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An Improved Method for Solving a System of Discrete-Time Generalized Riccati Equations

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Abstract. *We consider a set of discrete-time generalized Riccati equations that arise in quadratic optimal control of discrete-time stochastic systems subjected to both state-dependent noise and Markovian jumps. The iterative method to compute the maximal and stabilizing solution for a wide class of discrete-time nonlinear equations was derived by Dragan, Morozan and Stoica (International Journal of Control 83, (2010), 837-847). Here we modify this method and illustrate it in the computation of the maximal solution of a system of discrete-time generalized Riccati equations. Convergence properties of this method are analyzed. Numerical experiments are reported to estimate the effectiveness of this new iterative method.*

Key words : Set of Discrete–Time Riccati Equations, Jump Systems, Stochastic Systems, Optimal Control.

AMS Subject Classifications : 15A24, 15A45, 65F35

1. Introduction

In recent years, a special class of linear systems subject to abrupt changes in their structures have been investigated. This is the case of Markovian jump linear systems (MJLS), which comprise an important family of models subject to abrupt variations. There are many examples in the literature showing the importance of the different types of discrete-time Riccati equations involved in the construction of the optimal controls of different problems of robust control (see [1, 4, 7, 10] and literature therein). The properties and the numerical solution of different types of discrete-time Riccati equations have been intensively studied in many papers (5, 2, 12, 13, 15).

We consider the problem for computing a maximal symmetric solution to the following set of discrete-time generalized algebraic Riccati equations (DTGAREs):

$$X(i) = \mathcal{P}(i, \mathbf{X}) := \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{X}) A_l(i) + C^T(i)C(i)$$
$$- \left(\sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{X}) B_l(i) + L(i)\right) \left(R(i) + \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{X}) B_l(i)\right)^{-1}$$
$$\times \left(\sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{X}) A_l(i) + L(i)^T\right), \qquad i = 1, \ldots, N \tag{1}$$

where $\mathcal{E}(\mathbf{X}) = (\mathcal{E}_1(\mathbf{X}), \ldots, \mathcal{E}_N(\mathbf{X}))$ with $\mathbf{X} = (X(1), \ldots, X(N))$ and

$$\mathcal{E}_i(\mathbf{X}) = \sum_{j=1}^{N} p_{ij} X(j), \qquad X(j) \textit{ is an } n \times n \textit{ mtrix}, \;\; \textit{for } i = 1, \ldots, N.$$

In (1), $A_0(i), \ldots, A_r(i), B_0(i), \ldots, B_r(i), L(i), R(i) = R(i)^T$ and $Q(i) = C^T(i)C(t)$ $(i = 1, \ldots, N)$ are given real matrices of the appropriated dimensions. Normally the matrix $\begin{pmatrix} Q(i) & L(i) \\ L(i)^T & R(i) \end{pmatrix}$ is assumed to be positive semidefinite.

The sets of discrete-time generalized Riccati equations are investigated to find the maximal solution of linear-quadratic optimization problems for discrete-time stochastic systems with state-dependent noise and Markovian jumps (see Dragan and Morozan [6]). The optimization problem is described by

$$x(t+1) = \left[A_0(\eta_t) + \sum_{j=1}^{r} w_j(t) A_j(\eta_t) \right] x(t) + \left[B_0(\eta_t) + \sum_{j=1}^{r} w_j(t) B_j(\eta_t) \right] u(t),$$

and $x(0) = x_0$ where $x(t)$ is the state vector and $u(t)$ is the control. Here $w(t)^T = (w_1(t), w_2(t), \ldots, w_r(t))_{t \geq 0}$ is a sequence of an r-dimensional random vector on a given probability space $(\Omega, \mathcal{F}, P)$ with zero mean and covariance $E[w(t)\, w(t)^T] = I_r, t \geq 0$ and $\{\eta_t\}_{t \geq 0}$ is a homogeneous Markov chain with the state space the finite set $\mathcal{D} = \{1, 2, \ldots, N\}$ and the transition probability matrix $P = (p_{ij})_{i,j=1}^{N}$. In addition, it is assumed that the stochastic processes $\{w(t)\}_{t \geq 0}, \{\eta_t\}_{t \geq 0}$ are independent and $E[.]$ being the mathematical expectation. The objective is to minimize the cost functional

$$J(u) = \int_0^{\infty} E[|C(\eta_t)x(t)|^2 + u^T(t)R(\eta_t)u(t)]\, dt.$$

In order to minimize the above functional one has to solve the set of discrete time nonlinear equations (1). Necessary and sufficient conditions for the existence of the maximal solution and stabilizing solution of this kind of discrete-time nonlinear equations were presented in [8, 9], in terms of the concept of the stabilizability of a sequence of linear and positive operators.

Here we introduce a modification of the proposed iterative method from [9] to find the maximal solution and stabilizing solution of (1). We are going to prove the convergence properties to the proposed iteration under new assumptions described in terms of matrix inequalities. These properties are derived applying some matrix manipulations and facts from the matrix analysis. Finally, in order to demonstrate the efficiency of the algorithms, two examples are included.

The notation $\mathcal{H}^n$ stands for the linear space of symmetric matrices of size n over the field of real numbers. For any $X, Y \in \mathcal{H}^n$, we write $X > Y$ or $X \geq Y$ if $X - Y$ is positive definite or

$X - Y$ is positive semidefinite. The spectrum for any square real $n \times n$ matrix A will be denoted by $\sigma(A)$. A matrix A is said to be d-stable if the all eigenvalues of A lie in the open unit disk, i.e. $|\lambda_s(A)| < 1$ for $s = 1, \dots, n$, and almost d-stable if $|\lambda_s(A)| \leq 1$ for $s = 1, \dots, n$. The notations $\mathbf{X} = (X(1), X(2), \dots, X(N)) \in \mathcal{H}^n$ and the inequality $\mathbf{Y} \geq \mathbf{Z}$ mean that for $i = 1, \dots, N$, $X(i) \in \mathcal{H}^n$ and $Y(i) \geq Z(i)$, respectively. The linear space $\mathcal{H}^n$ is a Hilbert space with the Frobenius inner product $< X, Y > = trace(XY)$.

2. Preliminaries

We begin this section with some notations and useful statements. For the matrix function $\mathcal{P}(i, \mathbf{X})$ we introduce notations

$$S(i, \mathbf{Z}) = \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{Z}) B_l(i) + L(i)\,;$$
$$R(i, \mathbf{Z}) = R(i) + \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{Z}) B_l(i)\,;$$
$$F(i, \mathbf{Z}) = -R(i, \mathbf{Z})^{-1} S(i, \mathbf{Z})^T, \quad \left(\text{note that } \mathrm{S}(\mathrm{i}, \mathbf{Z}) = -\mathrm{F}(\mathrm{i}, \mathbf{Z})^{\mathrm{T}} \mathrm{R}(\mathrm{i}, \mathbf{Z})\right)$$

$$\begin{aligned} T(i, \mathbf{Z}) &= Q(i) + F(i, \mathbf{Z})^T L(i)^T + L(i)^T F(i, \mathbf{Z}) + F(i, \mathbf{Z})^T R(i) F(i, \mathbf{Z}) \\ &= \begin{pmatrix} I & F(i, \mathbf{Z})^T \end{pmatrix} \begin{pmatrix} Q(i) & L(i) \\ L(i)^T & R(i) \end{pmatrix} \begin{pmatrix} I \\ F(i, \mathbf{Z}) \end{pmatrix} \end{aligned}$$

and we present the set of equations (1) as follows:

$$X(i) = \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{X}) A_l(i) + C^T(i) C(i) - S(i, \mathbf{Z}) R(i, \mathbf{Z})^{-1} S(i, \mathbf{Z})^T,$$

with $i = 1, \dots, N$.
Then, for the matrix function $\mathcal{P}(i, \mathbf{X})$ we rewrite

$$\mathcal{P}(i, \mathbf{X}) = \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{X}) A_l(i) + C^T(i) C(i) - F(i, \mathbf{X})^T R(i, \mathbf{X}) F(i, \mathbf{X})\,. \tag{2}$$

and we will study the system $X(i) = \mathcal{P}(i, \mathbf{X})$ for $i = 1, \dots, N$. We start by some useful properties to $\mathcal{P}(i, \mathbf{X})$. For briefly we use the notation $\tilde{A}_l(i, \mathbf{Z}) = A_l(i) + B_l(i) F(i, \mathbf{Z})$ for some $\mathbf{Z} \in \mathcal{H}^n$.

Lemma 2. 1. *Assuming $Y \in \mathcal{H}^n$ and $Z \in \mathcal{H}^n$ are symmetric matrices, then the following properties of* $\mathcal{P}(i, \mathbf{X})$, $i = 1, \dots, N$,

$$\begin{aligned} \mathcal{P}_{\mathbf{Z}}(i, \mathbf{Y}) = & \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{Z})^T \mathcal{E}_i(\mathbf{Y}) \tilde{A}_l(i, \mathbf{Z}) + T(i, \mathbf{Z}) \\ & - (F(i, \mathbf{Y})^T - F(i, \mathbf{Z})^T) R(i, \mathbf{Y}) (F(i, \mathbf{Y}) - F(i, \mathbf{Z})), \end{aligned} \tag{3}$$

$$\begin{aligned} \mathcal{P}_{\mathbf{Z}}(i, \mathbf{Z}) - \mathcal{P}_{\mathbf{Z}}(i, \mathbf{Y}) = & \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{Z})^T \mathcal{E}_i(\mathbf{Z} - \mathbf{Y}) \tilde{A}_l(i, \mathbf{Z}) \\ & + (F(i, \mathbf{Y})^T - F(i, \mathbf{Z})^T) R(i, \mathbf{Y}) (F(i, \mathbf{Y}) - F(i, \mathbf{Z})), \end{aligned}$$

hold.

Proof. We write down

$$
\begin{aligned}
\mathcal{P}(i,\mathbf{Y}) &= \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{Y}) A_l(i) + C^T(i)C(i) - F(i,\mathbf{Y})^T R(i,\mathbf{Y})\, F(i,\mathbf{Y}) \\
&= \sum_{l=0}^{r} (A_l(i) + B_l(i)F(i,\mathbf{Z}))^T \mathcal{E}_i(\mathbf{Y})(A_l(i) + B_l(i)F(i,\mathbf{Z})) \; + C^T(i)C(i) \\
&\quad - F(i,\mathbf{Y})^T R(i,\mathbf{Y})\, F(i,\mathbf{Y}) - \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{Y})\, B_l(i)F(i,\mathbf{Z})) \\
&\quad - \sum_{l=0}^{r} F(i,\mathbf{Z})^T B_l(i)^T \mathcal{E}_i(\mathbf{Y})\, A_l(i) - \sum_{l=0}^{r} F(i,\mathbf{Z})^T B_l(i)^T \mathcal{E}_i(\mathbf{Y})\, B_l(i)F(i,\mathbf{Z}) \\
&= \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{Z})^T \mathcal{E}_i(\mathbf{Y}) \tilde{A}_l(i,\mathbf{Z}) - F(i,\mathbf{Y})^T R(i,\mathbf{Y})\, F(i,\mathbf{Y}) + T(i,\mathbf{Z}) \\
&\quad - \; F(i,\mathbf{Z})^T L(i)^T - L(i)\, F(i,\mathbf{Z}) - F(i,\mathbf{Z})^T R(i) F(i,\mathbf{Z}) \\
&\quad - \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{Y})\, B_l(i)F(i,\mathbf{Z}) \\
&\quad - \sum_{l=0}^{r} F(i,\mathbf{Z})^T B_l(i)^T \mathcal{E}_i(\mathbf{Y})\, A_l(i) - \sum_{l=0}^{r} F(i,\mathbf{Z})^T B_l(i)^T \mathcal{E}_i(\mathbf{Y})\, B_l(i)F(i,\mathbf{Z}))\,.
\end{aligned}
$$

Applying some matrix manipulations we obtain

$$
\begin{aligned}
\mathcal{P}(i,\mathbf{Y}) &= \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{Z})^T \mathcal{E}_i(\mathbf{Y}) \tilde{A}_l(i,\mathbf{Z}) - F(i,\mathbf{Y})^T R(i,\mathbf{Y})\, F(i,\mathbf{Y}) + T(i,\mathbf{Z}) \\
&\quad - F(i,\mathbf{Z})^T \left(L(i)^T + \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{Y})\, A_l(i) \right) \\
&\quad - \left(L(i) + \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{Y})\, B_l(i) \right) F(i,\mathbf{Z}) \\
&\quad - F(i,\mathbf{Z})^T \left(R(i) + \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{Y}) \right) F(i,\mathbf{Z})
\end{aligned}
$$

$$= \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{Z})^T \mathcal{E}_i(\mathbf{Y}) \tilde{A}_l(i,\mathbf{Z}) - F(i,\mathbf{Y})^T R(i,\mathbf{Y}) F(i,\mathbf{Y}) + T(i,\mathbf{Z})$$
$$- F(i,\mathbf{Z})^T S(i,\mathbf{Y})^T - S(i,\mathbf{Y}) F(i,\mathbf{Z}) - F(i,\mathbf{Z})^T R(i,\mathbf{Y}) F(i,\mathbf{Z})$$
$$= \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{Z})^T \mathcal{E}_i(\mathbf{Y}) \tilde{A}_l(i,\mathbf{Z}) - F(i,\mathbf{Y})^T R(i,\mathbf{Y}) F(i,\mathbf{Y}) + T(i,\mathbf{Z})$$
$$+ F(i,\mathbf{Z})^T R(i,\mathbf{Y}) F(i,\mathbf{Y}) + F(i,\mathbf{Y})^T R(i,\mathbf{Y}) F(i,\mathbf{Z})$$
$$- F(i,\mathbf{Z})^T R(i,\mathbf{Y}) F(i,\mathbf{Z}) .$$

Thus

$$\mathcal{P}(i,\mathbf{Y}) = \mathcal{P}_{\mathbf{Z}}(i,\mathbf{Y}) = \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{Z})^T \mathcal{E}_i(\mathbf{Y}) \tilde{A}_l(i,\mathbf{Z}) + T(i,\mathbf{Z})$$
$$- (F(i,\mathbf{Y})^T - F(i,\mathbf{Z})^T) R(i,\mathbf{Y}) (F(i,\mathbf{Y}) - F(i,\mathbf{Z})) .$$

The last equality expresses the map $\mathcal{P}(i,\mathbf{Y})$ through $\mathbf{Z}$, i.e. $\mathcal{P}_{\mathbf{Z}}(i,\mathbf{Y}) := \mathcal{P}(i,\mathbf{Y})$. In the special case of $\mathbf{Z} = \mathbf{Y}$ we have

$$\mathcal{P}_{\mathbf{Y}}(i,\mathbf{Y}) = \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{Y})^T \mathcal{E}_i(\mathbf{Y}) \tilde{A}_l(i,\mathbf{Y}) + T(i,\mathbf{Y}) . \tag{4}$$

Further on, we obtain:

$$\mathcal{P}_{\mathbf{Z}}(i,\mathbf{Z}) - \mathcal{P}_{\mathbf{Z}}(i,\mathbf{Y}) = \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{Z})^T \mathcal{E}_i(\mathbf{Z}-\mathbf{Y}) \tilde{A}_l(i,\mathbf{Z})$$
$$+ (F(i,\mathbf{Y})^T - F(i,\mathbf{Z})^T) R(i,\mathbf{Y}) (F(i,\mathbf{Y}) - F(i,\mathbf{Z})) .$$

This completes the proof of the lemma. ■

3. A New Method and its Properties

In this section, we modify the iterative procedure derived by Dragan, Morozan and Stoica [9] for computing the maximal solution of a set of nonlinear equations (1). A solution $\tilde{\mathbf{X}}$ of (1) is called maximal if $\tilde{\mathbf{X}} \geq \mathbf{X}$ for any solution $\mathbf{X}$. The proposed iteration, (4.7) of [9], is :

$$\begin{aligned} X(i)^{(k)} &= \mathcal{P}_{\mathbf{X}^{(k-1)}}(i,\mathbf{X}^{(k-1)}) \\ &= \sum_{l=0}^{r} [\tilde{A}_l(i,\mathbf{X}^{(k-1)})]^T \mathcal{E}_i(\mathbf{X}^{(k-1)}) [\tilde{A}_l(i,\mathbf{X}^{(k-1)})] \\ &\quad + T(i,\mathbf{X}^{(k-1)}) + \tfrac{\varepsilon}{k} I_n \end{aligned} \tag{5}$$

where $\tilde{A}_l(i,\mathbf{X}^{(k-1)}) = A_l(i) + B_l(i) F(i,\mathbf{X}^{(k-1)}) , \quad k = 1,2,3\ldots ,$

and ε is a small positive number. Note that iteration (5) is a special case of the general iterative method given in [9], Theorem 3.3. Based on the Gauss-Seidel technique our modification is

$$
\begin{aligned}
X(i)^{(k)} &= \sum_{l=0}^{r} [\tilde{A}_l(i, \mathbf{X}^{(k-1)})]^T \\
&= \times (\mathcal{E}_{i1}(\mathbf{X}^{(k)}) + p_{ii} X(i)^{(k-1)} + \mathcal{E}_{i2}(\mathbf{X}^{(k-1)})) \\
&\quad \times [\tilde{A}_l(i, \mathbf{X}^{(k-1)})] + T(i, \mathbf{X}^{(k-1)}), \\
&\quad i = 1, 2, \ldots, N, \quad k = 1, 2, 3 \ldots .
\end{aligned} \tag{6}
$$

where

$$
\mathcal{E}_{i1}(\mathbf{Y}) = \sum_{j=1}^{i-1} p_{ij} Y(j), \quad \textit{and} \quad \mathcal{E}_{i2}(\mathbf{Y}) = \sum_{j=i+1}^{N} p_{ij} Y(j). \tag{7}
$$

The method can be applied under the assumption that the matrix inequalities $\mathcal{P}(i, \mathbf{Z}) \geq Z(i)$ and $\mathcal{P}(i, \mathbf{Z}) \leq Z(i)$, $(i = 1, \ldots, N)$ are solvable. Under these conditions it will be shown that convergence of (6) takes place if the algorithm starts at any suitable initial point $\mathbf{X}^{(\mathbf{0})}$. The new iteration (6) can be considered as an accelerated modification to iteration (5). The convergence result is given by the theorem that follows.

Theorem 3.1. *Assume the existence of symmetric matrices $\hat{X} = (\hat{X}_1, \ldots, \hat{X}_N)$ and $X^{(0)} = (X_1^{(0)}, \ldots, X_N^{(0)})$ such that $(a)\, \mathcal{P}(i, \hat{X}) \geq \hat{X}(i)$; $(b)\, X^{(0)} \geq \hat{X}$; $(c)\, \mathcal{P}(i, X^{(0)}) \leq X(i)^{(0)}$ for $i = 1, \ldots, N$. Then for the matrix sequences $\{X(1)^{(k)}\}_{k=1}^{\infty}, \ldots, \{X(N)^{(k)}\}_{k=1}^{\infty}$, defined by (6), the following properties are satisfied:*
(i) we have $X^{(k)} \geq \hat{X}$, $\quad X^{(k)} \geq X^{(k+1)}$ and

$$
\mathcal{P}(i, \mathbf{X}^{(k)}) = X(i)^{(k+1)} + \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(k)})^T \mathcal{E}_{i1}(\mathbf{X}^{(k)} - \mathbf{X}^{(k+1)}) \tilde{A}_l(i, \mathbf{X}^{(k)}), \tag{8}
$$

Proof. Let $k = 0$. We will prove the inequality $\mathbf{X}^{(0)} \geq \mathbf{X}^{(1)}$. From iteration (6) for $k = 1$ and for each i we get:

$$
\begin{aligned}
X(i)^{(1)} &= \sum_{l=0}^{r} [\tilde{A}_l(i, \mathbf{X}^{(0)})]^T (\mathcal{E}_{i1}(\mathbf{X}^{(1)}) + p_{ii} X(i)^{(0)} + \mathcal{E}_{i2}(\mathbf{X}^{(0)})) \\
&\quad \times [\tilde{A}_l(i, \mathbf{X}^{(0)})] + T(i, \mathbf{X}^{(0)}).
\end{aligned} \tag{9}
$$

We may then derive an expression for $X(i)^{(0)} - X(i)^{(1)}$ under the assumption (c) of this theorem via

$$
\begin{aligned}
X(i)^{(0)} - \mathcal{P}(i, \mathbf{X}^{(0)}) & \\
&= X(i)^{(0)} - \sum_{l=0}^{r} [\tilde{A}_l(i, \mathbf{X}^{(0)})]^T (\mathcal{E}_i(\mathbf{X}^{(0)})[\tilde{A}_l(i, \mathbf{X}^{(0)})] - T(i, \mathbf{X}^{(0)}) \\
&= \mathcal{N}(i)^{(0)} \geq 0.
\end{aligned}
$$

Combination with (9) leads to

$$
X(i)^{(0)} - X(i)^{(1)} = \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(0)})^T (\mathcal{E}_{i1}(\mathbf{X}^{(0)} - \mathbf{X}^{(1)}))\tilde{A}_l(i, \mathbf{X}^{(0)}) + \mathcal{N}(i)^{(0)}.
$$

Since $X(1)^{(0)} - X(1)^{(1)} \geq 0$, we conclude for $i = 1$ that $X(1)^{(0)} - X(1)^{(1)} = \mathcal{N}(1)^{(0)} \geq 0$, and for $i = 2$ that

$$
X(2)^{(0)} - X(2)^{(1)} = \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(0)})^T p_{21}(X(1)^{(0)} - X(1)^{(1)})\tilde{A}_l(i, \mathbf{X}^{(0)}) + \mathcal{N}(2)^{(0)} \geq 0.
$$

The conclusion is the same for $i = 3, \ldots, N$.

Now, assume there exists a natural number p and the matrix sequences $\{X(1)^{(k)}\}_0^p, \ldots, \{X(N)^{(k)}\}_0^p$ are computed and property (i) is observed, i.e. for $i = 1, \ldots, N$ and $s = 0, \ldots, p-1$ we have $X(i)^{(s)} \geq \hat{X}(i)$, $X(i)^{(s)} \geq X(i)^{(s+1)}$ and (8) holds for $k = s$.
We will show that for $i = 1, \ldots, N$ the following statements are valid.

- $X(i)^{(p)} \geq \hat{X}(i)$,
- $X(i)^{(p)} \geq X(i)^{(p+1)}$,
- $\mathcal{P}(i, \mathbf{X}^{(p)}) = X(i)^{(p+1)} + \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p)})^T \mathcal{E}_{i1}(\mathbf{X}^{(p)} - \mathbf{X}^{(p+1)})\tilde{A}_l(i, \mathbf{X}^{(p)})$.

To prove $X(i)^{(p)} \geq \hat{X}(i)$ for $i = 1, \ldots, N$, we invoke $\mathcal{P}(i, \hat{\mathbf{X}}) \geq \hat{X}(i)$ from the transformation

$$
\begin{aligned}
X(i)^{(p)} - \hat{X}(i) = {} & \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T (\mathcal{E}_{i1}(\mathbf{X}^{(p)}) + p_{ii} X(i)^{(p-1)} + \mathcal{E}_{i2}(\mathbf{X}^{(p-1)})) \\
& \times \tilde{A}_l(i, \mathbf{X}^{(p-1)}) + T(i, \mathbf{X}^{(p-1)}) - \hat{X}(i).
\end{aligned}
$$

According to (3) we have

$$
\begin{aligned}
\mathcal{P}(i, \hat{\mathbf{X}}) &= \mathcal{P}_{\mathbf{X}^{(p-1)}}(i, \hat{\mathbf{X}}) = \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T \mathcal{E}_i(\hat{\mathbf{X}})\tilde{A}_l(i, \mathbf{X}^{(p-1)}) + T(i, \mathbf{X}^{(p-1)}) \\
& \quad -\left(F(i, \hat{\mathbf{X}})^T - F(i, \mathbf{X}^{(p-1)})^T\right) R(i, \hat{\mathbf{X}}) \left(F(i, \hat{\mathbf{X}}) - F(i, \mathbf{X}^{(p-1)})\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
X(i)^{(p)} - \hat{X}(i) - \mathcal{P}(i, \hat{\mathbf{X}}) & \\
= \sum_{l=0}^{r} & \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T \left(\mathcal{E}_{i1}(\mathbf{X}^{(p)} - \hat{\mathbf{X}}) + p_{ii}(X(i)^{(p-1)} - \hat{X}(i)) + \mathcal{E}_{i2}(\mathbf{X}^{(p-1)} - \hat{\mathbf{X}})\right) \\
& \times \tilde{A}_l(i, \mathbf{X}^{(p-1)}) - \hat{X}(i) \\
& + \left(F(i, \hat{\mathbf{X}})^T - F(i, \mathbf{X}^{(p-1)})^T\right) R(i, \hat{\mathbf{X}}) \left(F(i, \hat{\mathbf{X}}) - F(i, \mathbf{X}^{(p-1)})\right).
\end{aligned}
$$

We know that $\mathcal{P}(i, \hat{\mathbf{X}}) - \hat{X}(i) \geq 0$. Then $X(i)^{(p)} - \hat{X}(i) \geq 0$ for all $i = 1, \ldots, N$.

Next, we compute the matrices $X(i)^{(p+1)}$ from (6) and we will prove $X(i)^{(p)} \geq X(i)^{(p+1)}$ for $i = 1, \dots N$. From iteration (6) for $k = p + 1$ we deduce that

$$\begin{aligned} X(i)^{(p+1)} &= \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p)})^T (\mathcal{E}_{i1}(\mathbf{X}^{(p+1)}) + p_{ii} X(i)^{(p)} + \mathcal{E}_{i2}(\mathbf{X}^{(p)})) \\ &\quad \times \tilde{A}_l(i, \mathbf{X}^{(p)}) + T(i, \mathbf{X}^{(p)}) . \end{aligned} \tag{10}$$

From (3) we obtain

$$\begin{aligned} \mathcal{P}(i, \mathbf{X}^{(p)}) &= \mathcal{P}_{\mathbf{X}^{(p-1)}}(i, \mathbf{X}^{(p)}) \\ &= \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T \mathcal{E}_i(\mathbf{X}^{(p)}) \tilde{A}_l(i, \mathbf{X}^{(p-1)}) + T(i, \mathbf{X}^{(p-1)}) \\ &\quad - (F(i, \mathbf{X}^{(p)})^T - F(i, \mathbf{X}^{(p-1)})^T) R(i, \mathbf{X}^{(p)}) (F(i, \mathbf{X}^{(p)}) - F(i, \mathbf{X}^{(p-1)})), \end{aligned}$$

$$\begin{aligned} \mathcal{P}(i, \mathbf{X}^{(p)}) &= \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T \mathcal{E}_{i1}(\mathbf{X}^{(p)}) \tilde{A}_l(i, \mathbf{X}^{(p-1)}) + T(i, \mathbf{X}^{(p-1)}) \\ &\quad + \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T (p_{ii} X(i)^{(p)} + \mathcal{E}_{i2}(\mathbf{X}^{(p)})) \tilde{A}_l(i, \mathbf{X}^{(p-1)}) \\ &\quad - (F(i, \mathbf{X}^{(p)})^T - F(i, \mathbf{X}^{(p-1)})^T) R(i, \mathbf{X}^{(p)}) (F(i, \mathbf{X}^{(p)}) - F(i, \mathbf{X}^{(p-1)})) . \end{aligned}$$

Use now the iteration formula for $k = p$ to write

$$\begin{aligned} &\sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T \mathcal{E}_{i1}(\mathbf{X}^{(p)}) \tilde{A}_l(i, \mathbf{X}^{(p-1)}) + T(i, \mathbf{X}^{(p-1)}) \\ &= X(i)^{(p)} - \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T \times (p_{ii} X(i)^{(p-1)} + \mathcal{E}_{i2}(\mathbf{X}^{(p-1)})) \tilde{A}_l(i, \mathbf{X}^{(p-1)}) . \end{aligned}$$

We replace the last expression in $\mathcal{P}(i, \mathbf{X}^{(p)})$ to obtain

$$\begin{aligned} &\mathcal{P}(i, \mathbf{X}^{(p)}) = X(i)^{(p)} \\ &- \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T (p_{ii}(X(i)^{(p-1)} - X(i)^{(p)}) + \mathcal{E}_{i2}(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p)})) \\ &\times \tilde{A}_l(i, \mathbf{X}^{(p-1)}) - (F(i, \mathbf{X}^{(p)})^T - F(i, \mathbf{X}^{(p-1)})^T) R(i, \mathbf{X}^{(p)}) (F(i, \mathbf{X}^{(p)}) - F(i, \mathbf{X}^{(p-1)})) , \end{aligned}$$

and

$$X(i)^{(p)} = \mathcal{P}(i, \mathbf{X}^{(p)})$$
$$+\sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T (p_{ii}(X(i)^{(p-1)} - X(i)^{(p)}) + \mathcal{E}_{i2}(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p)}))$$
$$\times \tilde{A}_l(i, \mathbf{X}^{(p-1)}) + (F(i, \mathbf{X}^{(p)})^T - F(i, \mathbf{X}^{(p-1)})^T) R(i, \mathbf{X}^{(p)}) (F(i, \mathbf{X}^{(p)}) - F(i, \mathbf{X}^{(p-1)}))$$
$$X(i)^{(p)} = \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p)})^T \mathcal{E}_i(\mathbf{X}^{(p)}) \tilde{A}_l(i, \mathbf{X}^{(p)}) + T(i, \mathbf{X}^{(p)})$$
$$+\sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T (p_{ii}(X(i)^{(p-1)} - X(i)^{(p)}) + \mathcal{E}_{i2}(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p)})) \times \tilde{A}_l(i, \mathbf{X}^{(p-1)})$$
$$+(F(i, \mathbf{X}^{(p)})^T - F(i, \mathbf{X}^{(p-1)})^T) R(i, \mathbf{X}^{(p)}) (F(i, \mathbf{X}^{(p)}) - F(i, \mathbf{X}^{(p-1)}))\,.$$

Next, we form the difference $X(i)^{(p)} - X(i)^{(p+1)}$ using (10):

$$X(i)^{(p)} - X(i)^{(p+1)}$$
$$= \sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p)})^T \mathcal{E}_{i1}(\mathbf{X}^{(p)} - \mathbf{X}^{(p+1)}) \tilde{A}_l(i, \mathbf{X}^{(p)})$$
$$+\sum_{l=0}^{r} \tilde{A}_l(i, \mathbf{X}^{(p-1)})^T (p_{ii}(X(i)^{(p-1)} - X(i)^{(p)}) + \mathcal{E}_{i2}(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p)}))$$
$$\times \tilde{A}_l(i, \mathbf{X}^{(p-1)}) + (F(i, \mathbf{X}^{(p)})^T - F(i, \mathbf{X}^{(p-1)})^T) R(i, \mathbf{X}^{(p)}) (F(i, \mathbf{X}^{(p)}) - F(i, \mathbf{X}^{(p-1)}))\,.$$

For $i = 1$ we have

$$X(1)^{(p)} - X(1)^{(p+1)}$$
$$= +\sum_{l=0}^{r} \tilde{A}_l(1, \mathbf{X}^{(p-1)})^T (p_{11}(X(1)^{(p-1)} - X(1)^{(p)}) + \mathcal{E}_{12}(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p)})) \times \tilde{A}_l(1, \mathbf{X}^{(p-1)})$$
$$+(F(1, \mathbf{X}^{(p)})^T - F(1, \mathbf{X}^{(p-1)})^T) R(1, \mathbf{X}^{(p)}) (F(1, \mathbf{X}^{(p)}) - F(1, \mathbf{X}^{(p-1)})).$$

Thus $X(1)^{(p)} - X(1)^{(p+1)} \geq 0$.
For $i = 2$ we have

$$X(2)^{(p)} - X(2)^{(p+1)}$$
$$= +\sum_{l=0}^{r} \tilde{A}_l(2, \mathbf{X}^{(p)})^T p_{21}(X(1)^{(p)} - X(1)^{(p+1)}) \tilde{A}_l(2, \mathbf{X}^{(p)})$$
$$+\sum_{l=0}^{r} \tilde{A}_l(2, \mathbf{X}^{(p-1)})^T (p_{22}(X(2)^{(p-1)} - X(2)^{(p)}) + \mathcal{E}_{22}(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p)})) \times \tilde{A}_l(2, \mathbf{X}^{(p-1)})$$
$$+(F(2, \mathbf{X}^{(p)})^T - F(2, \mathbf{X}^{(p-1)})^T) R(2, \mathbf{X}^{(p)}) (F(2, \mathbf{X}^{(p)}) - F(2, \mathbf{X}^{(p-1)})).$$

Thus $X(2)^{(p)} - X(2)^{(p+1)} \geq 0$. In similar fashion we prove that $X(i)^{(p)} - X(i)^{(p+1)} \geq 0$ for $i = 3, \ldots, N$.
Moreover, we have to show that

$$\mathcal{P}(i,\mathbf{X}^{(p)}) = X(i)^{(p+1)} + \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{X}^{(p)})^T \mathcal{E}_{i1}(\mathbf{X}^{(p)} - \mathbf{X}^{(p+1)})\tilde{A}_l(i,\mathbf{X}^{(p)}),$$

for $i = 1,\dots,N$.
Clearly

$$\mathcal{P}(i,\mathbf{X}^{(p)}) = \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{X}^{(p)})^T \mathcal{E}_i(\mathbf{X}^{(p)}) \tilde{A}_l(i,\mathbf{X}^{(p)}) + T(i,\mathbf{X}^{(p)})$$

and

$$X(i)^{(p+1)} = \sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{X}^{(p)})^T \times \mathcal{E}_{i1}(\mathbf{X}^{(p+1)})$$

$$+p_{ii}X(i)^{(p)} + \mathcal{E}_{i2}(\mathbf{X}^{(p)})\tilde{A}_l(i,\mathbf{X}^{(p)}) + T(i,\mathbf{X}^{(p)}) .$$

By subtracting the last two equations we arrive at

$$\mathcal{P}(i,\mathbf{X}^{(p)}) \;=\; X(i)^{(p+1)} + \textstyle\sum_{l=0}^{r} \tilde{A}_l(i,\mathbf{X}^{(p)})^T \mathcal{E}_{i1}(\mathbf{X}^{(p)} - \mathbf{X}^{(p+1)})\tilde{A}_l(i,\mathbf{X}^{(p)}) ,$$

for $i = 1,\dots,N$. ■

4. Numerical Experiments

We investigate the numerical behavior of the considered iteration (5) and (6) for finding the maximal solution to the couple discrete time general Riccati equations (1). We will carry out some experiments for this purpose.

Our experiments are executed in MATLAB on a 1,7 GHz PENTIUM computer. We denote *tol*- a small positive real number denoting the accuracy of computation; $Error_s = \max_{i=1,\dots,N} \ \|\mathcal{P}(i,\mathbf{X}^{(s)}) - X(i)^{(s)}\|_2$, where $\|.\|_2$ is the spectral norm; *It*- number of iterations for which the inequality $Error_{It} \leq tol = 1e-12$ holds. The last inequality is used as a practical stopping criterion.

We use real coefficient matrices. For solving the above examples the suitable MATLAB procedures are used.

However, in order to apply the introduced iterations we have to comment how to choose the initial matrices $X(i)^{(0)}, i = 1,\dots,N$. The first approach is to follow the result proved by Dragan et al.[9], Proposition 4.1. One must solve a suitable system of coupled linear matrix inequalities given by (4.5) of [9] to determine a stabilizing feedback gain $(\tilde{F}_0(1),\dots,\tilde{F}_0(N))$. Next, it is necessary to compute a solution $X(i)^{(0)}$ of the corresponding LMI (4.6) of [9] and then we are ready to use the introduced iterations. The second approach to define the initial point $X(i)^{(0)}, i = 1,\dots,N$ is to choose $X(i)^{(0)} = \alpha I_n$, where α is an enough big positive number. Such approach is successfully applied (see for example [11, 13]). In this section we use the second approach to determine $X(i)^{(0)}$ for $i = 1,\dots,N$.

Example 4.1. In these experiments the coefficient matrices and the transition probability matrix (p_{ij}) are $(N = 3, r = 1, n = 2)$:

$$A_0(1) = \begin{bmatrix} 0.01 & 0.2 \\ 0.065 & 0.482 \end{bmatrix}, \; A_0(2) = \begin{bmatrix} 0.03 & 0.15 \\ 0.68 & 0.105 \end{bmatrix}, \; A_0(3) = \begin{bmatrix} 0.4 & 0.05 \\ 0.025 & 0.202 \end{bmatrix},$$

$$A_1(1) = \begin{bmatrix} 0 & 0.1 \\ 0.015 & 0.082 \end{bmatrix}, \; A_1(2) = \begin{bmatrix} 0 & 0.1 \\ 0.03 & 0.095 \end{bmatrix}, \; A_1(3) = \begin{bmatrix} 0 & 0.1 \\ 0.53 & 0.102 \end{bmatrix},$$

$$\text{and } (p_{ij}) = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.10 & 0.64 \end{bmatrix}.$$

The pertaining coefficient matrices are

$$B_0(1) = \begin{bmatrix} 0.10940 & 0.14894 \\ 0.05926 & 0.03779 \end{bmatrix}, \qquad B_1(1) = \begin{bmatrix} 0.13736 & 0.07370 \\ 0.03670 & 0.12512 \end{bmatrix},$$

$$B_0(2) = \begin{bmatrix} 3.5784 & -1.3499 \\ 2.7694 & 3.0349 \end{bmatrix}, \qquad B_1(2) = \begin{bmatrix} 0.7254 & 0.71474 \\ -0.063055 & -0.20497 \end{bmatrix},$$

$$B_0(3) = \begin{bmatrix} 0.6715 & 0.71724 \\ -1.2075 & 1.6302 \end{bmatrix}, \qquad B_1(3) = \begin{bmatrix} 0.48889 & 0.72689 \\ 1.0347 & -0.30344 \end{bmatrix},$$

and

$$L(1) = L(3) = 0, \qquad L(2) = \begin{bmatrix} 0.064 & -0.2 \\ 0 & 0.058 \end{bmatrix}.$$

$$Q(1) = \frac{1}{20}\begin{bmatrix} 3.6 & -3.8 \\ -3.8 & 4.87 \end{bmatrix}, \quad Q(2) = \frac{1}{100}\begin{bmatrix} 10 & -3 \\ -3 & 8 \end{bmatrix},$$

$$Q(3) = \frac{1}{100}\begin{bmatrix} 5 & -4.5 \\ -4.5 & 4.5 \end{bmatrix}, \quad R(1) = \mathrm{diag}\,(0.0126,\ 0.024),$$

$$R(2) = \mathrm{diag}\,(0.09,\ 0.012), \quad R(3) = \mathrm{diag}\,(0.12,\ 0.105).$$

We execute iteration (5) for computing the maximal solution of (1) and obtain the maximal solution $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ after 18 iterative steps with an initial point $X_1^{(0)} = X_2^{(0)} = X_3^{(0)} = 2I$. The computed solution $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ is

$$\tilde{X}_1 = \begin{pmatrix} 0.18067 & -0.18589 \\ -0.18589 & 0.2689 \end{pmatrix} \textit{with eigenvalues } \lambda_1 = 0.033736, \lambda_2 = 0.41584,$$

$$\tilde{X}_2 = \begin{pmatrix} 0.13869 & -0.036183 \\ -0.036183 & 0.079723 \end{pmatrix} \textit{with eigenvalues } \lambda_1 = 0.15588, \lambda_2 = 0.062533,$$

$$\tilde{X}_3 = \begin{pmatrix} 0.074639 & -0.041357 \\ -0.041357 & 0.047012 \end{pmatrix} \textit{with eigenvalues } \lambda_1 = 0.017223, \lambda_2 = 0.10443.$$

The maximal solution $\tilde{\mathbf{X}}$ is positive definite.
In an experiment with iteration (6) we obtain the maximal solution $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ after 17 iterative steps with an initial point $X_1^{(0)} = X_2^{(0)} = X_3^{(0)} = 2I$.

Example 4.2. A more general example is considered for $N = 3, r = 1$ and $n = 9,\ 10,\ 11,\ 12$, where the coefficient matrices $A_0(i),\ A_1(i),\ B_0(i),\ B_1(i),\ L(i),\ i = 1,2,3$ were constructed using theMATLAB notations:

$$A_0(1) = randn(n,n)/5;\ \ A_0(2) = randn(n,n)/5;\ \ A_0(3) = randn(n,n)/5;$$
$$A_1(1) = randn(n,n)/5;\ \ A_1(2) = randn(n,n)/5;\ \ A_1(3) = randn(n,n)/5;$$

$$B_0(1) = randn(n,2)/20;\quad B_0(2) = randn(n,2)/20;\quad B_0(3) = randn(n,2)/20;$$
$$B_1(1) = randn(n,2)/20;\quad B_1(2) = randn(n,2)/20;\quad B_1(3) = randn(n,2)/20;$$

$$L(1) = randn(n,2)/25;\quad L(2) = randn(n,2)/25;\quad L(3) = randn(n,2)/25.$$

$$Q(1) = 0.75 * eye(n,n),\quad Q(2) = 0.25 * eye(n,n),\quad Q(3) = 0.05 * eye(n,n).$$

The matrices $R(1), R(2)$ and $R(3)$ are the same as in Example 4.1.
All iterations are compared after introducing the following parameters
It_M - the biggest number of iteration steps while executing each iteration,
It_a - the average number of iteration steps while executing each iteration,
for each iteration, and the results are listed in Table 1.

Table 1. Comparison between iterations for 100 runs

	Method (5)		Method (6)		Speed up
n	It_M	It_a	It_M	It_a	$\frac{It_a \text{ for } (6)}{It_a \text{ for } (5)}$
9	118	70.3	101	59.4	0.84
10	188	93.8	165	79.0	0.84
11	348	130.0	295	107.6	0.83
12	1255	263.2	1031	216.9	0.82

5. Conclusion

We have illustrated that the maximal solution of (1) can be found in terms of decoupled formulas by using the proposed algorithm (6). The convergence properties of the proposed new method are proved in terms of facts from the matrix theory. The new algorithm has the advantage of requiring much less iterative steps, based on the Gauss-Seidel technique, to achieve the convergence stopping criterion. Thus it is faster than the original iteration (5). In addition, a new approach to determine the initial point has been implemented. The paper contains moreover some randomly chosen examples, where the effectiveness of this new iterative method is demonstrated.

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