***Journal of Numerical Mathematics and Stochastics*, 4** (1) : 26-39, 2012 © ***JNM@S***
http://www.jnmas.org/jnmas4-3.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

# Construction and Simulation of a Class of Random Metrics on Finite Metric Spaces

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**Abstract**. *This paper aims at proving that split metrics of finite sets* $\{1,2,\ldots,n\}$ *are extremal pseudometrics, and only for* $n \le 4$ *they are the unique extremal rays. Based on inherent properties, we propose algorithms for constructing a class of arbitrary (random) metrics. Moreover, we present some interesting analytical results on split metrics. The theoretical part is complemented by numerical simulations and geometric comparisons of random metrics to the Euclidean metric. The generalization of this work should be an interesting open problem.*

**Key words** : Metrics, Finite Metric Spaces, Split Metrics, *n* Points Space, Construction of Metrics, Random Metrics.

**AMS Subject Classifications** : 35B38, 05B25

## 1. Introduction

Mathematical modeling is one of the interesting methods for understanding and analyzing critical and asymptotic behaviors of several real life phenomena. In this respect, biological phenomena have motivated us to aim at related numerical constructions. For instance, in microbiology every strain of a collection of bacterial strains, considered as a finite metric space, can exhibit their dissimilarity via computing or by comparing the reaction of the considered strains with various tests. Comparing their DNA, and so on, can also serve the same purpose. For more details, we refer the reader to B. Sturmfels [8].

Another area of application is sociology, where a finite number of possible decisions of a group of agents represent the finite metric space. The measurement within two different decisions is given by a discrete metric on a finite set, see e.g. [4]. The numerical treatment of such models is till now realized only by using classical and deterministic metrics. In this work, we propose a new investigation in the pertaining numerical mathematics, by introducing the notion of random metrics on finite metric spaces. A part of this analysis and its use was our subject in [9]. We believe however that more interesting would be to envisage more importance added to this new notion. By considering an $n$ point metric space, studied for example in [1, 2, 3], we can, as in [2], define and use some characteristics of the so called cut or split metrics to construct arbitrary metrics. Bandelt and Dress introduce moreover in [2] an interesting result concerning metrics on finite sets. They prove that every metric of a finite metric space can be written as a sum of split metrics and a split-prime metric. In [5] the author presented the notion of a random distance on finite sets. Furthermore in [6, 7] the authors present some useful algorithms for binary trees. Using the fact that any linear combination of metrics is a metric, random metrics result from random choice of these metrics. In our paper, we provide some analysis and interesting algorithms for construction of a random metric derived from splitting of finite sets. It is important to note that random metric spaces are geometrically not trivial. Such spaces are characterized only by a metric on finite set. Hence the geometrical representation poses an interesting challenge.

Our paper is structured as follow. At first we analyze the notion of extremal metrics. This is followed by a proof of a necessary condition for an extremal metric. The second step is the analytical and numerical construction of random metrics. In the last step, we report some numerical simulations of random metrics in comparison to the Euclidean metric.

## 2. Extremal Pseudometrics

A pseudometric space is a generalization of a metric space in which we allow for the possibility that $d(x,y) = 0$ for distinct values of $x$ and $y$. Thus a pseudometric is a metric if $d(x,y) > 0$ for $x \neq y$. Let then $(X,d)$ be a finite pseudometric (resp. metric) space, and denote by $M$ the set of all metrics on $X$ to define $M_0$ as the set of all pseudometrics. We call $d_v$ a pseudometric vector, which is the $C_n^2$ dimensional vector set of vectors of the form $d_v = (d_{1,2},\dots,d_{1,n},d_{2,3},\dots,d_{2,n},\dots,d_{n-1,n})$, where $d_{i,j}$ are the entries of the corresponding pseudometric $d$. Let us conceive the polyhedron, given as the intersection of the half-spaces generated by pseudometric vectors, where the extremal pseudometric vectors are intersection of hyperplanes. We may consider then a pseudometric vector $d_v \in \mathbb{R}^m$. The triple $(d_{i,j},d_{j,k},d_{i,k}))$, for all $i,j,k = 1\dots,m$, is called pseudometric card and the triple $(x_i,x_j,x_k)$ is called an admissible card if and only if $x_i,x_j,x_k \geq 0$ and $x_i \leq x_j + x_k; \quad \forall i,j,k = 1,\dots,m$.

When $(X,d)$ is an $n$ points pseudometric space, we may define the sets $R$ and $I$ as:

$$R = \Big\{(i,j,k)|i,j,k = 1,\dots,n, \text{ pairwise distinct}\Big\}, \tag{1}$$

$$I = \{t|t = 1,\dots,m\}.$$

In other words, the set $R$ clearly represents the indices of all admissible cards, and satisfies the metric axioms. $I$ is the set of indices, such that the $t^{th}$ components $x_t$ represent a distance between two different points. For $n \in \mathbb{N}$, $m = C_n^2$, $t = 1,\dots,m$ and $i,j,k \in \{1,\dots,n\}$, the maps $l_t$ and $l_{i,j,k}$ defined viz

$$l_t,\ l_{i,j,k} : \mathbb{R}^m \to \mathbb{R}, \tag{2}$$

$$l_t(x) = l_i(x_1,\dots,x_m) = x_i,$$

$$l_{i,j,k}(x_1,\dots,x_m) = x_j + x_k - x_i.$$

are linear, for $t \in I$ and $(i,j,k) \in R$. Hence, we define the half-spaces $K_t$ and $K_{i,j,k}$ as follows.

$$K_t = \Big\{x \in \mathbb{R}^m|\ \ell_t(x) \geq 0\Big\}, \quad K_{i,j,k} = \Big\{x \in \mathbb{R}^m|\ \ell_{i,j,k}(x) \geq 0\Big\}. \tag{3}$$

The subsets

$$H_t = \{x \in \mathbb{R}^m | l_t(x) = 0\}, \quad H_{i,j,k} = \{x \in \mathbb{R}^m | l_{i,j,k}(x) = 0\}, \tag{4}$$

are hyperplanes in $\mathbb{R}^m$, defined by the extremal conditions on $H_t$ and $H_{i,j,k}$. In other words, the set $R$ represents the indices of all admissible cards, and satisfies the metric axioms. $I$ is the set of indices, such that the $i^{th}$ components $x_i$ represent a distance between two different vertices. Let $A$ be a subset of a finite set $X$. The split (or cut) metric is a mapping defined as

$$d(x,y) := \begin{cases} 0, & \text{if } (x,y) \in A \times A \text{ or } A^c \times A^c \\ 1, & \text{otherwise,} \end{cases}$$

where $A^c$ denote the complement set of $A$ in $X$.

**Example 2.1**. Let us consider two parallel and distinct real lines $P$ and $Q$ in $\mathbb{R}^2$, and define two points $p \in P$ and $q \in Q$ such that

$$d(p,q) = \inf_{y \in Q} d(p,y) = \inf_{y \in Q} d(x,q), \tag{5}$$

where $d$ is an Euclidean metric in $\mathbb{R}^2$. Now, we define the map $d_{pq}$ as

$$d_{pq} : P \cup Q \to \mathbb{R}^+; \quad d_{pq}(x,y) = \left| < \overrightarrow{xy}, \overrightarrow{pq} > \right|, \tag{6}$$

where $< \bullet, \bullet >$ is the usual scalar product in $\mathbb{R}^2$. If we suppose that $d_{pq}(p,q) = 1$, and since $\overrightarrow{pq}$ is orthogonal to $P$ and $Q$, then $d_{pq}$ is a pseudometric given as

$$d_{pq}(x,y) := \begin{cases} 0, & \text{if } (x,y) \in P^2 \text{ or } Q^2 \\ 1, & \text{otherwise.} \end{cases}$$

The axioms of a pseudometric are clearly satisfied.

**Example 2.2**. Let us denote by $EX_3$ the set of the standard pseudometrics $c_3^0,\ c_3^1,\ c_3^2,\ c_3^3$ of a three point pseudometric space. That is $EX_3 = \{(0,0,0);(0,1,1);(1,0,1);(1,1,0)\}$. Hence, we denote by $EX_n$ the set of all pseudometric vectors of all split metrics on the $n$ points set $X = \{1,2,\dots,n\}$.

A pseudometric $d$ is called extremal metric if for all $g,h \in M_0$, the following holds: $g + h = d$ implies $g = \nu d,\ h = \mu d$ with $\nu,\mu \geq 0$. If $d$ is extremal, then $(\alpha d)_{\alpha \in \mathbb{R}^+}$ is called extremal ray of $d$. It is important to note that the notion of extremal metric is equivalent to the notion of an extremal metric vector. This equivalence is easy to prove. Therefore, in the following we deal only with metric vectors.

**Lemma 2.1**. *An extremal pseudometric vector is given by the intersection of $m-1$ of the following hyperplanes*

$$H_t = \{x \in \mathbb{R}^m | l_t(x) = 0\},\ t \in I;\ H_{i,j,k} = \{x \in \mathbb{R}^m | l_{i,j,k}(x) = 0\},\ (i,j,k) \in R.$$

*Proof.* $l_t(x)$ and $l_{i,j,k}(x)$ represent the nonnegativity of a pseudometric and the triangular equation respectively. The set of all pseudometric vectors can be written as an intersection of half-spaces as known in linear programming. So the extremal rays of a pseudometric vector in our case is exactly the intersection line of $m-1$ hyperplanes, where $m = C_n^2$. Using the property that the metric vector of an arbitrary metric in a finite metric space is an intersection between hyperplanes, we propose subsequently a method for finding this intersection set. ■

**Lemma 2.2**. *Let $(X, d)$ be a finite points metric space. If $X$ is a three of four points metric space, then $d_v$ is an extremal if and only if $d_v \in \alpha EX_3$ resp. $d_v \in \alpha EX_4$ for $\alpha \in R^+$.*

*Proof*. As in linear programming, note that the intersection of the $m-1$ hyperplanes in lemma 2.1 are exactly the extremal pseudometric vectors. In our proof, we look for solutions in the form of a pseudometric vector. It is clear that the standard metric cards of a three point metric space satisfy the extremal: $\ell_i(x) = 0$ and $\ell_{i,j,k} = 0$. Moreover, $EX_3$ is a generator of all pseudometrics in a three points metric space. If $(d_{1,2}, d_{1,3}, d_{2,3})$ is an arbitrary pseudometric card in a three points metric space, then it can be written as the following linear combination:

$$\begin{aligned} d_v &= \sum_{i=0}^{4} \alpha_i c_3^1 \quad = (d_{1,2}, d_{1,3}, d_{2,3}) \\ &= 0c_3^0 + \tfrac{1}{2}[(d_{1,2} + d_{1,3} - d_{2,3})c_3^1 + (d_{1,2} - d_{1,3} + d_{2,3})c_3^2 + (-d_{1,2} + d_{1,3} + d_{2,3})]c_3^3, \end{aligned} \tag{7}$$

where $\alpha_i \geq 0$ for $i = 0, 1, 2, 3$. Thus, all pseudometrics on $X$ are linear combination of extremal rays of $EX_3$. Similar to the previous lemma every $d_v \in EX_4$ can be expressed as a linear combination of $EX_4$ viz

$$\begin{aligned} EX_4 &= \{c_4^0, c_4^1, c_4^2, c_4^3, c_4^4, c_4^5, c_4^6, c_4^7\} \\ &= \{(0,0,0,0,0,0); (1,1,1,0,0,0); (1,0,0,1,1,0); (0,0,1,0,1,1); \\ &\quad (0,1,0,1,0,1); (0,1,1,1,1,0); (1,0,1,1,0,1); (1,1,0,0,1,1)\}. \end{aligned} \tag{8}$$

It can easily be seen that the elements of $EX_4$ are solution of the hyperplane intersections given by lemma 2.1. Consider then a metric on $X$, $d_v = (d_{1,2}; d_{1,3}; d_{1,4}; d_{2,3}; d_{2,4}; d_{3,4})$, $\alpha_1, \ldots, \alpha_7 \in IR$, and compute the solution of the following linear system:

$$\sum_{i=0}^{7} \alpha_i c_n^i = d_v = (d_{1,2}, d_{1,3}, d_{1,4}, d_{2,3}, d_{2,4}, d_{3,4}). \tag{9}$$

Our next goal would be to check whether the system (9) has a positive and non-zero solution, namely $\alpha_1, \ldots, \alpha_7 \geq 0$ and $\sum_i \alpha_i > 0$. Indeed if we set $\alpha_0 = 0$ and $\alpha_4 = \alpha$, then the system of equations (9) has the following solution

$$\alpha_5 = \tfrac{1}{2}(d_{1,3} + d_{1,4} - d_{3,4}) - \alpha, \tag{10}$$

$$\alpha_6 = \tfrac{1}{2}(d_{1,2} + d_{1,4} - d_{2,4}) - \alpha, \tag{11}$$

$$\alpha_7 = \tfrac{1}{2}(d_{1,2} + d_{1,3} - d_{2,3}) - \alpha, \tag{12}$$

$$\alpha_1 = \alpha + \tfrac{1}{2}(d_{3,4} + d_{2,4}) - \tfrac{1}{2}(d_{1,3} + d_{1,2}), \tag{13}$$

$$\alpha_2 = \alpha + \tfrac{1}{2}(d_{2,3} + d_{3,4}) - \tfrac{1}{2}(d_{1,4} + d_{1,2}), \tag{14}$$

$$\alpha_3 = \alpha + \tfrac{1}{2}(d_{2,4} + d_{2,3}) - \tfrac{1}{2}(d_{1,4} + d_{1,3}). \tag{15}$$

We can show that $\alpha_5, \alpha_6$ and $\alpha_7$ are positive for all values of the metric vector $d_v$ if and only if $\alpha = 0$ or $2\alpha$ is smaller than the diference of all triangular inequalities. In the first case, the solution $(\alpha_1, \ldots, \alpha_7)$ will be an admissible solution if and only if at least one of the $\alpha_i$ is strictly positive. In the equations (10), (11) and (12) we can choose a positive $\alpha$ such that $\alpha_5, \alpha_6$ and $\alpha_7$

are positive. Once we have these conditions, we can also see that $\alpha_1, \alpha_2$ and $\alpha_3$ are positive too. Furthermore, since $d$ is a pseudometric vector, we have:

$$\alpha_1 = \frac{1}{2}(d_{3,4} + d_{2,4}) - \frac{1}{2}(d_{1,3} + d_{1,2}) \geq \frac{1}{2}(d_{3,4} + d_{2,4} - d_{2,3}) \geq 0, \tag{16}$$

$$\alpha_2 = \frac{1}{2}(d_{2,3} + d_{3,4}) - \frac{1}{2}(d_{1,4} + d_{1,2}) \geq \frac{1}{2}(d_{2,3} + d_{3,4} - d_{2,4}) \geq 0, \tag{17}$$

$$\alpha_3 = \frac{1}{2}(d_{2,4} + d_{2,3}) - \frac{1}{2}(d_{1,4} + d_{1,3}) \geq \frac{1}{2}(d_{2,4} + d_{2,3} - d_{3,4}) \geq 0. \tag{18}$$

In the second case we choose $2\alpha$ equal the minimum difference of the triangular inequalities give by (10)-(12). Thus, replacing $\alpha$ by its explicit value in (13)-(15), the vector $(\alpha_1, \ldots, \alpha_7)$ will be an admissible solution, namely if $2\alpha = d_{1,3} + d_{1,4} - d_{3,4}$, to yield

$$\alpha_1 = \frac{1}{2}(d_{1,4} + d_{2,4} - d_{1,2}) \geq 0, \tag{19}$$

$$\alpha_2 = \frac{1}{2}(d_{2,3} + d_{1,3} - d_{1,2}) \geq 0, \tag{20}$$

$$\alpha_3 = \frac{1}{2}(d_{2,4} + d_{2,3} - d_{3,4}) \geq 0, \tag{21}$$

where the same result can be reached if $2\alpha$ satisfies the other triangular inequalities given by equations(11) and (12). Conclusively, the linear system has a non zero and positive solution. Hence, the extremal metric vectors are exactly the vectors of $EX_4$. ■

**Theorem 2.1**. *Every split metric of an n points metric space is extremal if $n \leq 4$, and the unique extremal pseudometrics are the split metrics.*

*Proof*. The fact that the split metric are the unique extremal metrics is trivial if $n = 0, 1, 2$ and if $n = 3$ and $n = 4$ the result follows from lemma 2.2. Let us consider a subset $A$ of $X$ and define the following split metric:

$$d(x, y) := \begin{cases} 0, & \text{if } (x, y) \in A \times A \text{ or } A^c \times A^c \\ 1, & \text{otherwise.} \end{cases}$$

Denote by $n(A)$ the cardinal of $A$. Due to the existence of a bijection between the corresponding sets, we recall that the extremal pseudometric vector and extremal pseudometric are equivalent. Let us utilize then the result of lemma 2.2: $X$ is an $n$ points pseudometric space, then $d$ has exactly $C^3_{n(A)} + C^3_{n(A^c)}$ triangle of the form $c_0$ and $n(A^c)C^2_{n(A)} + n(A)C^2_{n(A^c)}$ triangle of the form $c_1$. Each triangle satisfies at least one of the following extremal conditions

$$l_t(x) = x_t = 0, \quad \text{or} \quad l_{i,j,k}(x) = d_{ij} - d_{ik} - d_{jk} = 0. \tag{22}$$

If we suppose then that $\sharp A = s$, then $n(A^c) = n - s$ and

$$C^3_{n(A)} + C^3_{n(A^c)} = C^3_s + C^3_{n-s},$$

$$n(A^c)C^2_{n(A)} + n(A)C^2_{n(A^c)} = (n - s)C^2_s + sC^2_{(n-s)}.$$

An extremal ray is the intersection of $m - 1 = C^2_n - 1$ hyperplanes. Accordingly, we need to prove that

$$C^3_s + C^3_{n-s} + (n - s)C^2_s + sC^2_{(n-s)} \geq C^2_n - 1, \quad \forall s = 0, \ldots, n. \tag{23}$$

The cases $n = 3$ and $n = 4$ of lemma 2.2 satisfy the condition (23). In a five points pseudometric space, for every $s = 0, \ldots, n$, we have $C^3_n$ as a total number of triangles in the form of $c^0_3$ and/or $c^1_3$, ( or $c^2_3$, $c^3_3$). Each triangle satisfying at least one intersection condition.

Since $C_n^3 \geq C_n^2 - 1,\ \forall n \geq 5,$ the corresponding pseudometric vector of $d$ is solution of the intersection of $C_n^2 - 1$ hyperplanes. Hence, $d$ is extremal. ∎

**Remark 2.1**. We conclude from the previous theorems that the split metrics are the unique extremal metric only if $n \leq 4$. Otherwise, for $n$ greater than 4, there exist other extremal metrics, which are not necessary split metrics. For more details see for example [1].

**Lemma 2.3**. *Let $(X, d)$ be an $n$ points metric space and if $EX_n$ is the set of all extremal pseudometric vectors, then $\sharp\{d | d \in EX_n\} = 2^{n-1}$.*

*Proof*. The power set $\mathcal{P}(X)$ of $X$ contains exactly $2^n$ sets. From theorem 2.1, we can utilize the sets of $\mathcal{P}(X)$ to construct all possible elements of $EX_n$. For any subset $A$ of $X$, of theorem 2.1, the subsets $A$ and $A^c$ define the same extremal pseudometric. Hence, the number of distinct extremal pseudometrics in $EX_n$ is exactly half number of subsets in $\mathcal{P}(X)$, namely

$$n(\{d | d \in EX_n\}) = 2^n \tfrac{1}{2} = 2^{n-1}. \tag{24}$$

∎

**Lemma 2.4**. *Let $(X, d)$ be an n points metric space. The following identity is true.*

$$n\Big(\{c_n^i |\ d(x,y) = 1;\ c_n^i \in EX_n\}\Big) = n\Big(\{c_n^i |\ c_n^i(x,y) = 0;\ d \in EX_n\}\Big) = 2^{n-2}, \tag{25}$$

*for an arbitrary $i \neq j \in \{1, \ldots, n\}$. (In other words, the number of zeros and ones in a fixed position in the extremal pseudometric vector are equals, and namely to $2^{n-2}$).*

*Proof*. The result of this lemma means that the number of pseudometric vectors, which has one of the properties (25) are the same and equal to $2^{n-2}$. According to theorem 2.3 we may fix two points $x$ and $y$ such that $x \neq y$ and compute all possible combination from $X \setminus \{x,y\}$. Hence, from $n(X \setminus \{x,y\}) = n - 2,$ it follows that $n(\mathcal{P}(X \setminus \{x,y\})) = 2^{n-2}$. This means that there exists exactly $2^{n-2}$ extremal pseudometric vectors where the component of $c_n(x,y)$ is exactly 1. ∎

**Corollary 2.1**. *Let $(X, d)$ be an n points metric space, then the following sum is satisfied.*

$$\sum_{i=1}^{2^{n-1}} c_n^i = 2^{n-2}(1, \ldots, 1),\ \textit{where}\ (1, \ldots, 1) \in \mathbb{R}^m,\ m = C_n^2. \tag{26}$$

*Proof*. Since the distance between two points using the $2^{n-1}$ extremal pseudometrics is in $2^{n-2}$ cases 0 and in $2^{n-2}$ cases 1, then the proof follows immediately from lemma 2.4. Here we propose a method to construct random metrics in a finite metric space. Namely, we use the set of all split metrics, which is extremal. Its cardinal number is $2^{n-1}$. Moreover, let us denote by $EX$ the set of all extremal metrics in an $n$ points metric space, where $EX_n \subset EX$. In order to build a random metric, we use the following constructions and algorithms.

For $\lambda_e \in \mathbb{R}^+$ and $\forall\ e \in EX,\ \alpha, \beta \in X$, the map $d$ , defined as

$$d(\alpha, \beta) := \sum_{e \in EX} \lambda_e e(\alpha, \beta), \tag{27}$$

is a pseudometric. And if $\lambda_e > 0$ for all $e \in EX$, then $d$ is a metric. Indeed since the positive scalar multiplication with a pseudometric is a pseudometric and the sum of pseudometrics is a pseudometric, the sum (27) is a pseudometric. We also note that the set of split metrics is a subset of all extremal metrics. Now suppose that $\lambda_i > 0$ for $i = 1\ldots, 2^{n-1}$, for all $\alpha, \beta \in X$, then the set $\{\alpha\}$ is a subset of $X$ and $X \setminus \{\alpha\}$ is the complement set of $\{\alpha\}$. Since $\alpha \neq \beta$, it follows that $\beta \in X \setminus \{\alpha\}$. We may define according to the previous the following extremal pseudometric

$$d(x,y) := \begin{cases} 0, & \text{if } (x,y) \in \{\alpha\} \times \{\alpha\} \text{ or } X \setminus \{\alpha\} \times X \setminus \{\alpha\} \\ 1, & \text{otherwise.} \end{cases} \tag{28}$$

As $d$ is an element of $EX_n$, so there exists an $i$, such that $c_n^i(\alpha, \beta) = 1$. Hence for all $\alpha, \beta \in X$ there exists $c_n^i$ such that $c_n^i(\alpha, \beta) > 0$, and if $\lambda_i > 0$ then $d(\alpha, \beta) > 0$, which means that $d$ is a metric.

In the previous notation, $d$ is called a random metric, whenever the choice of $\lambda_e$ and/or the choice of $e$ is random. For our numerical simulation of random metrics, we use the following lemmas to construct a class of random metrics from the class of cut metrics. As before $(X, d)$ is an $n$ points pseudometric space. If $\lambda_i$, for $i = 1\ldots, 2^{n-1}$, is a sequence of independent and identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$ with realization in $]0, \infty[$, then the map $d(\bullet, \bullet)(\omega)$ defined as

$$d(\alpha, \beta)(\omega) := \sum_{i=1}^{2^{n-1}} \lambda_i(\omega) c_n^i(\alpha, \beta), \quad c_n^i \in EX_n, \tag{29}$$

is a class of random metrics extracted from the set of the cut metrics. For $I \subset \{1\ldots, 2^{n-1}\}$, $\lambda_i \in \mathbb{R}^+$ , $i \in I$, and $\lambda > 0$, the map $d$ defined as

$$d(\alpha, \beta) := \lambda\delta(\alpha, \beta) + \sum_{i \in I} \lambda_i c_n^i(\alpha, \beta), \quad c_n^i \in EX_n, \tag{30}$$

is a pseudometric, and if $\lambda$ is strictly positive, $d$ is a metric. Here, $\delta$ is the discrete metric given by

$$\delta(x,y) := \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

Since the sum of pseudometrics is a pseudometric for all $\lambda_i$ and for all $\lambda$ the sum (30) is a pseudometric. If $\lambda > 0$, then for all $\alpha, \beta \in X$ the distance $d(\alpha, \beta)$ is strict positive since $\lambda(\alpha, \beta)$ is strictly positive. Moreover For $p \in X$, the following pseudometric is extremal

$$d_p(x,y) := \begin{cases} 1, & \text{if } (x = p \text{ and } y \neq p) \text{ or } (x \neq p \text{ and } y = p) \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

■

**Remark 2.2**. The construction of a random metric is based on: (*i*) random choice of the coefficients $\lambda_e$, (*ii*) random choice of the linearly independent hyperplane equations or (*iii*) all assumptions together.

For our numerical simulations of random numbers, we use two random variables, the first one is uniformly distributed in any positive set of $\mathbb{R}^+$. The second one is normally distributed. The metric is then constructed using one of the algorithms presented bellow.

## 2.1. **Algorithms for construction of random metrics**

In this sub-section we present some algorithms to construct metrics and random metrics.

**Algorithm 1**. According to formula (28) the following algorithm generates a class of random metrics.
1. Set $d = 0$ zero metric and $p = 1$.
2. Define $d_p$ as $d_p(x,y) := \begin{cases} 1, & \text{if } (x = p \text{ and } y \neq p) \text{ or } (x \neq p \text{ and } y = p) \\ 0, & \text{otherwise.} \end{cases}$
3. Choose randomly $a_p > 0$; Set $d = d + a_p\, d_p$.
4. If $p = n$, Return $d$; Stop. Else, Set $p = p + 1$, Go to 2. End.

**Algorithm 2**. Relation (29) leads to the following algorithm for constructing only a class of random metrics that are extractable from split metrics.
1. Set $d = 0$ zero metric, $p = 1$ and $j = 1$.
2. Define $d_p$ a cut metric of $\{1,2,3,\ldots,n\}$ as,
3. For $p = 1$ to $2^{n-1}$ construct the cut matrics $d_p$.
4. For $j = 1$ to $2^{n-1}$ do choose randomly $a_j > 0$; Set $d = d + a_j\, d_j$, Return $d$, End.

**Algorithm 3**. This is another algorithm for constructing random metrics, as in the preceding algorithm, from the same class of split metrics.
1. Set $d = 0$ zero metric, $M$ is a large integer.
2. For $j = 1$ to $M$ choose randomly a cut matric $d_j$ and $a_j > 0$; Set $d = d + a_j\, d_j$,
3. Choose randomly $a > 0$; Set $d = a + d$, Return $d$, End.

**Algorithm 4**. More generally the following algorithm can be used to construct any random metric from a convex combination of extremal pseudometrics.
1. Let $s(d) = Sum\ d(x,y)$, Set $d = 0$ zero metric, $p = 0$.
2. Choose randomly $C_n^2 - 1$ for an equation of type: $d(x,y) = 0$ and/or $d(x,y) = d(x,z) + d(z,y)$, Solve $d'$ in $s(d') = 1$.
3. For a random number $a \in (0,1)$; Set $d = ad + (1-a)d'$, and $p = p + 1$.
4. If $p = C_n^2 + 1$, Return $d$, Stop. Else, Go to 2. End.

## 2.2. **Numerical simulation of random metrics**

Let us generate numerical simulations of random metrics and compare them with the the Euclidean metric. For $NP = 8$, we present in matrix form two examples of random metrics and for $NP = 25, 50$ grid points, representing the finite metric space, we plot the geometrical behavior of the corresponding random metrics. This is done by plotting the two dimensional function $(x,y) \mapsto d_k(x,y)$ with $k = e, r$, where $d_e, d_r$ represent the Euclidean and random metrics respectively. The used random numbers in our algorithms are uniformly distributed. The first matrix refers to the Euclidean metric for an equidistant discretization, the second and the third examples refer obviously to random matrices.

$$d_e = \begin{pmatrix} 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 \\ 0.1 & 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 \\ 0.2 & 0.1 & 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.2 & 0.1 & 0 & 0.1 & 0.2 & 0.3 \\ 0.5 & 0.4 & 0.3 & 0.2 & 0.1 & 0 & 0.1 & 0.2 \\ 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 & 0 & 0.1 \\ 0.7 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 & 0 \end{pmatrix}$$

$$d_r = \begin{pmatrix} 0 & 0.5062 & 0.4627 & 0.5693 & 0.2864 & 0.4828 & 0.3735 & 0.3238 \\ 0.5062 & 0 & 0.634 & 0.7406 & 0.4577 & 0.654 & 0.5447 & 0.4951 \\ 0.4627 & 0.634 & 0 & 0.6971 & 0.4142 & 0.6106 & 0.5013 & 0.4516 \\ 0.5693 & 0.7406 & 0.6971 & 0 & 0.5208 & 0.7172 & 0.6079 & 0.5582 \\ 0.2864 & 0.4577 & 0.4142 & 0.5208 & 0 & 0.4343 & 0.325 & 0.2753 \\ 0.4828 & 0.654 & 0.6106 & 0.7172 & 0.4343 & 0 & 0.5214 & 0.4717 \\ 0.3735 & 0.5447 & 0.5013 & 0.6079 & 0.325 & 0.5214 & 0 & 0.3624 \\ 0.3238 & 0.4951 & 0.4516 & 0.5582 & 0.2753 & 0.4717 & 0.3624 & 0 \end{pmatrix}$$

$$d_r = \begin{pmatrix} 0 & 0.4717 & 0.5044 & 0.4676 & 0.5394 & 0.5018 & 0.5124 & 0.5446 \\ 0.4717 & 0 & 0.4621 & 0.4253 & 0.4971 & 0.4595 & 0.4701 & 0.5023 \\ 0.5044 & 0.4621 & 0 & 0.458 & 0.5298 & 0.4923 & 0.5029 & 0.5351 \\ 0.4676 & 0.4253 & 0.458 & 0 & 0.493 & 0.4554 & 0.466 & 0.4982 \\ 0.5394 & 0.4971 & 0.5298 & 0.493 & 0 & 0.5272 & 0.5378 & 0.57 \\ 0.5018 & 0.4595 & 0.4923 & 0.4554 & 0.5272 & 0 & 0.5003 & 0.5325 \\ 0.5124 & 0.4701 & 0.5029 & 0.466 & 0.5378 & 0.5003 & 0 & 0.543 \\ 0.5446 & 0.5023 & 0.5351 & 0.4982 & 0.57 & 0.5325 & 0.543 & 0 \end{pmatrix}$$

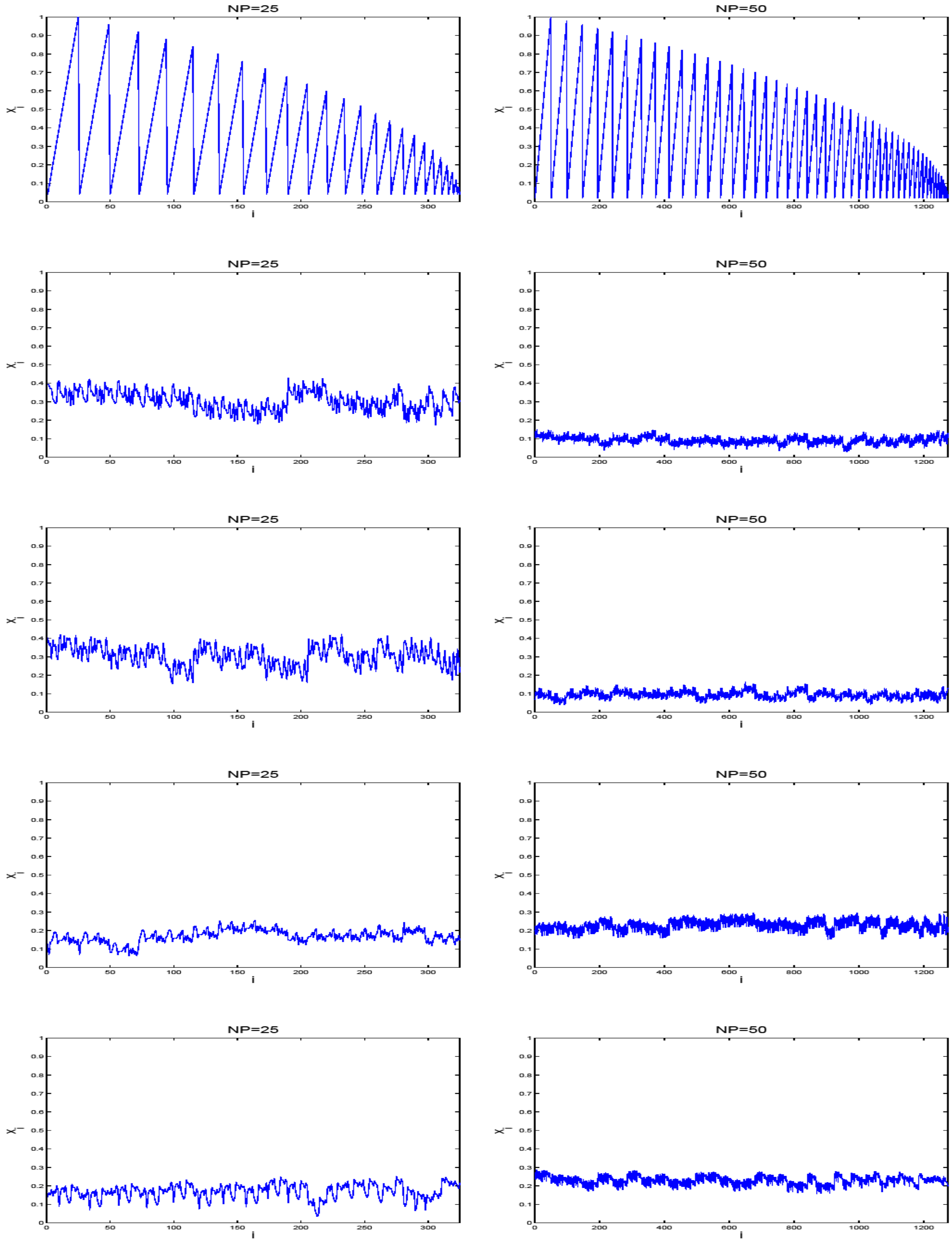


Figure 1: Euclidean metric vector (first row) and simulation of four random metric vectors for each finite metric space with $NP = 25$ and $NP = 50$.

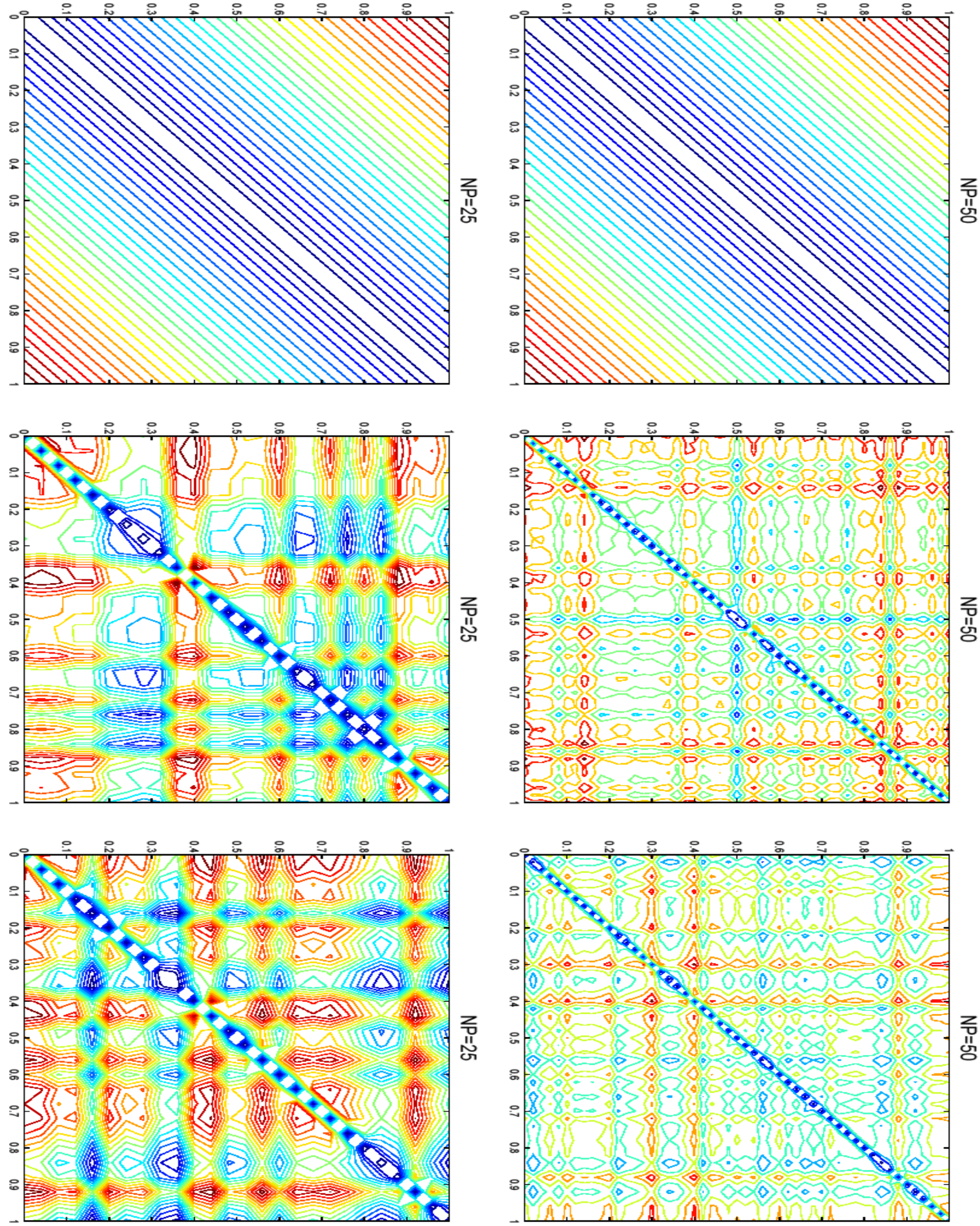


Figure 2: Contour plot for the Euclidean metric matrix(first row) and simulation of random metric matrices for each finite metric space with $NP = 25$ and $NP = 50$ (second and third rows).

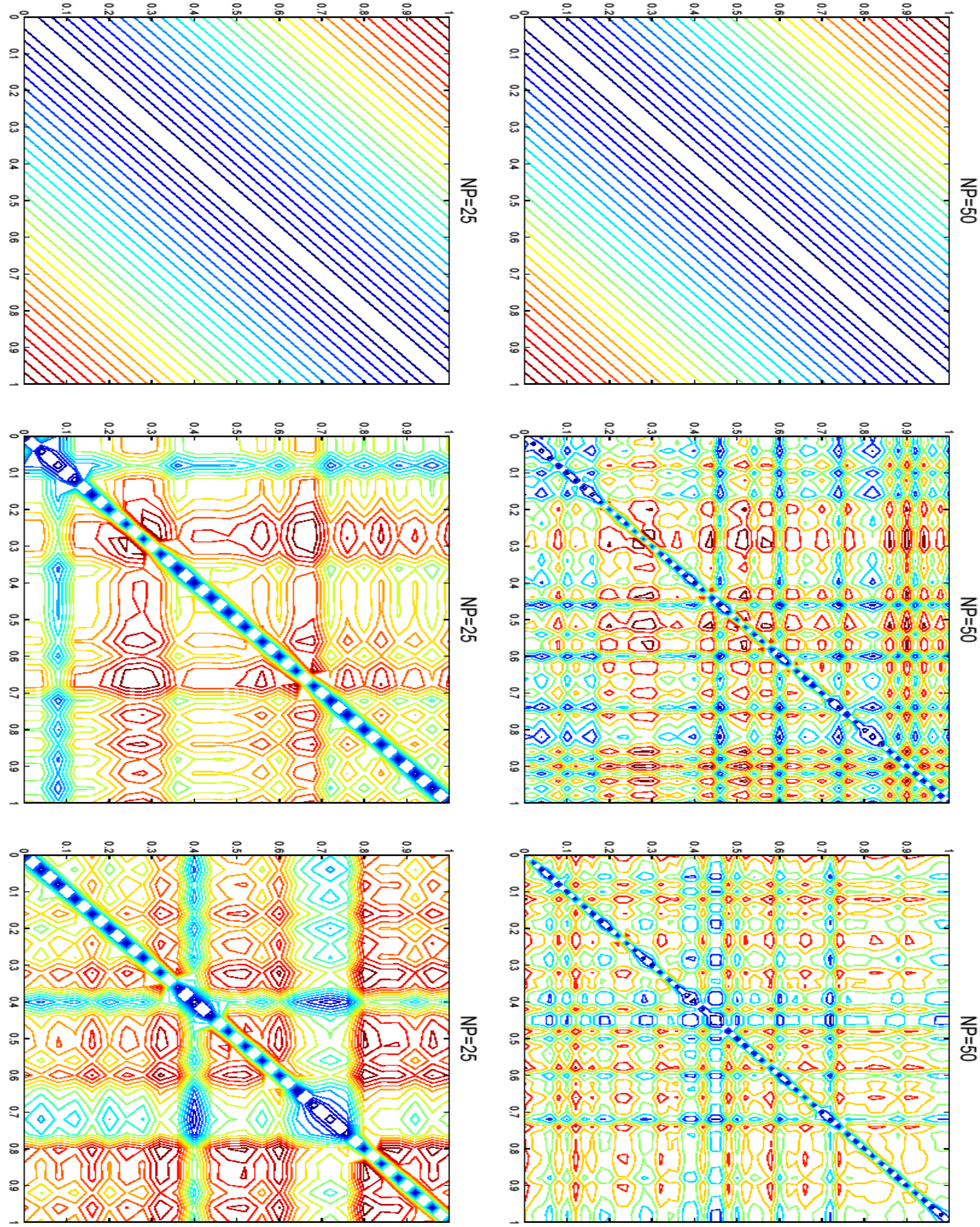


Figure 3: Contour plot for the Euclidean metric matrix(first row) and simulation of random metric matrices for each finite metric space with $NP = 25$ and $NP = 50$ (second and third rows).

In order to compare the entries of the considered metrics, we use the metric vector $d_v = x = \left(x_1, \ldots, x_{C_n^2}\right)$, as it is denoted in our simulation.

On a finite metric space, with $NP = 25$ and $NP = 50$ respectively, we plot in Figure 1 the Euclidean metric vector (first row) of an equidistant discretization of the unit interval to generate four random vector metrics for each number of points *N*.

Using a uniformly distributed random number generator and a random choice of split metrics, the time series plot in Figure 1 clearly shows the difference between the Euclidean metric vector and the random vector. We have to note that the notion of a distribution is not reflected by the generated metrics. Work in this direction can therefore be quite valuable in the future.

Since the metric is defined only on a finite metric space, we have interpolated the graph to get a continuous representation. Note that the Euclidean metric has the same behavior for $NP = 25$ and $NP = 50$ and thus for any number of points, but the random metric vector behaviors are different. This is due to the choice of the number of points ($NP = 25$ and $NP = 50$) and to the random effect of the the random generator. More interesting is the symmetric behavior and the zero diagonal of all simulated random metrics that is illustrated by Figures 2 and 3. We remark moreover that, for both simulations, the Euclidean metric is smooth and totally different from the other generated metrics.

## 3. Concluding Remarks

Generating random metrics leads to the construction of abstract (finite) metric spaces. Its illustration is an interesting open question. Despite the fact that the reported results pertain only to academic examples, the method can apparently be extended to more realistic problems. For instance DNA-problems. Even more interesting should be such constructions on continuous metric spaces. And to upgrade this method, one has to characterize extremal metrics on arbitrary spaces. For the considered *n* points sets, the number $2^{n-1}$ increases exponentially, therefore the cost of the method is that of a problem of *NP* –hardness. The choice of different random distributions could reflect well on the numerical and graphical difference between the simulated extremal metrics.

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Article history: Submitted February, 19, 2012; Accepted April, 10, 2012.