Journal of Numerical Mathematics and Stochastics, **3** (1) : 80-95, 2011 © ***JNM@S***
http://www.jnmas.org/jnmas3-9.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

Hybrid of Sigmoidal Transformations and Collocation Method for a Generalized Airfoil Equation

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Abstract. *This paper considers the generalized airfoil equation (GAE) with weakly singular kernel, which has a particular Hadamard finite-part and a Cauchy principal value integrals. Through the use of Sigmoidal transformations and a collocation method we derive sufficiently good approximations to the solution of this equation. Convergence analysis for this method is investigated and the a priori error estimation is computed. The paper contains also illustrative numerical examples.*

Key words : Integral Equations, Weakly singular equations, Sigmoidal transformations, Numerical analysis, Cauchy integral equation.

AMS Subject Classifications : 65M12, 65M15, 65M60, 45G05, 45E10

1. Introduction

To invoke the generalized airfoil equation (GAE) when $x \in \Gamma$ we introduce the class of operators

$$S[\phi(x)] = f(x), \tag{1}$$

where

$$S[\phi(x)] = \overline{a(x)}\phi(x) + \frac{b_1(x)}{\pi} C_1[\phi(x)] + \frac{b_2(x)}{\pi} C_2[\phi(x)] + \int_\Gamma k_0(x,t)\phi(t)dt,$$

$f(x)$, b_1, b_2, $\bar{a}$ and k_0 are given Hölder continuous functions and the unknown function $\phi(x) \in L_1(\Gamma)$, is analytic in D, expect possibly for a finite number of points at which ϕ has poles of order one. Here, we may assume that Γ is an arc in the complex plane $\mathbb{R}^2$ or $\mathbb{R}$, and let **A** and **B** denote the endpoints of Γ. It is also assumed that neither **A** nor **B** belongs to Γ. D is a domain containing Γ such that **A** and **B** are boundary points of D. Furthermore, $C_1[\ ]$ denotes a

Cauchy principal value integral operator, and if $\Gamma = (0,1)$ then this operator is denoted as follows:

$$C_1[\phi(x)] = \lim_{\epsilon\to 0}\left(\int_0^{x-\epsilon} + \int_{x+\epsilon}^1\right)\frac{\phi(t)dt}{t-x}. \quad (2)$$

The limit exists when ϕ is Hölder continuous on $\Gamma = (0,1)$. Finally, $C_2[\phi]$ is the Hadamard finite-part integral, which is obtained by the recursive definition:

$$C_{n+1}[\phi(x)] = \frac{1}{n!}\frac{d^n}{dx^n}C_1[\phi(x)], \quad (3)$$

for $n \in \mathbf{N}$, $0 < x < 1$. $C_2[\phi]$ also exists if $\phi(x)$ is Hölder continuous on Γ. The above formula is the well-known Hadamard finite-part integral. The kernel $k_0(z,t)$ has the form

$$k_0(x,t) = k(x-t)a(x,t).$$

These kernel functions have an infinite singularity, and the most important examples are $\log|x-t|$, $|x-t|^{\gamma-1}$ for some $\gamma > 0$ and variants of them. Eq. (1) occurs in wide variety of mathematical physics and engineering problems (see [1], [3], [4], [7], [14], [19]). In the vast literature we meet special cases of (1) (see [10], [15], [20], [24], [25]). The novelty of this paper is related to the implementation of Sigmoidal transformations (see [11], [12]) with the collocation method ([24], [25]) for the GAE. The outline of this paper is as follows. In Section 2, we give a smoothness space solution for (1). In Section 3, we investigate a new version of finding approximations for $C_1[\ ]$ and $C_2[\ ]$. Based on the collocation idea and Sigmoidal transformations we introduce in Section 4 a class of numerical solutions for (1), (we call them Sigmoidal collocations). Convergence analysis of Sigmoidal collocations is given also in this section. Section 5 is devoted to illustrative numerical examples. Finally, concluding remarks are provided in section 6.

2. Properties of Solution Space and Regularity

Consider Eq. (1) where

$$a(x,y) \in C^{m_0}(\Gamma\times\Gamma), \quad k \in C^{m_0-1}(\Gamma), \quad m_0 \geq 1. \quad (4)$$

Differentiation of a weakly singular kernel increases the order of the singularity, e.g. $D_x^\beta|x-y|^{-\nu}$ $(\nu > 0)$ behaves as $|x-y|^{-\nu-|\beta|}$. This observation motivates the following smoothness assumption about the kernel. The kernel $k_0(x,y) = k(x-y)a(x,y)$ is m_0 times $(m_0 \geq 1)$ continuously differentiable on $(\Gamma\times\Gamma)\backslash\{x=y\}$ and there exists a real $\nu \in (-\infty, d)$ (d is a real number) so that the estimate

$$|D_x^\alpha D_{x+y}^\beta k(x-y)| \leq c\begin{cases} 1, & \nu+|\alpha| < 0 \\ 1+|\log|x-y||, & \nu+|\alpha| = 0 \\ |x-y|^{-\nu-|\alpha|}, & \nu+|\alpha| > 0 \end{cases} \quad (5)$$

with c as a constant, $x,y \in \Gamma$, is valid for $|\alpha|+|\beta| \leq m_0$. Here $D_x = \frac{\partial}{\partial x}$, $D_{x+y} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and

$D^{\beta}_{x+y} = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^{\beta})$. Putting $|\alpha| = |\beta| = 0$, provides for a weak singularity ($\nu < d$) in $k(x-y)$. In case $\nu < 0$, the kernel is bounded but its derivatives may be singular. As a solution space let us introduce the following Sobolev space

$$E^{\beta,\Gamma}_{\tau} = \left\{ \phi \in C(\Gamma) \cap C^{m_0}(\Gamma) | \sum_{\beta \leq m_0} \sup_{x \in \Gamma} \frac{|D^{\beta}_x \phi(x)|}{|x|^{-\beta} + |\tau - x|^{-\beta}} < \infty \right\},$$

equipped with the norm

$$\|\phi\|_{E^{\beta,\Gamma}_{\tau}} = \max_{x \in \Gamma} |\phi(x)| + \sum_{\beta \leq m_0} \sup_{x \in \Gamma} \frac{|D^{\beta}_x \phi(x)|}{|x|^{-\beta} + |\tau - x|^{-\beta}}$$

where τ is a constant and it belongs in $\Gamma = [0,1]$. Throughout this paper we shall consider $\tau = 1$. Now we may define the following function space:

(i) $\phi(x)$ is Hölder continuous on every closed subinterval of $\Gamma = (0,1)$,
(ii) near $x = 0$, we have $|\phi(x)| \leq C_1 x^{-\gamma_1}$, such that $0 \leq \gamma_1 < \frac{1}{p_1}$ and $p_1 \in \mathbf{N}$,
(iii) near $x = 1$, we have $|\phi(x)| \leq C_2 (1-x)^{-\gamma_2}$, such that $0 \leq \gamma_2 < \frac{1}{p_2}$ and $p_2 \in \mathbf{N}$,
(iv) near $x = 0$, we have $|\phi^{(2)}(x)| \leq C_3 x^{-\gamma_3}$, such that $0 \leq \gamma_3 < \frac{1}{p_3}$ and $p_3 \in \mathbf{N}$,
(v) near $x = 1$, we have $|\phi^{(2)}(x)| \leq C_4 (1-x)^{-\gamma_4}$, such that $0 \leq \gamma_4 < \frac{1}{p_4}$ and $p_4 \in \mathbf{N}$,

where $C = max\{C_i\}_{i=1}^{4}$ is some constant. We denote this space on $\Gamma = (0,1)$ by $H(p_1,p_2,p_3,p_4) \subseteq H^q_{\rho}(\Gamma)$ (see Remark 2.1). Hence the space of solutions, ϕ, to (1) is:

$$E^{\beta,\Gamma}_{\tau}(p_1,p_2,p_3,p_4) = E^{\beta,\Gamma}_{\tau} \bigcap H(p_1,p_2,p_3,p_4),$$

which is a Banach functional space endowed with the norm $\|.\|_{E^{\beta,\Gamma}_{\tau}}$.

Remark 2.1. Let $L^2_{\rho}(\Gamma)$ be the Hilbert space of all periodic and square integrable functions on Γ, with respect to the weight $\rho(x)$, endowed with scalar product

$$(u,v)_{\rho} = \int_{\Gamma} u(x)v(x)\rho(x)d\Gamma,$$

and the norm $\|u\| = (u,u)^{\frac{1}{2}}_{\rho}$. Let $H^q_{\rho}(\Gamma)$ be the set of all functions ϕ that are k-times differentiable with $\phi^{(k)} \in L^2_{\rho}(\Gamma)$ (when $q \geq 0$ is not an integer, we have $q = k + \alpha_0$, $0 < \alpha_0 < 1$ and if $q \geq 0$ is an integer $q = k$. Therefore, $H^q_{\rho}(\Gamma)$ is the set of all functions $\phi \in H^k_{\rho}(\Gamma)$). We can prove that $H^q_{\rho}(\Gamma)$ follows from a type of Hölder condition with exponent α_0 for $\phi^{(k)}$ (see [8]). Also, if $q > p \geq 0$ then we have the following identity and compact operator:

$$I : H^q_{\rho}(\Gamma) \to H^p_{\rho}(\Gamma),$$

that is $I(\phi) = \phi$.

Moreover, the smoothness of the solution to (1) is demonstrated by the following proposition.

Proposition 2.1. *Let conditions (4) and (5) be satisfied and let* $f(x) \in E^{\beta,\Gamma}_{\tau}(p_1,p_2,p_3,p_4)$. *If Eq.(1) has an integrable solution* ϕ, *then* $\phi \in E^{\beta,\Gamma}_{\tau}(p_1,p_2,p_3,p_4)$ *when* $|\bar{a}|, |b_1|, |b_2| \leq 1$.

Proof. We have to prove that any solution $\phi \in L_{\infty}(\Gamma)$ of Eq. (1) on a small open subset $\Omega \subseteq \Gamma$ so that $(\bar{a}I - A_{\Omega})[\quad]$, defined by

$(\bar{a}I - A_\Omega)[\phi] = S_\Omega[\phi] = \overline{a(x)}\phi(x) + \frac{b_1(x)}{\pi}C_1[\phi(x)] + \frac{b_2(x)}{\pi}C_2[\phi(x)] + \int_\Omega k_0(x,t)\phi(t)dt = f$ is invertible in both spaces $L_\infty(\Omega)$ and $E_\tau^\beta(p_1,p_2,p_3,p_4)$. If the operator A_Ω maps $E_\tau^{\beta,\Omega}(p_1,p_2,p_3,p_4)$ into $E_\tau^{\beta,\Omega}(p_1,p_2,p_3,p_4)$ for ϕ then

$$\|A_\Omega[\phi]\|_{L_\infty(\Gamma)} \leq C_0\|\phi\|_{L_\infty(\Gamma)}. \tag{6}$$

Clearly therefore $(\bar{a}I - A_\Omega)[\ \ ]$ is invertible in $L_\infty(\Gamma)$ (see [8]). We need next to prove the invertibility in $E_\tau^{\beta,\Omega}(p_1,p_2,p_3,p_4)$. Let us introduce with $E_\tau^{\beta,\Omega}(p_1,p_2,p_3,p_4)$ a provisional new norm

$$\|\phi\|'_{E_\tau^{\beta,\Omega}} = 4\|\phi\|_{L_\infty(\Omega)} + \|\phi\|_{E_\tau^{\beta,\Omega}}$$

which is equivalent to the old one:

$$\|\phi\|_{E_\tau^{\beta,\Omega}} \leq \|\phi\|'_{E_\tau^{\beta,\Omega}} \leq 4\|\phi\|_{E_\tau^{\beta,\Omega}}.$$

Using (6) we find that

$$\|A_\Omega[\phi]\|' = 4\|A_\Omega[\phi]\|_{L_\infty(\Omega)} + \|A_\Omega[\phi]\|_{E_\tau^{\beta,\Omega}} \leq$$

$$\leq 4C_0\|\phi\|_{E_\tau^{\beta,\Omega}} + \max_{x\in\Omega}|A_\Omega[\phi]| + \sum_{\beta\leq m_0}\sup_{x\in\Gamma}\frac{|D^\beta A_\Omega[\phi(x)]|}{|x|^{-\beta}+|\tau-x|^{-\beta}} \leq$$

$$\leq (4C_0 + \tfrac{2}{\pi} + 1)\|\phi\|_{L_\infty(\Omega)} + \sum_{\beta\leq m_0}\sup_{x\in\Gamma}\frac{|D^\beta\phi(x)|}{|x|^{-\beta}+|\tau-x|^{-\beta}} \leq$$

$$\leq c_1\|\phi\|'_{E_\tau^{\beta,\Omega}},$$

where C_0 and c_1 are constants. A consequence of this is that $(\bar{a}I - A_\Omega)[\ \ ]$ is invertible in $L_\infty(\Omega)$ and $E_\tau^{\beta,\Omega}$. If $\phi_0 \in L_\infty(\Omega)$ is a solution to Eq. (1), we have to prove that $\phi_0 \in E_\tau^{\beta,\Gamma}$. Note that the restriction of ϕ_0 to Ω implies satisfaction of

$$S_\Omega[\phi(x)] = f_\Omega(x), \qquad x \in \Omega, \tag{7}$$

where $f_\Omega(x) = f(x) - S_{\Gamma\setminus\Omega}[\phi(x)]$, and $x \in \Omega$. An important relevant observation here is that $f_\Omega(x) \in E_\tau^{\beta,\Omega}$. On the other hand, we see that $S_{\Gamma\setminus\Omega}[\phi(x)]$ belongs in $E_\tau^{\beta,\Gamma}$ and

$$\|f_\Omega\|_{E_\tau^{\beta,\Omega}} \leq \|f\|_{E_\tau^{\beta,\Gamma}} + c\|f_0\|_{L_\infty(\Omega)},$$

where c = constant. Thus Eq. (7) is uniquely solvable in $E_\tau^{\beta,\Gamma}$. It is clear then that Eq. (7) is uniquely solvable in $L_\infty(\Gamma)$. Since the solution to Eq.(7) is the restriction to Ω of the solution of (1) and $E_\tau^{\beta,\Omega} \subset L_\infty(\Omega)$, then we conclude that $\phi \in E_\tau^{\beta,\Gamma}$. ■

3. The New Version of Euler-Maclaurin Summation Formula for Solving $C_i[\ ], i = 1,2$

Our goal in this section is to provide the best approximations to $C_2[\phi(x)]$ and $C_1[\phi(x)]$ for $\phi \in E_\tau^{\beta,\Gamma}(p_1,p_2,p_3,p_4)$. Hence, we start with the following modified proposition.

Proposition 3.1. *Let $\phi \in E_\tau^{\beta,\Gamma}(p_1, p_2, p_3, p_4)$ and $p = \max_{i=1}^4 p_i$ (such that $p_i \in N$, $i = 1,2,3,4$) then for*

$$\psi(x,c) = \begin{cases} \pi\phi(c)\cot(\pi(x-c)) & x-c \notin \mathbf{Z} \\ 0 & x-c \in \mathbf{Z} \end{cases} \tag{8}$$

and $t_\nu = \frac{\nu+1}{2}$, $t_\nu - mc \notin Z$ we have:

$$C_1[\phi(x)] = Q_{1,1}^{[m,\nu]}\phi(x) - \sum_{q=1}^{p-1} \frac{B_q(t_\nu)}{q!m^q}\left[\frac{d^{q-1}}{dy^{q-1}}\left(\frac{\phi(y)}{y-x}\right)_{y=1} - \frac{d^{q-1}}{dy^{q-1}}\left(\frac{\phi(y)}{y-x}\right)_{y=0}\right]$$

$$-\frac{1}{m^p}\int_0^1 \frac{d^p}{dy^p}\left(\frac{\phi(y)}{y-x} - \psi(y,x)\right)\left(\frac{B_p(t_\nu) - \overline{B_p(t_\nu - my)}}{p!}\right)dy \tag{9}$$

and

$$C_2[\phi(x)] = \frac{d}{dx}C_1[\phi(x)].$$

Here $\overline{B_p}$ denotes the corresponding Bernoulli function having a period (coinciding with B_p on $(0,1)$). Also, if $t_\nu = (\nu+1)/2$ for $-1 < \nu < 1$ and $t_\nu - mx$ not an integer, then we have

$$Q_{1,1}^{[m,\nu]}\phi(x) = \frac{1}{m}\sum_{j=0}^{m-1} \frac{\phi((j+t_\nu)/m)}{(j+t_\nu)/m - x} - \pi\phi(x)\cot\pi(t_\nu - mx).$$

If we put $\nu = 1$, we have

$$Q_{1,1}^{[m,1]}\phi(x) = \frac{1}{m-1}\sum_{j=0}^{m-1}{}'' \frac{\phi(j/m)}{j/m - x} - \pi\phi(x)\cot(\pi mx),$$

where $\sum''$ denotes the sum where the first and last terms are halved such that the following distinct points in $(0,1)$ are considered:

$$\begin{cases} v_i = \left(\frac{1}{2}\right)\left(\frac{i}{m}\right)^{r_0}, \\ v_{i+m} = 1 - v_{m-i}, \\ i = 0,1,\ldots,m \qquad n = 2m. \end{cases}$$

Also, r_0 is a constant and $r_0 > 1$.

Proof. As in [9], [10], [13] and [22] we may develop this proof. Consider a uniform partition $x_i = a + ih$, $i = 0,1,\ldots,m-1$ and $x_{m-1} = b$ of $[a,b]$, and recall the Euler-Maclaurin Summation:

$$Q_{1,1}^{[m,\nu]}\phi = I\phi + \sum_{q=1}^{p-1} \frac{B_q(a+t_\nu)}{q!}\frac{\phi^{(q-1)}(b) - \phi^{(q-1)}(a)}{m^q}(b-a) +$$

$$+\frac{b-a}{m^p}\int_a^b \phi^{(m)}(x)\frac{B_p(a+t_\nu) - B_p(t_\nu - mx + a)}{m!}dx, \tag{10}$$

where $I\phi = \int_a^b \phi(x)dx$, $B_s(x)$ is the Bernoulli polynomial of degree s and $Q_{1,1}^{[m,v]}$ is given by:

$$Q_{1,1}^{[m,v]}\phi = \frac{1}{m}\sum_{j=0}^{m-1}\phi(a + (j + t_v)/m), \tag{11}$$

with $t_v = \frac{v-1}{-2}a + \frac{v+1}{2}b$, $-1 < v < 1$. On another note, it is clear that if ϕ satisfies the properties: $\phi^{(q-1)}(0) = \phi^{(q-1)}(1) = 0$ for $q = 1,\ldots,p-1$, then

$$C_1[\phi(x)] = Q_{1,1}^{[m,v]}\phi(x) + O(\tfrac{1}{m^p}).$$

If x is close to an abscissa $\frac{(j+t_v)}{m}$, a minor perturbation in x can give a major perturbation to a corresponding term in the sum. However, the perturbation is precisely balanced by that in the term $\pi\phi(x)\cot(\pi(t_v - mx))$ so that $Q_{1,1}^{[m,v]}\phi(x)$ suffers only a minor perturbation.The same proof is also, this proof is true for $C_2[\phi(x)]$. Hence the proof follows for integral equation (1) from Proposition 2.1. ■

In order to have an integral with vanishing derivatives at the end points we introduce the idea of a Sigmoidal transformation. This idea has been investigated by many authors, see for example [11], [12], [21], [22] and [23].

Definition 3.1. A Sigmoidal transformation γ_r of order $r \geq 1$ is a real valued function with the following properties:
(i) $\gamma_r \in C^1[0,1] \cap C^\infty(0,1)$ with $\gamma_r(0) = 0$,
(ii) γ_r is strictly increasing on $[0,1]$,
(iii) $\gamma_r(x) + \gamma_r(1-x) = 1$ for all $x \in [0,1]$,
(iv) $\gamma_r^{(1)}$ is strictly increasing on $[0,1/2]$ with $\gamma_r^{(1)}(0) = 0$,
(v) $\gamma_r^{(j)}(x) = O(x^{r-j})$ near $x = 0$, for all $j \in \mathbf{N} \cup \{0\}$.

Remark 3.1. For $r > 1$, the graph of γ_r is S-shaped, hence the adjective "Sigmoidal" to describe the transformation γ_r. A particular example of such a transformation is given by

$$\gamma_r(x) = \frac{x^r}{x^r + (1-x)^r},$$

such that $x \in [0,1)$, and $r > 0$.

It can be is easily verified that conditions (i)-(iv) of Definition 3.1 are satisfied by γ_r. For this transformation we have $\gamma_r(\gamma_{1/r}) = x$ so as $\gamma_r^{-1} = \gamma_{1/r}$. For a brief discussion on other Sigmoidal transformations, we refer to [5], [6], [16], [17], [18], [22]. This is the starting point for this paper towards finding the best approximations to $C_1[\phi(x)]$ and $C_2[\phi(x)]$. To this end we formulate the following new proposition.

Proposition 3.2. *Let $\phi \in E_\tau^{\beta,\Gamma}(p_1,p_2,p_3,p_4)$, where p_i, $i = 1,2,3,4$ is defined as follows:*
i) $p_i = n$ if $B_n(t_v) \neq 0$
ii) $p_i = n+1$ if $B_n(t_v) = 0$,
and we assume that near $x = 0$ and $x = 1$ we respectively have $\phi(x) \sim c_0x^{n_0}$ and $\phi(x) \sim c_1(1-x)^{n_1}$, where n_0, n_1 are non-negative integers, c_0 , c_1 are non-zero and γ_r be a

Sigmoidal of order r, (where $2 \leq r \in N$ *), $n = min\{r(1+n_0), r(1+n_1)\}$. If* $c = \gamma_r(s)$, $0 < s < 1$ *and* $m \in N$, *then*

$$C_1[\phi(x)] = Q_{1,r}^{[m,v]}\phi(x) + O(1/m^{p_0}), \tag{12}$$

and

$$C_2[\phi(x)] = Q_{2,r}^{[m,v]}\phi(x) + O(1/m^{\overline{p_1}}), \tag{13}$$

where, provided that $t_v - ms$ *is not an integer,*

$$Q_{1,r}^{[m,v]}\phi(x) = \frac{1}{m}\sum_{j=0}^{m-1}\frac{\phi(\gamma_r(j+t_v)/m)\gamma_r^{(1)}(j+t_v)/m}{\gamma_r((j+t_v)/m)-x} - \pi\phi(x)\cot(\pi(t_v - ms)).$$

Also, $p_0 = p$ *if* $r \in Z^+$ *and otherwise* $p_0 = min\{r(1+n_0), r(1+n_1)\}$, *to write*

$$Q_{2,r}^{[m,v]}\phi(x) = \frac{1}{m}\sum_{j=0}^{m-1}\frac{\phi(\gamma_r(j+t_v)/m)\gamma_r^{(1)}(j+t_v)/m}{(\gamma_r((j+t_v)/m)-x)^2} + \frac{\phi(x)}{m\gamma_r^{(1)}(s)}\left(\frac{\gamma_r^{(3)}(s)}{6\gamma_r^{(1)}(s)} - \frac{1}{4}\left(\frac{\gamma_r^{(2)}(s)}{\gamma_r^{(1)}(s)}\right)^2 - \frac{m^2\pi^2}{3}\right) +$$

$$+\frac{\phi^{(1)}(x)\gamma_r^{(1)}(s)}{2m\gamma_r^{(1)}(s)} + \frac{\phi^{(2)}(x)\gamma_r^{(1)}(s)}{2m}.$$

If $0 < r \in Z$ *then* $\overline{p_1} = p$ *and for* $0 < r \notin Z$ *we have* $\overline{p_1} = min\{r(1+n_0), r(1+n_1)\}$.

Proof. Returning back to Proposition 3.1, Definition 3.1, and expanding [10], are essential ingredients of this proof. The order of the Sigmoidal transformation is r and it can be an integer or not. If we put $x = \gamma_r(\sigma)$ in (2), we may rewrite it as follows:

$$C_1[\phi(x)] = \lim_{\epsilon\to 0}\left(\int_0^{x-\epsilon} + \int_{x+\epsilon}^1\right)\frac{\phi(\gamma_r(\sigma))\gamma_r^{(1)}(\sigma)d\sigma}{\gamma_r(\sigma)-x}. \tag{14}$$

Letting $x = \gamma_r(s)$ where $s \in (0,1)$, allows rewriting (14) as

$$C_1[\phi(x)] = \lim_{\epsilon\to 0}\left(\int_0^{x-\epsilon} + \int_{x+\epsilon}^1\right)\frac{h_r(\sigma,s)}{\sigma - s}ds, \tag{15}$$

where

$$h_r(\sigma,s) = \begin{cases} \frac{\phi(\gamma_r(\sigma))\gamma_r^{(1)}(\sigma)}{\gamma_r(\sigma)-\gamma_r(s)} & \sigma \neq s \\ \phi(\gamma_r(s)) = \phi(s) & \sigma = s. \end{cases} \tag{16}$$

We find, after some algebra, following the works [20] and [22], the terms of $C_1[\phi(x)]$. It is well known, on one hand, that for some Sigmoidal transformations, all derivatives will be zero and we shall then have an exponential rate of convergence. On the other hand, the truncation error is given by the Euler-Maclaurin expansion which assumes a knowledge of the integrand and its derivatives at the end points 0 and 1. As we have observed, a suitably chosen Sigmoidal transformation of the variable of integration, would, in general, allow for the vanishing of an arbitrary number of these derivatives at the endpoints. Thus the convergence rate of the quadrature consequently improves. Also, following [10] and the combination theorems of [22] makes this approach analogous to the proof for (12). This is achieved by differentiating

$C_1[\phi(x)]$ partially with respect to x, thereby completing the proof for this proposition for computing $C_2[\phi(x)]$. ■

In the next section, we shall to utilize the above idea for solving Eq. (1) in a framework that is based on a collocation method.

4. Sigmoidal Collocation and Convergence Analysis

4.1. Convergence analysis

A natural and well -studied approach to the approximate solution ϕ of (1) is based on $\phi_n \in \Phi_n \subseteq E_\tau^{\beta,\Gamma}(p_1,p_2,p_3,p_4)$. More specifically, consider a sequence of elements ϕ_{ni} which is closed in the normed linear spaced $E_\tau^{\beta,\Gamma}(p_1,p_2,p_3,p_4)$ and let Φ_n denotes the span of $\{\phi_{n1},\ldots,\phi_{nn}\}$. We now look for an approximation $\phi_n \in \Phi_n$ of the form

$$\phi_n(x) = \sum_{i=1}^{n} \alpha_i \phi_{ni}(x), \tag{17}$$

where the coefficients α_i are unknown. Also, let $\phi_n(x) = \tau_n\phi(x)$ be a trigonometric polynomial of degree n defined as a linear combination of the following functions that are elements of

$$Æ = \{1,\cos x,\sin x,\cos^2 x,\cos x\sin x,\sin^2 x,\ldots,\cos^n x,\cos^{n-1}x\sin x,\ldots,\sin^n x\}$$

or, equivalently, a linear combination of the functions in the set

$$Œ = \{1,\cos x,\sin x,\cos 2x,\sin 2x,\ldots,\cos nx,\sin nx\}.$$

Moreover, we can write:

$$r_n(\alpha) = S[\phi_n] - f(x), \tag{18}$$

with the pertaining inequality,

$$\|\phi_n - \phi\|_{E_\tau^{\beta,\Omega}} \le \|S^{-1}\|_{E_\tau^{\beta,\Omega}} \, \|r_n\|_{E_\tau^{\beta,\Omega}}.$$

The above inequality requires information about properties of S^{-1} (a bounded inverse if S, which is difficult to observe). Alternatively, we prove the convergence analysis for this method via the proposition that follows.

Proposition 4.1. *Assumption in (1) that $\overline{a(x)}$, $b_1(x)$, $b_1(x)$, $f(x) \in E_\tau^{\beta,\Gamma}(p_1,p_2,p_3,p_4)$, $k(x,y) \in L_\rho^2$ and $p > q > \frac{1}{2}$, leads to*

$$|C_i[\phi] - Q_{i,r}^{m,\nu}\phi| \le |C_i[\phi] - Q^{m,\nu}\phi| \le \frac{2\sqrt{\zeta(2q)}\,\|\phi\|_{E_\tau^{q,\Gamma}}}{m^q}, \qquad m \ge 1, \qquad i = 1,2,$$

and

$$\lim_{n\to\infty} \|r_n(\alpha)\|_{E_\tau^{r,\Gamma}} = 0,$$

that is ϕ_n converges to ϕ. Also, if $\tau_n\phi$ is a trigonometric interpolation polynomial, then the following inequality

$$\|\phi - \tau_n\phi\|_{E_\tau^{r,\Gamma}} \le \frac{C}{n^{q-r}} \|\phi\|_{E_\tau^{q,\Gamma}}, \tag{19}$$

for $0 \leq r \leq q$ *and* $n > 1$*, holds. Here* ζ *denotes the Zeta function and C is constant.*

Proof. Consider Proposition 2.1 to conclude that $\phi \in H^q_\rho(\Gamma)$. Also, if we put $S[\phi] = f$, then we can write the following inequality:

$$\|r_n(\alpha)\|_{E^{r,\Gamma}_\tau} = \|S[\phi] - S[\phi_n]\|_{E^{r,\Gamma}_\tau} \leq \|\overline{a(x)}\|_{E^{r,\Gamma}_\tau} \frac{C}{n^{q-r}} \|\phi\|_{E^{q,\Gamma}_\tau} + \frac{\|b_1(x)\|_{E^{r,\Gamma}_\tau}}{\pi} \left(\frac{2\sqrt{\zeta(2q)}}{m^q} + O\left(\frac{1}{m^{p_0}}\right)\right) + $$
$$+ \frac{\|b_2(x)\|_{E^{r,\Gamma}_\tau}}{\pi} \left(\frac{2\sqrt{\zeta(2q)}}{m^q} + O\left(\frac{1}{m^{\overline{p_1}}}\right)\right) + \|k_0(x,y)\|_{E^{q,\Gamma}_\tau} \frac{C}{n^{q-r}} \|\phi\|_{E^{q,\Gamma}_\tau} \leq O(n^{-\bar{k}}),$$

where C is constant. Clearly as $n \to \infty$, $\|r_n(\alpha)\| \to 0$. Moreover, applying the Cauchy - Schwartz inequality to the Fourier series together with (3.5), (3.6) (by using [8], [9]), ends up with the above inequalities. Therefore, we can show that the *a priori* error estimation is of order $O(n^{-\bar{k}})$, such that $\bar{k} = max\{q - r, p_0, \overline{p_1}, q\}$. This is the optimal rate of convergence of this method for solving this equation. ■

4.2. **Sigmoidal collocation**

In a Sigmoidal collocation method for (1) we write the following equation:

$$[\xi_n(\overline{a(x)}) + \frac{1}{\pi}\xi_n(b_1(x)C_1[\ \ ]) + \frac{1}{\pi}\xi_n(b_2(x)C_2[\ \ ]) + \xi_n\chi[\ \ ]]\phi_n = \xi_n f \tag{20}$$

where $\chi[\phi] = \int_\Gamma k_0(x,t)\phi(t)dt$ and ξ_n is a bounded projection operator. We can moreover consider the following bound:

$$\|\phi_n\|_\infty \leq \|(\xi_n(\overline{a(x)}) + \frac{1}{\pi}\xi_n(b_1(x)C_1[\ \ ] + \frac{1}{\pi}\xi_n(b_2(x)C_2[\ \ ]) + \xi_n\chi[\ \ ])^{-1}\| \ \|\xi_n\| \ \|f\|_\infty. \tag{21}$$

The collocation method (see [2], [3], [13]) and the relations mentioned above lead to the system:

$$\mathbf{A_n}\alpha = b, \tag{22}$$

where $\mathbf{A_n} = (a_{ij})$, $b = f(x_i)$ are known and $\alpha = (\alpha_j)$ is the unknown vector. Observe that $\forall i,j \in \{1, 2, \ldots, n\}$,

$$a_{ij} = S[\phi_{nj}(x_i)] = \overline{a(x_i)}\phi_{nj}(x_i) + \frac{b_1(x_i)}{\pi}\left(Q^{[m,v]}_{1,r}\phi_{nj}(x_i) + O\left(\frac{1}{m^{p_0}}\right)\right) +$$
$$+ \frac{b_2(x_i)}{\pi}\left(Q^{[m,v]}_{2,r}\phi_{nj}(x_i) + O\left(\frac{1}{m^{\overline{p_1}}}\right)\right) + \int_0^1 k_0(x_i,t)\phi_{nj}(t)dt.$$

In the next section we use the system (22) and investigate the bound of Cond$(\mathbf{A_n}) = \|\mathbf{A_n}\| \ \|\mathbf{A_n^{-1}}\|$. Let us focus here on measuring the size of $\|\mathbf{A_n^{-1}}\|$. It is usually straightforward to find a bound for $\|\mathbf{A_n}\|$, and it seldom becomes large. The solution α of the linear system is related to ϕ_n at the node points by

$$\phi_n(x_j) = \sum_{i=1}^n \alpha_i\phi_{ni}(x_j), \qquad j = 1, \ldots, n.$$

In matrix form, we have

$$\Gamma_n\alpha = \underline{\phi_n}, \tag{23}$$

where $\underline{\phi_n} = [\phi_{ni}(x_1), \ldots, \phi_{ni}(x_n)]$ and $\Gamma_n = [\phi_{ni}(x_j)]^n_{i,j=1}$ such that $det(\Gamma_n) \neq 0$. Then from (21) and (23) we have:

$$\|\alpha = \mathbf{A_n^{-1}}b\|_\infty \leq \|\Gamma_n^{-1}\| \ \ \|(\xi_n(\overline{a(x)}) + \frac{1}{\pi}\xi_n(b_1(x)C_1[\ \ ]) + \frac{1}{\pi}\xi_n(b_2(x)C_2[\ \ ])$$
$$+ \xi_n\chi[\ \ ])^{-1}\| \ \ \|\xi_n\| \ \ \|b\|_\infty. \tag{24}$$

Also

$$\|\mathbf{A_n^{-1}}\| \leq \|\Gamma_n^{-1}\| \ \ \|(\xi_n(\overline{a(x)}) + \frac{1}{\pi}\xi_n(b_1(x)C_1[\ \ ]) + \frac{1}{\pi}\xi_n(b_2(x)C_2[\ \ ])$$
$$+ \xi_n\chi[\ \ ])^{-1}\| \ \ \|\xi_n\|. \tag{25}$$

The term $\|(\xi_n(\overline{a(x)}) + \frac{1}{\pi}\xi_n(b_1(x)C_1[\ \ ]) + \frac{1}{\pi}\xi_n(b_2(x)C_2[\ \ ]) + \xi_n\chi[\ \ ])^{-1}\|$ is approximately $\|(\overline{a(x)} + \frac{1}{\pi}b_1(x)C_1[\ \ ] + \frac{1}{\pi}b_2(x)C_2[\ \ ] + \chi[\ \ ])^{-1}\|$ for large n, and we do not consider it any further. The terms $\|\Gamma_n^{-1}\|$ and $\|\xi_n\|$ are both important, and must be examined for each collocation method.

Therefore, the previous results for the Sigmoidal collocation method carry across to linear systems viz

i-For piecewise linear interpolation, we can conclude that $\|\xi_n\| = 1$, $\|\Gamma_n\| \leq 1$ and $\|\mathbf{A_n}\| \leq O(n)$ therefore,

$$Cond(\mathbf{A_n}) = O(n) \ \ \|(\xi_n(\overline{a(x)}) + \frac{1}{\pi}\xi_n(b_1(x)C_1[\ \ ]) + \frac{1}{\pi}\xi_n(b_2(x)C_2[\ \ ]) + \xi_n\chi[\ \ ])^{-1}\|.$$

ii-For trigonometric interpolation, we can conclude that $\|\xi_n\| = O(\log n)$, $\|\Gamma_n\| \leq 1$ and $\|\mathbf{A_n}\| = O(n)$ therefore,

$$Cond(\mathbf{A_n}) = O(n\log n) \ \ \|(\xi_n(\overline{a(x)}) + \frac{1}{\pi}\xi_n(b_1(x)C_1[\ \ ]) + \frac{1}{\pi}\xi_n(b_2(x)C_2[\ \ ]) + \xi_n\chi[\ \ ])^{-1}\|.$$

With other forms of piecewise polynomial interpolation of higher degree, the results will be much the same. However, if we choose monomials as our basis the then Γ_n is a Vandermonde matrix and the matrix $\mathbf{A_n}$ is likely to be ill-conditioned.

5. Numerical Examples

Here we report on three numerical examples for testing the convergence rates of this method. For the sake of simplicity, we consider in Eq.(1), $\Gamma = (0, 1)$. An AMD Opteron computer, with 15 Gigabytes RAM memory with 2.2 GHz CPU, has been used for these experiments. We have used *Æ* and *Œ* in examples 5.2 and 5.3, respectively. Solution of (22) is made with CG and PCG algorithms. In all these examples, the zero vector is used for an initial guess. The stopping criterion is $\|\overline{r_k}\|/\|\overline{r_0}\| < 10^{-7}$, where $\overline{r_k}$ is the residual vector of CG and PCG methods after k iterations. Moreover, we apply base functions over nonuniform spatial knots. The distinct points in $(0, 1)$ are introduced as follows:

$$\begin{cases} x_i = (\frac{1}{2})(\frac{i}{m})^{r_0}, \\ x_{i+m} = 1 - x_{m-i}, \\ i = 0, 1, \ldots, m \end{cases} \tag{26}$$

where $n = 2m$ and $r_0 > 1$. The figures to follow contain graphs of the logarithms of the error= $\|\phi - \phi_n\|_\infty$ as r increases.

Example 5.1. We assume that $\bar{a}(x) = b_1(x) = b_2(x) = 1$ and $k_0(x,t) = \frac{1}{(x-t)^{\frac{1}{2}}}$. If $r_0 = 3$, $\phi(x) = \sin x$ to yield $f(x)$. Using B-spline of order one as base functions in (17) over nonuniform spatial knots for (n,r) we obtain the results exhibited in Tables 5.1 and 5.2.

Table 5.1. Results for Example 5.1 for $r_0 = 3$, without preconditioner

n	r	$\|\phi-\phi_n\|_\infty$	r	$\|\phi-\phi_n\|_\infty$	r	$\|\phi-\phi_n\|_\infty$	r	$\|\phi-\phi_n\|_\infty$
16	3	$0.134e-2$	5	$0.966e-3$	7	$0.034e-2$	9	$0.856e-2$
26	3	0.115e-2	5	0.153e-3	7	0.725e-4	9	0.113e-4
30	3	0.543e-5	5	0.838e-7	7	0.541e-8	9	0.928e-9

Table 5.2. Results for Example 5.1 for $r_0 = 3$, with preconditioner

n	r	$\|\phi-\phi_n\|_\infty$	r	$\|\phi-\phi_n\|_\infty$	r	$\|\phi-\phi_n\|_\infty$	r	$\|\phi-\phi_n\|_\infty$
16	3	$0.144e-3$	5	$0.342e-4$	7	$0.387e-3$	9	$0.546e-3$
26	3	0.025e-2	5	0.763e-4	7	0.315e-4	9	0.543e-5
30	3	0.721e-6	5	0.437e-7	7	0.005e-8	9	0.0348e-9

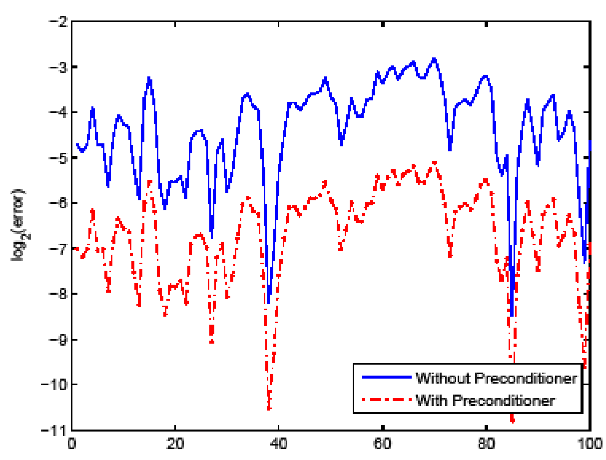

Figure 1. Behavior of logarithm of error $\|\phi-\phi_n\|_\infty$ for $r_0 = 9$ and $n = 30$ in Example 5.1.

Example 5.2. Using trigonometric functions as base functions for various (n,r) we obtain the results shown in Tables 5.3 and 5.4, under all the all assumptions of Example 5.1.

Table 5.3. Results for Example 5.2 for $r_0 = 3$, without preconditioner

n	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$
16	3	0.254e-3	5	0.243e-4	7	0.132e-4	9	0.521e-5
26	3	0.113e-3	5	0.124e-5	7	0.515e-5	9	0.613e-6
30	3	0.501e-5	5	0.698e-7	7	0.143e-8	9	0.908e-7

Table 5.4. Results for Example 5.2 for $r_0 = 3$, with preconditioner

n	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$
16	3	0.983e-4	5	0.652e-5	7	0.934e-5	9	0.134e-5
26	3	0.432e-4	5	0.543e-6	7	0.110e-5	9	0.942e-7
30	3	0.104e-5	5	0.112e-7	7	0.874e-9	9	0.128e-7

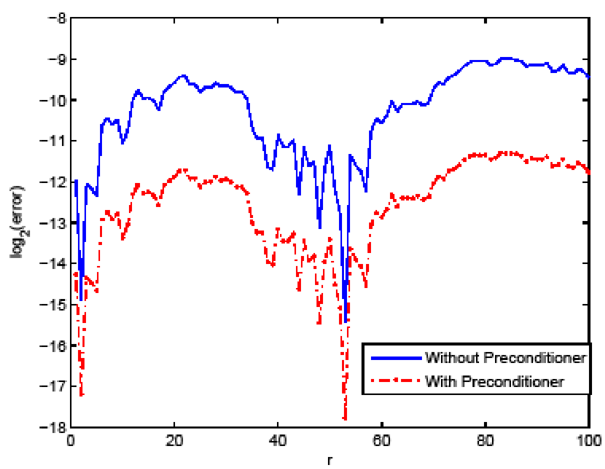

Figure 2. Behavior of logarithm of error $\|\phi - \phi_n\|_\infty$ for $r_0 = 9$ and $n = 30$ in Example 5.2.

Example 5.3. Assume that $\overline{a(x)} = b_1(x) = b_2(x) = x$ and $k_0(x,t) = \log(x-t)$. If $r_0 = 3$, $\phi(x) = \sqrt{x}$ then define $f(x)$. Using trigonometric functions as base functions for various (n, r) we obtain the results shown in Tables 5.5 and 5.6.

Table 5.5. Results for Example 5.3 for $r_0 = 3$, without preconditioner

n	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$
16	3	0.451e-3	5	0.641e-4	7	0.732e-4	9	0.421e-4
26	3	0.123e-3	5	0.174e-5	7	0.455e-5	9	0.013e-4
30	3	0.651e-4	5	0.257e-6	7	0.523e-6	9	0.208e-5

Table 5.6. Results for Example 5.3 for $r_0 = 3$, with preconditioner

n	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$	r	$\|\phi - \phi_n\|_\infty$
16	3	0.114e-3	5	0.241e-4	7	0.892e-5	9	0.851e-5
26	3	0.533e-4	5	0.011e-5	7	0.213e-5	9	0.765e-6
30	3	0.953e-5	5	0.997e-7	7	0.123e-6	9	0.127e-6

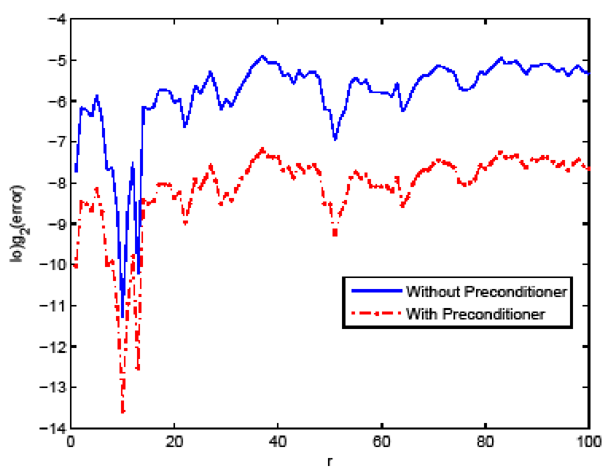

Figure 3. Behavior of logarithm of error $\|\phi - \phi_n\|_\infty$ for $r_0 = 9$ and $n = 30$ in Example 5.3.

Remark 5.1. In the all examples for solving the linear system (20), a preconditioned conjugate gradient method is used.

The two Examples 5.1 and 5.2 are meant to demonstrate the proposed method. However, they seem insufficient because they are given with just constant coefficients and the solution is always a sine. Example 5.3 is a suppliment to these examples and a pointer towards the theoretical machinery for examples with variable coefficients.

6. Concluding Remarks

Based on the use of Sigmoidal transformations and the collocation method we have reported on a simple algorith for solving the GAE. Of course, this method is not formulated without inspiration from other works in this field. Since the accuracy of the computed solution is strongly tied to the used base functions, r, n and r_0, a thorough further investigation of the role of these parameters on accuracy is a wortwhile project for any future work on this subject.

Acknowledgements
Professor Mohmmad Asadzadeh of Chalmers University found many typographical and other errors in this paper The author appreciates this help and apologies to numerous authors of works related to this subject who are not mentioned in the reference list. He is also grateful to an anonymous referee for his critical reading of the original typescript.

References

[1] R. Duduchave, D. Elliott, and W. L. Wendland, *The Spline Collocation Method for Mellin Convolution Integral Equations*, Universitat Stuttgart, Sonderforschungsbereich 404, Bericht, 1996.

[2] D. Elliott, and S. Prissdorf, An algorithm for the approximate solution of integral equations of Mellin type, *Numerical Mathematics* **70**, (1995), 427-452.

[3] A. Erdelyi, Ed. *Tables of Integral Transforms* (Bateman Manuscript Project), McGraw-Hill, New York, 1954.

[4] L. S. Gradshteyn, and I. M. Ryzhik, *Table of Integrals, Series and Products*, 5th Edition, Edited by A. Jeffrey, Ed., Academic Press, Boston, 1994.

[5] M. Iri, S. Moriguti, and Y. Takasawa, On a certain quadrature formula, *Journal of Computational and Applied Mathematics* **17**, (1995), 3-20.

[6] I. Jeon, A Nyström method for boundary integral equations on domains with a piecewise smooth boundary, *Integral Equations and Applications* **5**, (1993), 221-242.

[7] N. M. Korobev, *Numerical Theoretic Methods of Approximate Analysis*, GIFL, Moscow, 1963.

[8] R. Kress, *Linear Integral Equations*, Springer Verlag, Berlin, 1989.

[9] D. P. Laurie, Periodical transformations for numerical integration, *Journal of Computational and Applied Mathematics* **66**, (1996), 337-344.

[10] J. Lyness, The Euler Machaurin expansion for the Cauchy principal value integral,

Numerical Mathematics **45**, (1985), 611-622.

[11] G. Monegato, and J. N. Lyness, The Euler-Maclaurin expansion and finite part integrals, *Numerical Mathematics* **81**, (1998), 273-291.

[12] G. Monegato, and L. Scuderi, High order methods for weakly singular integral equations with non-smooth input functions, *Numerical Mathematics* **67**, (1998), 1493-1515.

[13] M. Mori, An IMT-type double exponential formula for numerical integration, *Publications of RIMS, Kyoto University* **14**, (1976), 713-729.

[14] F. E. J. Olver, *Asymptotic and Special Functions*, Academic press, New York and London,1974.

[15] S. Prossdorf, and A. Rathsfeld, On an integral equation of the first kind arising from a cruciform crack problem, in: *Integral Equations and Inverse Problems*,V. Pethov, and R. Lazarov (Eds), Longman, Coventry, 1991.

[16] T.W. Sag, and G. Szekeres, Numerical evaluation of high - dimensional integrals, *Mathematics of Computation* **18**, (1946), 245-253.

[17] A. Sidi, A new variable transformation for numerical integration, in: *Numerical Integration TV*, H. Brass, and G. Hammerlin (Eds), ISNM **112**, (1993), 359-373.

[18] B. Bertraml, and O. Ruchr, Product integration for finite-part singular integral equations: Numerical asymptotic and convergence acceleration, *Journal of Computational and Applied Mathematics* **71**, (1992), 163-176.

[19] N. Pukhteer, *Exact Methods for Calculating Cauchy Type Integrals*, Nauka, Novosibirsk, 1980.

[20] B. I. Yun, Ageneralized non-linear transformation for evaluating singular integrals, *International Journal of Numerical Methods in Engineering* **65**, (2006), 1947-1969.

[21] B. I. Yun, and P. Kim, A new Sigmoidal transformation for weakly singular integrals in the boundary element method, *SIAM Journal on Computing* **24**, (2003), 1203-1217.

[22] D. Elliott, and E. Venturino, Sigmoidal transformations and Euler-Maclaurin expansion for evaluating certain Hadamard finite-part integrals, Numerical Mathematics **77**, (1997), 453-465.

[23] D. Rostamy, M. Rostamy, and S. Khaloghli, Improvement of the Michell method based on Sigmoidal transformation for solving waves, in Press.

[24] R. P. Srivastav, and E. Jen, Numerical solution of singular integral equations using Gauss type formulate-I. Quadrature and collocation on Chebyshev nodes, *IMA Journal of Numerical*

Analysis **3**, (1983), 305-318.

[25] R. P. Srivastav, and E. Jen, Numerical solution of singular integral equations using Gauss type formulate-II. Quadrature and collocation on Chebyshev nodes, *IMA Journal of Numerical Analysis* **3**, (1983), 319-325.

Article history: Submitted April 23, 2011 ; Accepted June, 18, 2011.