \sin^{\gamma p} t\, dt\right\}^{1/p} = O\Big(\frac{1}{n+1}\Big), \tag{8}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \Big(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\Big)^p dt\right\}^{1/p} = O((n+1)^\delta), \tag{9}$$

where δ is an arbitrary number such that $s(1-\delta)-1>0$, $1/p+1/s=1$, $1\le p\le\infty$, when conditions (8) and (9) hold uniformly in x and $(EC)_n^{q;\alpha,\beta}$ is $(E,q)(C_n^{\alpha,\beta})$ means of the series (1).

Proof. We start from relation (3) of theorem 2.1:

$$(EC)_n^{(q;\alpha,\beta)}(f;x) - f(x) = \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right)\phi(t)K_n^{q;\alpha,\beta}(t)dt := S_1 + S_2. \tag{10}$$

As in [10], it is easy to prove the implication $f\in W(L_p,\xi(t)) \Rightarrow \phi\in W(L_p,\xi(t))$. Moreover, using Hölder's inequality and the fact that $\phi\in W(L_p,\xi(t))$, condition (8), $\sin t \ge \frac{2t}{\pi}$, lemma 1.1, and second mean value theorem for integrals, we have

$$|S_1| \le \left\{\int_0^{\frac{1}{n+1}} \Big(\frac{t|\phi(t)|}{\xi(t)}\Big)^p \sin^{\gamma p} t\, dt\right\}^{1/p} \left\{\int_0^{\frac{1}{n+1}} \Big(\frac{\xi(t)|K_n^{q;\alpha,\beta}(t)|}{t\sin^\gamma t}\Big)^s dt\right\}^{1/s}$$
$$= O\Big(\frac{1}{n+1}\Big)\left\{\int_0^{\frac{1}{n+1}} \Big(\frac{(n+1)\xi(t)}{t^{1+\gamma}}\Big)^s dt\right\}^{1/s}$$
$$= O\Big(\frac{1}{n+1}\Big)\left\{\int_\varepsilon^{\frac{1}{n+1}} \frac{1}{t^{(1+\gamma)s}}dt\right\}^{1/s}, \qquad \Big(0<\varepsilon<\frac{1}{n+1}\Big)$$
$$= O\Big((n+1)^{\gamma+\frac{1}{p}}\ \xi\Big(\frac{1}{n+1}\Big)\Big), \quad \textit{where} \quad 1/p+1/s=1. \tag{11}$$

Again, using Hölder's inequality, $|\sin t| \leq 1$, $\sin t \geq \frac{2t}{\pi}$, conditions (7) and (9), lemma 1.2, and second mean value theorem, we obtain

$$\begin{aligned}
|S_1| &\leq \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p \sin^{\gamma p} t\,dt\right\}^{1/p}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)|K_n^{q;\alpha,\beta}(t)|}{t^{-\delta} sin^{\gamma} t}\right)^s dt\right\}^{1/s} \\
&= O((n+1)^{\delta})\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{\gamma+1-\delta}}\right)^s dt\right\}^{1/s} \\
&= O((n+1)^{\delta})\left\{\int_{\frac{1}{\pi}}^{n+1}\left(\frac{\xi(1/u)}{u^{\delta-1-\gamma}}\right)^s \frac{du}{u^2}\right\}^{1/s} \\
&= O((n+1)^{\delta}\,\xi(\tfrac{1}{n+1}))\left\{\int_{\frac{1}{\pi}}^{n+1}\frac{du}{u^{s(\delta-1-\gamma)+2}}\right\}^{1/s} \\
&= O((n+1)^{\delta}\,\xi(\tfrac{1}{n+1}))\left\{\frac{(n+1)^{s(\gamma+1-\delta)-1}-\pi^{s(\delta-1-\gamma)+1}}{s(\gamma+1-\delta)-1}\right\}^{1/s} \\
&= O((n+1)^{\delta}\,\xi(\tfrac{1}{n+1}))\{(n+1)^{\gamma+1-\delta-1/s}\} \\
&= O\left((n+1)^{\gamma+\frac{1}{p}}\,\xi(\tfrac{1}{n+1})\right), \quad where \quad 1/p+1/s=1. \qquad (12)
\end{aligned}$$

Finally, by inserting (11) and (12) into 10) we obtain

$$|(EC)_n^{(q;\alpha,\beta)}(f;x)-f(x)| = O\left((n+1)^{\gamma+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right),$$

and thus

$$\begin{aligned}
\|(EC)_n^{(q;\alpha,\beta)}(f;x)-f(x)\|_p &= \left\{\int_0^{2\pi}\int_0 |(EC)_n^{(q;\alpha,\beta)}(f;x)-f(x)|^p dx\right\}^{1/p} \\
&= O\left\{\left[\int_0^{2\pi}\left((n+1)^{\gamma+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right)^p dx\right]^{1/p}\right\} \\
&= O\left((n+1)^{\gamma+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right),
\end{aligned}$$

which completes the proof. ■

3. Conclusion

In this section we give some direct consequences of the main results. The $(E,q)(C,\alpha,\beta)$ means can be reduced to the following means:

1. If $\beta=0$ then $(E,q)(C,\alpha,\beta)\equiv(E,q)(C,\alpha,0)\equiv(E,q)(C,\alpha)$
2. If $\alpha=1$ then $(E,q)(C,\alpha,\beta)\equiv(E,q)(C,1,\beta)$
3. If $\beta=0$, $q=1$ then $(E,q)(C,\alpha,\beta)\equiv(E,1)(C,\alpha,0)\equiv(E,1)(C,\alpha)$
4. If $\alpha=1$, $q=1$ then $(E,q)(C,\alpha,\beta)\equiv(E,1)(C,1,\beta)$
5. If $\alpha=q=1$, $\beta=0$ then $(E,q)(C,\alpha,\beta)\equiv(E,1)(C,1)$.

Denoting $(E,q)(C,\alpha)$, $(E,q)(C,1,\beta)$, $(E,1)(C,\alpha)$, $(E,1)(C,1,\beta)$, $(E,1)(C,1)$ means of $s_n(f;x)$, respectively, by $(EC)_n^{(q;\alpha,0)}(f;x)$, $(EC)_n^{(q;1,\beta)}(f;x)$, $(EC)_n^{(1;\alpha,0)}(f;x)$, $(EC)_n^{(1;1,\beta)}(f;x)$, and $(EC)_n^{(1;1,0)}(f;x)$ then from theorems 2.1 and 2.2 lots of corollaries can be derived. We shall formulate below only two of them.

Corollary 3.1. *If $\alpha = q = 1$, $\beta = 0$, and f is a 2π periodic function that belongs in the* Lipα *class, then its degree of approximation is given by*

$$\|(EC)_n^{(1;1,0)}(f) - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), \quad 0 < \alpha < 1.$$

Corollary 3.2. *If f is a 2π periodic function that belongs in the weighted $W(L_p, \xi(t))$ class, then its degree of approximation is given by*

$$\|(EC)_n^{(1;1,0)}(f) - f\|_p = O\left((n+1)^{\gamma+\frac{1}{p}}\, \xi\left(\frac{1}{n+1}\right)\right),$$

provided that $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \quad \text{is a decreasing sequence,}$$

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^p \sin^{\gamma p} t\, dt\right\}^{1/p} = O\left(\frac{1}{n+1}\right), \tag{13}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O\left((n+1)^\delta\right), \tag{14}$$

where δ is an arbitrary number such that $s(1-\delta) - 1 > 0$, $1/p + 1/s = 1$, $1 \le p \le \infty$, when conditions (13) and (14) hold uniformly in x and $(EC)_n^{(1;1,0)}$ is the $(E,1)(C,1)$ means of the series (1).

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