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Backward Doubly Stochastic Differential Equations Driven by Levy Process : The Case of Non-Liphschitz Coefficients

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Abstract. *In this work we deal with a Backward doubly stochastic differential equation (BDSDE) associated to a random Poisson measure. We establish existence and uniqueness of the solution in the case of non-Lipschitz coefficients.*

Key words : Backward Doubly Stochastic Differential Equations, Random Poisson Measure, Itô's Representation Formula, Gronwall Lemma.

AMS Subject Classifications : 60H05, 60G44

1. Introduction

It is well known that Backward stochastic differential equations (BSDEs in short) provides a stochastic representation of solutions of semilinear Partial differential equations (PDEs). As far as we know, in these works, Lipschitz or at least a monotonicity condition is required on the drift of the BSDEs. Recently several authors investigate sucessfully in weakening these conditions (see among others [1],[5]). The assumption usually satisfied by the drift is replaced by a rather smooth one which ensures existence and uniqueness result. Inspired by the method developped in [5], Sow [4], extended Wang and Huang's result to BSDE with jumps and proved a large deviation principle of such family of equations.

Backward doubly stochastic differential equations (BDSDE in short) appears as a natural extension of backward stochastic differential equations (BSDE). Their link with stochastic partial differential equations (SPDEs) in the case of Lipchitzian drift was established by Pardoux and Peng [3]. In this paper, our aim is to generalize the result established in [4] to BDSDE driven by a Lévy process. The two independent Brownian motions are coupled with a independent Poisson random measure given in [4]. We study first solvability of our equation in the case of Lipschtzian coefficients. Using this result and an efficient iterative procedure, we prove existence and uniqueness of solution with coefficients satisfying rather weaker conditions.

2. Backward Doubly SDE and Poisson Random Measure

2.1. Deinitions and notations

Let Ω be a non-empty set, $\mathcal{F}$ a σ –algebra of sets of Ω and $\mathbf{P}$ a probability measure defined on $\mathcal{F}$. The triplet $(\Omega,\mathcal{F},\mathbf{P})$ defines a probability space, which is assumed to be complete. For a fix real $T>0$, we assume given three mutually independent processes :

- a ℓ –dimensional Brownian motion $(B_t)_{0\leq t\leq T}$,
- a d –dimensional Brownian motion $(W_t)_{0\leq t\leq T}$,
- a random Poisson measure μ on $E\times\mathbf{R}_+$.

The space $E=\mathbf{R}^\ell-\{0\}$ is equipped with its Borel field $\mathfrak{E}$ with compensator $\nu(dt,de)=\lambda(de)dt$ such that $\{\widetilde{\mu}([0,t]\times A)=(\mu-\nu)[0,t]\times A\}$ is a martingale for any $A\in\mathfrak{E}$ satisfying $\lambda(A)<\infty$. λ is a σ –finite measure on $\mathfrak{E}$ and satisfies

$$\int_E(1\wedge|e|^2)\lambda(de)<\infty.$$

We consider the family $(\mathcal{F}_t)_{0\leq t\leq T}$ given by

$$\mathcal{F}_t=\mathcal{F}_t^W\vee\mathcal{F}_{t,T}^B\vee\mathcal{F}_t^\mu,\quad 0\leq t\leq T,$$

where for any process $\{\eta_t\}_{t\geq0}$, $\mathcal{F}_{s,t}^\eta=\sigma\{\eta_r-\eta_s,\ s\leq r\leq t\}\vee\mathcal{N}$, $\mathcal{F}_t^\eta=\mathcal{F}_{0,t}^\eta$. $\mathcal{N}$ denotes the class of $\mathbf{P}$ –null sets of $\mathcal{F}$. Note that $(\mathcal{F}_t)_{0\leq t\leq T}$ does not constitute a classical filtration.
Let $g:\Omega\times[0,T]\times\mathbf{R}^k\times\mathbf{R}^{k\times d}\times\mathbf{R}^k\to\mathbf{R}^{k\times\ell}$ and $f:\Omega\times[0,T]\times\mathbf{R}^k\times\mathbf{R}^{k\times d}\times\mathbf{R}^k\to\mathbf{R}^k$ be jointly measurable. Given ξ a $\mathcal{F}_T$ –measurable $\mathbf{R}^k$ valued random variable, we are interested in the backward doubly stochastic differential equation with random Poisson measure (BDSDEP in short)

$$Y_t=\xi+\int_t^T f(r,\Theta_r)dr+\int_t^T g(r,\Theta_r)dB_r-\int_t^T Z_r dW_r-\int_t^T\int_E U_r(e)\widetilde{\mu}(dr,de),\ 0\leq t\leq T,\qquad(1)$$

where $\Theta_r=(Y_r,Z_r,U_r)$.
For $Q\in\mathbf{N}^*$, $|\ .\ |$ and $\langle\cdot\rangle$ stand for the euclidian norm and the inner product in $\mathbf{R}^Q$.
We consider the following sets (where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}$):

- $S^2_{[0,T]}(\mathbf{R}^Q)$ the space of $\mathcal{F}_t$ –adapted càdlàg processes

$$\Psi:[0,T]\times\Omega\longrightarrow\mathbf{R}^Q,\ \|\Psi\|_2^2=\mathbf{E}\left(\sup_{0\leq t\leq T}|\Psi_t|^2\right)<\infty.$$

- $H^2_{[0,T]}(\mathbf{R}^Q)$ the space of $\mathcal{F}_t$ –progressively measurable processes

$$\Psi:[0,T]\times\Omega\longrightarrow\mathbf{R}^Q,\ \|\Psi\|^2=\mathbf{E}\int_0^T|\Psi_t|^2\,dt<\infty.$$

- $L^2_{[0,T]}(\widetilde{\mu},\mathbf{R}^Q)$ the space of mappings $U:\Omega\times[0,T]\times E\longrightarrow\mathbf{R}^Q$ which are $\mathcal{P}\otimes\mathfrak{E}$-measurable s.t. $\|U\|^2 = \mathbf{E}\int_0^T\int_E |U_t(e)|^2\,\lambda(de)dt < \infty$, where $\mathcal{P}\otimes\mathfrak{E}$ denotes the σ –algebra of predictable sets of $\Omega\times[0,T]$.

Notice that the space $\mathcal{B}^2_{[0,T]}(\mathbf{R}^Q) = S^2_{[0,T]}(\mathbf{R}^Q)\times H^2_{[0,T]}(\mathbf{R}^Q)\times L^2_{[0,T]}(\widetilde{\mu},\mathbf{R}^Q)$ is a Banach space.

Definition 2.1. A triplet of processes $(Y_t,Z_t,U_t)_{0\le t\le T}$ is called a solution to eq. (1), if $(Y_t,Z_t,U_t)\in B^2_{[0,T]}(R^k)$ and satisfies (1).

2.2. **The case of Lipschitz coefficients**

We say that the coefficients f and g satisfy assumption (**H1**) if the following hold:

(**H1.1**) : For all $(y,z,u)\in\mathbf{R}^k\times\mathbf{R}^{k\times d}\times\mathbf{R}^k$, $g(\bullet,y,z,u)\in H^2_{[0,T]}(\mathbf{R}^k)$ and $f(\bullet,y,z,u)\in H^2_{[0,T]}(\mathbf{R}^k)$.

(**H1.2**) : There exists two constants $c>0$ and $0<\alpha<1$ s. t. for $0\le t\le T$, $(y,y')\in(\mathbf{R}^k)^2$, $(z,z')\in(\mathbf{R}^{k\times d})^2$, $(u,u')\in(\mathbf{R}^k)^2$,

$$|f(t,y,z,u)-f(t,y',z',u')|^2 \le c(|y-y'|^2+|z-z'|^2+|u-u'|^2),$$

$$|g(t,y,z,u)-g(t,y',z',u')|^2 \le c|y-y'|^2+\alpha(|z-z'|^2+|u-u'|^2).$$

We may then state a result that follows.

Theorem 2.1. *Assume that* $(H1)$ *is in force. Then equation (1) admits a unique solution* $(Y,Z,U)\in B^2_{[0,T]}(R^k)$.

Before proving this theorem, let us establish the corresponding result in case f and g do not depend on Y,Z and U. So we consider the BDSDEP

$$Y_t = \xi+\int_t^T f(r)dr+\int_t^T g(r)dB_r-\int_t^T Z_r dW_r-\int_t^T\int_E U_r(e)\widetilde{\mu}(dr,de),\quad 0\le t\le T, \tag{2}$$

to introduce a proposition.

Proposition 2.1. *There exists a unique triplet* $(Y,Z,U)\in B^2_{[0,T]}(R^k)$ *which solves equation (2)*.

Proof. (Uniqueness) Let (Y,Z,U) and $(\widetilde{Y},\widetilde{Z},\widetilde{U})$ be two solutions of (2). By denoting $\bar{Y}=Y-\widetilde{Y}$, $\bar{Z}=Z-\widetilde{Z}$, $\bar{U}=U-\widetilde{U}$, we deduce from (2) that

$$\bar{Y}_t+\int_t^T\bar{Z}_r\,dW_r+\int_t^T\int_E\bar{U}_r(e)\,\widetilde{\mu}(dr,de)=0,\quad 0\le t\le T.$$

Itô’s formula yields

$$\mathbf{E}|\bar{Y}_t|^2+\xi+\int_0^T f(r)dr+\int_0^T g(r)dB_r\int_t^T|\bar{Z}_r|^2dr+\mathbf{E}\int_t^T\int_E|\bar{U}_r(e)|^2\lambda(de)dr=0,\quad 0\le t\le T,$$

which leads to the required result.

(Existence) Let us consider the filtration $(\mathcal{G}_t)_{0\le t\le T}$ given by

$$\mathcal{G}_t=\mathcal{F}^W_t\vee\mathcal{F}^B_T\vee\mathcal{F}^\mu_t,\quad 0\le t\le T,$$

and the $\mathcal{G}_t$ –square integrable martingale

$$M_t = \mathbf{E}^{\mathcal{G}_t}\Big[\xi + \int_0^T f(r)dr + \int_0^T g(r)dB_r\Big].$$

An extension of Itô's representation theorem proves the existence of $\mathcal{G}_t$ –progressively measurable pair of processes (Z,U) with values in $H^2_{[0,T]}(\mathbf{R}^k) \times L^2_{[0,T]}(\widetilde{\mu},\mathbf{R}^k)$ such that

$$M_t = M_0 + \int_0^t Z_r\, dW_r - \int_0^t \int_E U_r(e)\widetilde{\mu}(dr,de), \quad 0 \le t \le T.$$

This implies

$$M_T = M_t + \int_t^T Z_r\, dW_r - \int_t^T \int_E U_r(e)\widetilde{\mu}(dr,de), \quad 0 \le t \le T.$$

Hence putting

$$Y_t = \mathbf{E}^{\mathcal{G}_t}\Big[\xi + \int_t^T f(r)dr + \int_t^T g(r)dB_r\Big],$$

we deduce that the triplet (Y,Z,U) solves (2). ■

Next, let us establish the following result which will be useful in the sequel.

Lemma 2.1. *Let* $X \in S^2_{[0,T]}(R^k)$, $\vartheta \in H^2_{[0,T]}(R^k)$, $\zeta \in H^2_{[0,T]}(R^k)$, $\pi \in H^2_{[0,T]}(R^k)$ *and* $\phi \in L^2_{[0,T]}(\widetilde{\mu},R^k)$ *be such that*

$$X_t = X_0 + \int_0^t \vartheta_r dr + \int_0^t \zeta_r dB_r + \int_0^t \pi_r dW_r + \int_0^t \int_E \phi_r(e)\widetilde{\mu}(dr,de), \quad 0 \le t \le T.$$

Then for any $0 \le t \le T$,

$$\begin{aligned} |X_t|^2 &= |X_0|^2 + 2\int_0^t \langle X_r,\vartheta_r\rangle dr + 2\int_0^t \langle X_r,\zeta_r dB_r\rangle + 2\int_0^t \langle X_r,\pi_r dW_r\rangle \\ &\quad + 2\int_0^t\int_E \langle X_r,\phi_r(e)\widetilde{\mu}(dr,de)\rangle - \int_0^t |\zeta_r|^2 dr + \int_0^t |\pi_r|^2 dr + \int_0^t\int_E |\phi_r(e)|^2\lambda(de)dr, \end{aligned}$$

$$\mathbf{E}|X_t|^2 = \mathbf{E}|X_0|^2 + 2\mathbf{E}\int_0^t \langle X_r,\vartheta_r\rangle dr - \mathbf{E}\int_0^t |\zeta_r|^2 dr + \mathbf{E}\int_0^t |\pi_r|^2 dr + \mathbf{E}\int_0^t\int_E |\phi_r(e)|^2\lambda(de)dr.$$

Proof. The Proof is omitted since it is a straightforward adaptation of [3, Lemma 1.3]. ■

Proof of Theorem 2.1. Let (Y,Z,U) and $(\widetilde{Y},\widetilde{Z},\widetilde{U})$ be two solutions of (1) and for $0 \le s \le T$,

$$\Delta f(s) = f(s,Y_s,Z_s,U_s) - f(s,\widetilde{Y}_s,\widetilde{Z}_s,\widetilde{U}_s), \;\; \Delta g(s) = g(s,Y_s,Z_s,U_s) - g(s,\widetilde{Y}_s,\widetilde{Z}_s,\widetilde{U}_s). \tag{3}$$

The triplet $(\overline{Y},\overline{Z},\overline{U})$ (defined in the proof of Proposition 2.1) solves the BDSDEP

$$\overline{Y}_t = \int_t^T \Delta f(r)dr + \int_t^T \Delta g(r)dB_r - \int_t^T \overline{Z}_r\, dW_r - \int_t^T\int_E \overline{U}_r(e)\,\widetilde{\mu}(dr,de), \;\; 0 \le t \le T.$$

Applying Lemma 2.1, we deduce that

$$\mathbf{E}|\overline{Y}_t|^2 + \mathbf{E}\int_t^T |\overline{Z}_r|^2 dr + \mathbf{E}\int_t^T\int_E |\overline{U}_r(e)|^2\lambda(de)dr = 2\mathbf{E}\int_t^T \langle \overline{Y}_r,\Delta f(r)\rangle dr + \mathbf{E}\int_t^T |\Delta g(r)|^2 dr.$$

Using assumptions (H1), we have

$$|\Delta g(r)|^2 \le c|\overline{Y}_r|^2 + \alpha[|\overline{Z}_r|^2 + |\overline{U}_r|^2],$$

$$\langle \overline{Y}_r,\Delta f(r)\rangle \;\le\; \sqrt{c}\,|\overline{Y}_r|^2 + \tfrac{c}{1-\alpha}|\overline{Y}_r|^2 + \tfrac{1-\alpha}{2}[|\overline{Z}_r|^2 + |\overline{U}_r|^2].$$

Hence putting pieces together, there exists a positive constant $K(c,\alpha)$ depending on c and α

such that

$$\mathbf{E}|\overline{Y}_t|^2 + \frac{1-\alpha}{2}\mathbf{E}\int_t^T|\overline{Z}_r|^2dr + \frac{1-\alpha}{2}\mathbf{E}\int_t^T\int_E|\overline{U}_r(e)|^2\lambda(de)dr \le K(c,\alpha)\mathbf{E}\int_t^T|\overline{Y}_r|^2dr.$$

It remains to exploit Gronwall's lemma to deduce that $\overline{Y} = 0$. This implies $\overline{Z} = 0$ and $\overline{U} = 0$. Uniqueness follows.

To prove existence of solutions, we consider the sequence $\Theta^n = (Y^n, Z^n, U^n)_{n\in\mathbf{N}}$ defined by

$$\begin{cases} Y^0 = 0, \quad Z^0 = 0, \quad U^0 = 0, \\ Y_t^{n+1} = \xi + \int_t^T f(r,\Theta_r^n)dr + \int_t^T g(r,\Theta_r^n)dB_r - \int_t^T Z_r^{n+1}dW_r - \int_t^T\int_E U_r^{n+1}(e)\widetilde{\mu}(dr,de). \end{cases} \tag{4}$$

Since for a fix $n \in \mathbf{N}$, the coefficients f and g of the BDSDEP (4) do not depend on the solution $(Y^{n+1}, Z^{n+1}, U^{n+1})$, it follows from Proposition 2.1 that the sequence $(\Theta^n)_{n\in\mathbf{N}}$ is well defined. Our strategy consists in proving that $(\Theta^n)_{n\in\mathbf{N}}$ is a Cauchy sequence in $\mathcal{B}^2_{[0,T]}(\mathbf{R}^k)$. For this end, let us define

$$\overline{Y}_t^{n+1} = Y_t^{n+1} - Y_t^n, \quad \overline{Z}_t^{n+1} = Z_t^{n+1} - Z_t^n \quad \text{and} \quad \overline{U}_t^{n+1} = U_t^{n+1} - U_t^n, \quad 0 \le t \le T.$$

Fix $\beta \in \mathbf{R}$. The same computations as in the proof of uniqueness combining by an integration by parts yield

$$\mathbf{E}|\overline{Y}_t^{n+1}|^2e^{\beta t} + \beta\mathbf{E}\int_t^T|\overline{Y}_r^{n+1}|^2e^{\beta r}dr + \mathbf{E}\int_t^T|\overline{Z}_r^{n+1}|^2e^{\beta r}dr + \mathbf{E}\int_t^T\int_E|\overline{U}_r^{n+1}(e)|^2e^{\beta r}\lambda(de)dr$$

$$= 2\mathbf{E}\int_t^T e^{\beta r}\langle\overline{Y}_r^{n+1}, f(r,\Theta_r^n) - f(r,\Theta_r^{n-1})\rangle dr + \mathbf{E}\int_t^T e^{\beta r}|g(r,\Theta_r^n) - g(r,\Theta_r^{n-1})|^2dr.$$

Using standard estimates, the right hand side is less than

$$\gamma\mathbf{E}\int_t^T|\overline{Y}_r^{n+1}|^2e^{\beta r}dr + \mathbf{E}\int_t^T[2c|\overline{Y}_r^n|^2 + \frac{1+\alpha}{2}(|\overline{Z}_r^n|^2 + |\overline{U}_r^n|^2)]e^{\beta r}dr$$

where γ is positive constant. Now choosing $\beta = \gamma + \bar{c}$ (where $\bar{c} = \frac{4c}{1+\alpha}$), we deduce that

$$\mathbf{E}\int_t^T\Big[\bar{c}|\overline{Y}_r^{n+1}|^2 + |\overline{Z}_r^{n+1}|^2 + |\overline{U}_r^{n+1}|^2\Big]e^{\beta r}dr \le \left(\frac{1+\alpha}{2}\right)^n\mathbf{E}\int_t^T\Big[\bar{c}|\overline{Y}_r^1|^2 + |\overline{Z}_r^1|^2 + |\overline{U}_r^1|^2\Big]e^{\beta r}dr$$

Since $\frac{1+\alpha}{2} < 1$, we deduce that $(\Theta_t^n)_{n\in\mathbf{N}}$ is a Cauchy sequence in $\mathcal{B}^2_{[0,T]}(\mathbf{R}^k)$. Let $n \to +\infty$ in (4) and by virtue of

$$\Theta_t = \lim_{n\to+\infty}\Theta_t^n, \quad 0 \le t \le T,$$

is it easy to see that $(\Theta_t)_{0\le t\le T} = (Y_t, Z_t, U_t)_{0\le t\le T}$ solves (1). ■

We are now in position to move on to study our main subject.

2.3. **The case of non-Lipschitz coefficients**

In the following we assume that f and g satisfy assumptions (**H2**) :

(**H2.1**) : (H1.1) holds.

(**H2.2**) : There exists $c > 0$ and $0 < \alpha < 1$ s. t. for $0 \le t \le T$, $(y,y') \in (\mathbf{R}^k)^2$, $(z,z') \in (\mathbf{R}^{k\times d})^2$, $(u,u') \in (\mathbf{R}^k)^2$,

$$|f(t,y,z,u) - f(t,y',z',u')|^2 \le \rho(t,|y-y'|^2) + c(|z-z'|^2 + |u-u'|^2)$$

$$|g(t,y,z,u) - g(t,y',z',u')|^2 \le \rho(t,|y-y'|^2) + \alpha(|z-z'|^2 + |u-u'|^2)$$

where $\rho(\cdot,\cdot) : [0,T] \times \mathbf{R}^+ \to \mathbf{R}^+$ satisfies

- For fixed $t \in [0,T]$, $\rho(t,\cdot)$ is a continuous, concave and nondecreasing s.t. $\rho(t,0) = 0$.
- For any $\Gamma > 0$, the ordinary differential equation

$$v' = -\Gamma\rho(t,v), \quad v(T) = 0, \tag{5}$$

has a unique solution $v(t) = 0, \quad 0 \leq t \leq T$.

- There exists $a(\cdot), b(\cdot) : [0,T] \to \mathbf{R}^+$ such that $\rho(t,v) \leq a(t) + b(t)v$ and $\int_0^T [a(t) + b(t)]dt < \infty$.

Let us recall that some examples satisfying assumptions (H2) are given in [5]. Our strategy for the proof of existence is to use the Picard approximate sequence. We consider now the sequence $(Y_t^n, Z_t^n, U_t^n)_{n\geq 0}$ given by

$$\begin{cases} Y^0 = 0; \\ Y_t^n = \xi + \int_t^T f(s, Y_s^{n-1}, Z_s^n, U_s^n)ds + \int_t^T g(s, Y_s^{n-1}, Z_s^n, U_s^n)dB_s \\ \qquad - \int_t^T Z_s^n dB_s - \int_t^T \int_E U_s^n(e)\widetilde{\mu}(ds,de), \quad n \geq 1. \end{cases} \tag{6}$$

It follows from Theorem 2.1 that this sequence is well defined. The key point is to prove that $(Y_t^n, Z_t^n, U_t^n)_{n\geq 0}$ is a Cauchy sequence. To this end we need two lemmata.

Lemma 2.2. *Assume that $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ and (H2) is in force. There exists a constant $C_{(\alpha,c)} > 1$ depending only α and c such that for $n, \ m \geq 1$ we have*

$$\mathbf{E}|Y_t^{n+m} - Y_t^n|^2 \leq C_{(\alpha,c)} e^{(T-t)C_{(\alpha,c)}} \int_t^T \rho(s, \mathbf{E}|Y_s^{n+m-1} - Y_s^{n-1}|^2)ds, \quad 0 \leq t \leq T.$$

Proof. Using Lemma 2.1, we have for $n,m \geq 1$ and $0 \leq t \leq T$,

$$\mathbf{E}|Y_t^{n+m} - Y_t^n|^2 + \mathbf{E}\int_t^T |Z_s^{n+m} - Z_s^n|^2 ds + \mathbf{E}\int_t^T \int_E |U_s^{n+m}(e) - U_s^n(e)|^2 \lambda(de)ds$$
$$= 2\mathbf{E}\int_t^T \langle Y_s^{n+m} - Y_s^n, \Delta f^{(n,m)}(s)\rangle ds + \mathbf{E}\int_t^T |\Delta g^{(n,m)}(s)|^2 ds, \tag{7}$$

where for a function $h, \Delta h^{(n,m)}(s) = h(s, Y_s^{n+m-1}, Z_s^{n+m}, U_s^{n+m}) - h(s, Y_s^{n-1}, Z_s^n, U_s^n), \quad 0 \leq s \leq T$.
Using standard estimates and assumption (H2.2), we have

$$2\langle Y_s^{n+m} - Y_s^n, \Delta f^{(n,m)}(s)\rangle \ \leq \ \tfrac{2c}{1-\alpha} + \tfrac{1-\alpha}{2c}|\Delta f^{(n,m)}(s)|^2$$
$$\leq \tfrac{2c}{1-\alpha}|Y_s^{n+m} - Y_s^n|^2 + \tfrac{1-\alpha}{2c}\rho(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2)$$
$$+ \tfrac{1-\alpha}{2}|Z_s^{n+m} - Z_s^n|^2 + \tfrac{1-\alpha}{2}|U_s^{n+m} - U_s^n|^2,$$
$$|\Delta g^{(n,m)}(s)|^2 \leq \rho(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2) + \alpha|Z_s^{n+m} - Z_s^n|^2 + \alpha|U_s^{n+m} - U_s^n|^2.$$

By plugging these two inequalities in (7), we deduce that

$$\mathbf{E}|Y_t^{n+m} - Y_t^n|^2 + \tfrac{1-\alpha}{2}\mathbf{E}\int_t^T |Z_s^{n+m} - Z_s^n|^2 ds + \tfrac{1-\alpha}{2}\mathbf{E}\int_t^T \int_E |U_s^{n+m}(e) - U_s^n(e)|^2 \lambda(de)ds$$
$$\leq C_{(\alpha,c)}\mathbf{E}\int_t^T |Y_s^{n+m} - Y_s^n|^2 ds + C_{(\alpha,c)}\mathbf{E}\int_t^T \rho(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2)ds,$$

where $C_{(\alpha,c)} = 2c/(1-\alpha) + 1 + [2c/(1-\alpha)]^{-1} > 1$. Hence Gronwall's lemma implies that

$$\mathbf{E}|Y_t^{n+m} - Y_t^n|^2 \leq C_{(\alpha,c)} e^{(T-t)C_{(\alpha,c)}} \int_t^T \rho(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2)ds, \qquad 0 \leq t \leq T, \tag{8}$$

which is the required result. ■

Lemma 2.3. *Assume that* $\xi \in L^2(\Omega, F_T, P)$ *and (H2) is in force. Then there exists* $0 \le T_1 < T$ *not depending on* ξ *and a constant* $M \ge 0$ *such that*

$$\mathbf{E}|Y_t^n|^2 \le M, \qquad T_1 \le t \le T.$$

Proof. Using again Lemma 2.1, we deduce that for $n \ge 1$,

$$\mathbf{E}|Y_t^n|^2 + \mathbf{E}\int_t^T |Z_s^n|^2 ds + \mathbf{E}\int_t^T\int_E |U_s^n(e)|^2\lambda(de)ds \le \mathbf{E}|\xi|^2 + 2\mathbf{E}\int_t^T \langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n, U_s^n)\rangle ds$$
$$+\mathbf{E}\int_t^T |g(s, Y_s^{n-1}, Z_s^n, U_s^n)|^2 ds, \quad 0 \le t \le T.$$

By assumption (H2.2) and standard estimates, we realize for any $\varepsilon > 0$ that

$$2\langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n, U_s^n)\rangle \le \frac{2C}{\varepsilon}|Y_s^n|^2 + \frac{\varepsilon}{2C}|f(s, Y_s^{n-1}, Z_s^n, U_s^n)|^2$$
$$\le \frac{\varepsilon}{C}\rho(s, |Y_s^{n-1}|^2) + \varepsilon(|Z_s^n|^2 + |U_s^n|^2) + \frac{\varepsilon}{C}|f(s,0,0,0)|^2$$

and

$$|g(s, Y_s^{n-1}, Z_s^n, U_s^n)|^2 \le (2-\alpha)|g(s, Y_s^{n-1}, Z_s^n, U_s^n) - g(s,0,0,0)|^2 + \frac{2-\alpha}{1-\alpha}|g(s,0,0,0)|^2$$
$$\le (2-\alpha)\rho(s, |Y_s^{n-1}|^2) + (2-\alpha)\alpha(|Z_s^n|^2 + |U_s^n|^2) + \frac{2-\alpha}{1-\alpha}|g(s,0,0,0)|^2.$$

Hence there exists a positive constant $K(\alpha, c)$ depending on α and c such that

$$\mathbf{E}|Y_t^n|^2 + ((\alpha-1)^2 - \varepsilon)\left[\mathbf{E}\int_t^T |Z_s^n|^2 ds + \mathbf{E}\int_t^T\int_E |U_s^n(e)|^2\lambda(de)ds\right] \le \mathbf{E}|\xi|^2 + K(\alpha,c)\mathbf{E}\int_t^T |Y_s^n|^2 ds$$
$$+\mathbf{E}\int_t^T (|f(s,0,0,0)|^2 + |g(s,0,0,0)|^2)ds + K(\alpha,c)\int_t^T \rho(s, \mathbf{E}|Y_s^{n-1}|^2)ds.$$

In particular with the choice $\varepsilon = (\alpha-1)^2/2$, we deduce that

$$\mathbf{E}|Y_t^n|^2 \le \mathbf{E}|\xi|^2 + K(\alpha,c)\mathbf{E}\int_t^T |Y_s^n|^2 ds$$
$$+\mathbf{E}\int_t^T (|f(s,0,0,0)|^2 + |g(s,0,0,0)|^2)ds + K(\alpha,c)\int_t^T \rho(s, \mathbf{E}|Y_s^{n-1}|^2)ds.$$

Gronwall's lemma implies

$$\mathbf{E}|Y_t^n|^2 \le \left[\mathbf{E}|\xi|^2 + \mathbf{E}\int_t^T (|f(s,0,0,0)|^2 + |g(s,0,0,0)|^2)ds\right] e^{(T-t)K(\alpha,c)}$$
$$+ K(\alpha,c)e^{(T-t)K(\alpha,c)}\int_t^T \rho(s, \mathbf{E}|Y_s^{n-1}|^2)ds. \tag{9}$$

For notational simplicity, we note $K = K(\alpha, c)$. Putting $\overline{T}_1 = \sup(T - \frac{\ln K}{K}, 0)$, we deduce from (9) that

$$\mathbf{E}|Y_t^n|^2 \le \mu_t + K^2\int_t^T \rho(s, \mathbf{E}|Y_s^{n-1}|^2)ds, \qquad \overline{T}_1 \le t \le T, \tag{10}$$

where $\mu_t = K\left[\mathbf{E}|\xi|^2 + \mathbf{E}\int_t^T (|f(s,0,0,0)|^2 + |g(s,0,0,0)|^2)ds\right]$. Finally let

$$M = 2\mu_0 + 2\int_0^T a(s)ds, \tag{11}$$

then use the arguments developed in [5] to complete the proof. ■

Lemma 2.3 ensures that the sequence $(Y_t^n)_{n\ge1}$ is uniformly bounded in mean square on $[T_1, T]$. It is enough to prove that $(Y_t^n)_{n\ge1}$ is a Cauchy sequence in $S^2_{[T_1,T]}(\mathbb{R}^k)$ to deduce that the limiting process belongs in this space. Lemma 2.2 will help us to achieve this goal.We are able then to make the claim that follows.

Theorem 2.2. *Let $\xi \in L^2(\Omega, F_T, P)$. Under (H2) equation (1)is has a unique solution $(Y_t, Z_t, U_t)_{0\le t\le T}$.*

Proof. Using the constant M given by (11) we consider the sequence $(\varphi_n)_{n\geq 1}$ given by

$$\varphi_0(t) = \int_t^T \rho(s, M)ds, \qquad \varphi_{n+1}(t) = \int_t^T \rho(s, \varphi_n(s))ds, \quad n \geq 0\,, \quad T_1 \leq t \leq T.$$

Exploiting the argument developed in [5,Theorem1], we prove that the sequence $(Y_t^n)_{n\geq 1}$ is a Cauchy sequence in $S^2_{[T_1,T]}(\mathbf{R}^k)$, $(Z_t^n)_{n\geq 1}$ is a Cauchy sequence in $H^2_{[T_1,T]}(\mathbf{R}^k)$ and $(U_t^n)_{n\geq 1}$ is a Cauchy sequence in $L^2_{[T_1,T]}(\widetilde{\mu}, \mathbf{R}^k)$, where T_1 is given by Lemma 2.3. Letting $n \to \infty$ in (6), we obtain

$$Y_t = \xi + \int_t^T f(s, \Theta_s)ds + \int_t^T g(s, \Theta_s)dB_s - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(e)\widetilde{\mu}(ds, de), T_1 \leq t \leq T. \tag{12}$$

Hence $\Theta_t = (Y_t, Z_t, U_t)$ satisfies (1) on $[T_1, T]$. Moreover by virtue of (H2), we have $(Z_t, U_t) \in H^2_{[T_1,T]}(\mathbf{R}^{k\times d}) \times L^2_{[T_1,T]}(\widetilde{\mu}, \mathbf{R}^k)$. As a consequence, we deduce by Doob's inequality that

$$\begin{aligned} &\mathbf{E}\int_t^T (|f(s, \Theta_s)|^2 + |g(s, \Theta_s)|^2)ds + \mathbf{E}\left(\sup_{T_1\le t\le T} \left|\int_{T_1}^t Z_s dB_s\right|^2\right) \\ &\qquad + \mathbf{E}\left(\sup_{T_1\le t\le T} \left|\int_{T_1}^t \int_E U_s(e)\widetilde{\mu}(de, ds)\right|^2\right) < \infty. \end{aligned}$$

This implies essentially that $\mathbf{E}(\sup_{T_1\le t\le T} |Y_t|^2) < \infty$. Thus the triplet (Y_t, Z_t, U_t) solves (1) on $[T_1, T]$. Using Lemma 2.3, one can deduce existence of a solution on $[T_2, T_1]$, with $0 \leq T_2 < T_1$. Hence by iteration we prove existence of solution on $[0, T]$.

Let us prove uniqueness. Let (Y_t, Z_t, U_t) and $(\widetilde{Y}_t, \widetilde{Z}_t, \widetilde{U}_t)$ be two solutions of (1). Applying Lemma 2.1, where $\Delta f(s)$ and $\Delta g(s)$ are given by (3), we have

$$\begin{aligned} &\mathbf{E}|Y_t - \widetilde{Y}_t|^2 + \mathbf{E}\int_t^T |Z_s - \widetilde{Z}_s|^2 ds + \mathbf{E}\int_t^T \int_E |U_s(e) - \widetilde{U}_s(e)|^2 \lambda(de)ds = 2\mathbf{E}\int_t^T \langle Y_s - \widetilde{Y}_s,\ \Delta f(s)\rangle ds \\ &\qquad + \mathbf{E}\int_t^T |\Delta g(s)|^2 ds, \quad 0 \leq t \leq T. \end{aligned}$$

Using the same computations as in the proof of Lemma 2.2, we deduce, for any $0 \leq t \leq T$, that

$$\begin{aligned} &\mathbf{E}|Y_t - \widetilde{Y}_t|^2 + \frac{1-\alpha}{2}\mathbf{E}\int_t^T |Z_s - \widetilde{Z}_s|^2 ds + \frac{1-\alpha}{2}\mathbf{E}\int_t^T \int_E |U_s(e) - \widetilde{U}_s(e)|^2 \lambda(de)ds \\ &\leq C_{(\alpha,c)}\mathbf{E}\int_t^T |Y_s - \widetilde{Y}_s|^2 ds + C_{(\alpha,c)}\mathbf{E}\int_t^T \rho\Big(s, |Y_s - \widetilde{Y}_s|^2\Big)ds. \end{aligned} \tag{13}$$

Hence by applying Gronwall's Lemma, we arrive at

$$\mathbf{E}|Y_t - \widetilde{Y}_t|^2 \leq C_{(\alpha,c)} e^{(T-t)C_{(\alpha,c)}} \int_t^T \rho\Big(s, \mathbf{E}|Y_s - \widetilde{Y}_s|^2\Big) ds, \ 0 \leq t \leq T. \tag{14}$$

Let $\delta = [C_{(\alpha,c)}]^{-1} \ln C_{(\alpha,c)} > 0$ (since $C_{(\alpha,c)} > 1$) and $m = [T/\delta] + 1$. If $(t_j)_{0 \leq j \leq m}$ denotes the uniform subdivision of $[0, T]$ given by $T_0 = 0, \quad T_j = T - (m-j)\delta, \quad j \geq 1$, we have

$$\mathbf{E}|Y_t - \widetilde{Y}_t|^2 \leq C^2_{(\alpha,c)} \int_t^T \rho\Big(s, \mathbf{E}|Y_s - \widetilde{Y}_s|^2\Big) ds, \ T_{m-1} \leq t \leq T.$$

This implies from the comparison theorem of ordinary differential equation, $\mathbf{E}|Y_t - \widetilde{Y}_t|^2 \leq r(t)$ where $r(t)$ is the maximum of solution of (5) (with $\Gamma = C^2_{(\alpha,c)}$). As a consequence, we have $Y_t = \widetilde{Y}_t, \quad T_{m-1} \leq t \leq T$. From (13), we deduce that $Z_t = \widetilde{Z}_t$ and $U_t = \widetilde{U}_t, \quad T_{m-1} \leq t \leq T$. Using the same procedure, we prove uniqueness on $[T_j, T_{j+1}], \quad j = 0, \ldots, m-2$. This completes the proof. ■

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References

[1] X. Mao, Adapted solution of Backward stochastic differential equations with non-Lipschitz coefficients, *Stochastic Processes and their Applications* **58**, (1995), 281–292.

[2] M. N'zi, and J. M. Owo, Backward doubly stochastic differential equations with discontinuous coefficients, *Statistics and Probability Letters* **79**, (2009), 920-926.

[3] É. Pardoux, and S. Peng, Backward doubly stochastic differential equations and semilinear PDEs, *Probability Theory and Related Fields* **98**, (1994), 209-227.

[4] A. B. Sow, BSDE with jumps and non Lipschitz coefficients: Application to large deviations, *Global Journal of Pure and Applied Mathematics,* to appear.

[5] Y. Wang, and Z. Huang, Backward stochastic differential equations with non Lipschitz coefficients equations, *Statistics and Probability Letters* **79**, (2009), 1438-1443.

[6] Y. Wang, and X. Wang, Adapted solutions of backward stochastic differential equations with non Lipschitz coefficients equations, *Chinese Journal of Applied Probability and Statistics* **19**, (2003), 245-251.

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