Journal of Numerical Mathematics and Stochastics, **3** (1) : 46-56, 2011 © ***JNM@S***
http://www.jnmas.org/jnmas3-6.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

The Large-Time Behavior of Stochastic 2D-Navier-Stokes Equations

D. C. FERNÁNDEZ

Departamento de Ingeniería Industrial, Instituto Technológico y de Estudios Superiores de Monterrey Campus Santa Fe, México, D. F., C.P.01389, México, E-mail: david.castillo@itesm.mx

Abstract. *In this paper, we shall primarily investigate large time behavior and exponential behavior of a weak solution to stochastic 2D Navier–Stokes equations. To be precise, we introduce a concept of an almost sure large time behavior of a solution process and establish it for such class of equations. This is carried out by formulating the stochastic Navier-Stokes equation as a semilinear stochastic evolution equation in Hilbert spaces.*

Key words : Stochastic Navier–Stokes Equations, Exponential Behavior of Weak Solution, Large-Time Behavior of Weak Solutions.

AMS Subject Classifications : 60H15, 35R50, 76D05

1. Introduction

The mathematical theory of the Navier-Stokes equation is of fundamental importance to a deep understanding, prediction and control of turbulence in nature and in technological applications such as combustion dynamics and manufacturing processes. The incompressible Navier–Stokes equation has a long history (e.g., see [5], [14],[15], for some earlier studies) as a model to understand external random forces. In aeronautical applications random forcing of the Navier–Stokes equation models structural vibrations and, in atmospheric dynamics, unknown external forces such as sun heating and industrial pollution can be represented as random forces. In addition to the above reasons there is a mathematical reason for studying stochastic Navier–Stokes equations. A rigorous theory of the stochastic Navier–Stokes equation has been a subject of several papers. Several approaches have been proposed in [1], [2], [4] ,[6], [8],[9], from the classic paper by Bensoussan and Temam [2], Bensoussan [1], to some more recent results e.g., by Ferrario [8]-[9] and Cutland [6].

The long–time behavior and the exponential stability of flows are very interesting and important problems in the theory of fluid dynamics, as the vast literature shows (see Temam

[14], among others, and the references therein). In the deterministic case, it has been known for a long time that the solutions of $2D$–Navier–Stokes equation tend to a stationary one (unique in fact) when time goes to infinity, see Temam [14]. On the other hand, another interesting question is to analyze the effects produced on a deterministic system by some stochastic or random disturbances appearing in the problem. In the paper of Caraballo Langa and Taniguchi [3], they consider the stochastic disturbances in the Itô sense, so the stabilization results proved should be interpreted in a suitable sense (see also [3]). These facts have motived the present work whose main objective is to show some aspects of the effects produced in the long–time behavior of the solution for a two dimensional Navier–Stokes equation under the presence of stochastic disturbances in the Itô sense and the stabilization interpreted in a suitable sense (Mao and Liu [12]). To be precise, we consider

$$
\begin{aligned}
&dx = [\nu\Delta x - \langle x, \Delta\rangle x + \Delta p + F(x)]dt + G(t,x)dW(t), \quad t > 0;\\
&div\, x = \nabla x = 0 \quad \text{in} \quad [0,\infty)\times D,\\
&x = 0 \qquad \text{on} \qquad [0,\infty)\times\Gamma,\\
&x(0,x) = x_0(x) \quad \text{in} \quad x \in D. \qquad (1)
\end{aligned}
$$

where D is a regular open bounden domain of $\mathbb{R}^2$ with boundary Γ, x is the velocity field of the fluid, p the pressure, ν the kinematic viscosity, x_0 the initial velocity field, F the external force field, and $G(t,x)dW(t)$ the random field where $W(t)$ is an infinite dimensional Wiener process. The above classical form of the Navier–Stokes equation can be re–written in the following abstract form

$$
\begin{aligned}
&dx = [\nu Ax(t) + B(x(t)) + F(x(t))]dt + G(t,x(t))dW(t),\\
&x_0 = x(0); \qquad (2)
\end{aligned}
$$

see section 2 for details.

The format of the rest of this paper is as follows. In the second section, we give the preliminaries containing several definitions and a Lemma. In the third section we prove some results on the large time behavior. relying on a basic robustness analysis, the criteria for almost certain asymptotic stability of the stochastic 2D–Navier–Stokes equation are outlined in Theorem 3.2. In the fourth section, we discuss the exponential behavior of $2D$–Navier–Stokes Equations.

2. Preliminaries

Throughout this paper, we work in the frameworks used by Caraballo, Langa, Taniguchi [2], Liu [4]. We will also make use of notions and notations as in these papers. So we start by introducing the following Hilbert spaces:

$$H = \text{the closure of the set } \{C_0^\infty(D;\mathbb{R}^2) : \nabla u = 0\}$$

in $L^2(D;\mathbb{R}^2)$ with the norm $\|u\|_H = (u,u)^{1/2}$, where for $u \in L^2(D;\mathbb{R}^2)$,

$$(u,v) = \sum_{j=1}^{2}\int_D u^j(x)v^j(x)dx.$$

$$V = \text{the closure of the set } \{u \in C_0^\infty(D,\mathbb{R}^2) : \nabla u = 0\}$$

in $H_0^{'}(D;\mathbb{R}^2)$ with the norm $\|u\|_V = ((u,u))^{1/2}$, where for $u,v \in H_0^{'}(D;\mathbb{R}^2)$,

$$((u,v)) = \sum_{j=1}^{2}\Big(\tfrac{\partial u}{\partial u_j}, \tfrac{\partial v}{\partial v_j}\Big).$$

Then it follows that H and V are separable Hilbert with associated inner products $(\bullet,\bullet)$ and $((\bullet,\bullet))$ and the following is satisfied

$$V \subset H \equiv H' \subset V',$$

where the injections are dense, continuous and compact. Let P_H be an orthogonal projection from $L^2(D;\mathbb{R}^2)$ onto H. Using this, we define the operator
$A : L(D;\mathbb{R}^2) \to H$ by $Au = P_H\Delta u$.
Next, we define an operator $B : V \times V \to V'$ by

$$\langle B(u,v), w\rangle = b(u,v,w) \quad \forall u,v,w \in V,$$

where $\langle\bullet,\bullet\rangle$ denotes the duality $\langle V', V\rangle$ and b is the trilinear form defined by

$$b(u,v,w) = \sum_{i,j=1}^{2}\int_D u^j(x)\tfrac{\partial v^j}{\partial x_i}(x)w^j(x)dx.$$

Let us set $B(u) = B(u,u), \forall u \in V$ according to the model of equation (2).
Furthermore, we shall need some properties of the trilinear form b, and we list below the ones that we need, see Temam [6],

$$|b(u,v,w| \leq c_1\|u\|_H^{1/2}\|u\|_V^{1/2}\|v\|_V\|w\|_H^{1/2}\|w\|_V^{1/2}, \quad \forall u,v,w \in V, \tag{3}$$

$$b(u,v,v) = 0, \quad \forall u,v \in V, \tag{4}$$

$$b(u,v,v) = 0, \quad \forall u,v \in V, \tag{5}$$

$$b(u,u,v-u) - b(v,v,v-u) = -b(v-u,u,v-u), \quad \forall u,v,w \in V. \tag{6}$$

Let $(\Omega, B, \{B_t\}, P)$ be a probability space on which an increasing and right continuous family $\{B_t\}_{t\geq 0}$ of complete sub σ–algebras of $\{B_t\}_{t\geq 0}$ is defined. Then let $\beta_n(t)$ $(n = 1,2,3,\ldots.)$, be a sequence of real–valued one dimensional standard Brownian motions independent of $(\Omega, B, \{B_t\}, P)$. Moreover, set

$$W(t) = \sum_{n=1}^{\infty}\sqrt{\lambda_n^{'}}\,\beta_n(t)e_n, \quad t \geq 0,$$

where $\lambda_n' \geq 0$ $(n = 1,2,3,\ldots)$ are non–negative real numbers such that $\sum_{n=1}^{\infty}\lambda_n' < \infty$, and $\{e_n\}$ $(n = 1,2,3,\ldots)$ is a complete orthonormal basis in the real and separable Hilbert space K. Moreover, let $Q_W \in L(K,H)$ be the operator defined by $Q_We_n = \lambda_n'e_n$. The last K–valued stochastic process $W(t)$ is called a Q_W–Wiener process. We are now prepared to state equation (2) precisely when $F : V \to V'$ and $G : (0,\infty) \times V \to L(K,H)$ are continuous functions.

Definition 2. 1. An adapted process $x : [0,T] \times \Omega \to H$ is a weak–solution to the stochastic Navier–Stokes equation (2) if
(i) $x(t)$ is $\mathcal{B}_t$ –adapted
(ii) $x(t) \in L^{\infty}(([0,T];H) \cap L^2([0,T],V)$ almost surely (a.s.) for all $T > 0$,

(iii) the following equation holds as an identity in V' (a.s.), for $t \in [0,\infty)$

$$x(t) = x(0) + \int_0^t [-\nu Ax(s) - B(x(s)) + F(x(s))]ds + \int_0^t G(s,x(s))dW(s).$$

We will assume the existence of such weak solutions (see, for instance Bensoussan [1]). Finally, we conclude this section by stating the following results which are fundamental in the development of this article.

Lemma 2. 2. [6] *Assume $x(t)$ is the weak solution of (2). For the previous $G(s,x(s))$* and *$V(s,x(s))$ when T,α,β are any positive numbers, there holds*

$$P\left[\sup_{0\le t\le T}\left[\int_0^t (V_x(s,x(s)),G(s,x(s))dW(s)) - \tfrac{\alpha}{2}\int_0^t QV(s,x(s))ds\right] > \beta\right] \le e^{-\alpha\beta},$$

where

$$QV(s,x(s)) = tr[V_x \otimes V_x(t,x(t))(G(t,x(t))Q_W G^*(t,x(t)))].$$

Now we establish an Itô´s formula (see Pardoux ([13]) which is going to be necessary later on in this paper . Let $C^{1,2}([0,\infty)\times H,\mathbb{R}^+)$ denote the space of all $\mathbb{R}^+$-valued functions Υ, defined on $(0,\infty)\times H$ with the following properties:
(1) $\Upsilon(t,x)$ is differentiable in $t \in [0,\infty)$ and twice Frechet differentiable in x with $\Upsilon_t(t,\bullet),\Upsilon_x(t,\bullet)$ and $\Upsilon_{xx}(t,\bullet)$ locally bounded on H,
(2) $\Upsilon(t,\bullet),\Upsilon(t,\bullet)$ and $\Upsilon_x(t,\bullet)$ are continuous on H,
(3) for all trace class operators R, $tr(\Upsilon_{xx}(t,\bullet)R)$ is continuous from H in $\mathbb{R}$,
(4) if $v \in V$ then $\Upsilon_x(t,v) \in V$, and $u \to <\Upsilon_x(t,u),v^*>$ is continuous for each $v^* \in V'$,
(5) $\|\Upsilon_x(t,v)\| \le C_0(t)(1+\|v\|), C_0(t) > 0,$ for all $v \in V$.

Theorem 2. 3. (Itô formula) *Let $\Upsilon \in C^{(1,2)}([0,\infty)\times H,R^+)$. If the stochastic process $x(t)$ is a weak solution to (2), then it holds that*

$$\begin{aligned}\Upsilon(t,x(t)) &= \Upsilon(0,x(0)) + \int_0^t L\Upsilon(s,x(s))ds\\ &= +\int_0^t (\Upsilon_x(s,x(s)),G(x,x(s))dW(s),\end{aligned} \tag{7}$$

where

$$\begin{aligned}L\Upsilon(s,x(s)) &= \Upsilon_t(s,x(s))\\ &+ < -\nu Ax(s) - B(x(s)) + F(x(s)),\Upsilon_x(s,x(s) >\\ &+ \tfrac{1}{2}tr(\Upsilon_{xx}(s,x(s))G(s,x(s))QG(s,x(s))^*).\end{aligned} \tag{8}$$

3. Large-Time Asymptotic Behavior

Our purpose here is to introduce the large time behavior of a weak solution to equation (2) . Let $\alpha_1 > 0$ be the first eigenvalue of A. We remark that $\|v\|_V^2 \ge \alpha_1\|v\|_H^2, \forall v \in V$ to investigate the existence of the stationary solution to the equation

$$\nu Au + B(u) = F(u). \tag{9}$$

Throughout this section we will utilize the following condition:

$(A1)$ There exists $\beta > 0$ such that
$\|F(u) - F(v)\|_{V'} \leq \beta\|u - v\|, \beta > 0, u, v \in V.$

Lemma 3. 1. [2],[6] *Suppose that condition $A1$ is satisfied and $F(v_m)$ converges to $F(v)$ weakly in V' whenever $\{v_m\} \subset V$ converges to $v \in V$ weakly in V and strongly in $A1$. Then,*
(a) *if $\nu > \beta$, there exists a stationary solution $u_\infty \in V$ to (9),*
(b) *furthermore, if* $\nu > \dfrac{c_1\|F(0)\|_{V'}}{\sqrt{\alpha_1}(\nu-\beta)} + \beta,$
then the stationary solution to (9) is unique.

In the rest of the article when we assume $x_\infty \in V$, we will understand that the stationary solution to (9) is unique. Let us consider now the large–time asymptotic behavior of stochastic $2D$–Navier–Stokes equations. For this purpose we need to utilize the following additional conditions:
(L1) Let $\varphi_1(t), \varphi_2(t)$ be continuous non–negative functions, such that:
$\|G(t, x(t))\|^2_{L_2^0} + Q\xi^{-1}(t)\|x(t) - x_\infty\|^2_H \leq \varphi_1(t)\|x(t) - x_\infty\|^2_H + \varphi_2(t),$
(L2) Suppose $\lambda(t) \uparrow +\infty$, $t \to \infty$, be a positive increasing function. Furthermore, there exist positive constants $p, \theta, \mu, \ell > 0$ and a positive function $\xi(t) \downarrow 0$, $t \to \infty$ such that:

$$\limsup_{k\to\infty} \frac{\xi(k-1)}{\xi(k)} < p < \infty, \quad \sum_{k=1}^{\infty} \frac{1}{\lambda(k)^2} < \infty, \quad k = 1, 2\ldots,$$

$$\limsup_{k\to\infty} \frac{\log\log\lambda(k)}{\log\log\lambda(k-1)} \leq \theta, \quad k = 1, 2\ldots, \quad \limsup_{t\to\infty} \frac{\xi(t)\int_0^t \xi^{-1}(s)\varphi_1(s)ds}{\log\log\lambda(t)} \leq \ell,$$

$$\limsup_{t\to\infty} \frac{\log\xi(t)\int_0^t \varphi_2(s)ds}{\log\log\lambda(t)} \leq \mu, \quad \limsup_{t\to\infty} \frac{\xi(t)\int_0^t \xi^{-1}(s)ds}{\log\log\lambda(t)} \leq \eta,$$

$$\limsup_{t\to\infty} \frac{\xi(t)\int_0^t \frac{d\xi^{-1}(s)ds}{ds}}{\log\log\lambda(t)} \leq r.$$

We also adhere to the notation: $x^*(t) = \sup_{0\leq s\leq t} \|x(s) - x_\infty\|_H.$

Theorem 3. 2. *Let $x_\infty \in V$ and suppose that (L1) and (L2) are satisfied. Then, there exists a constant random variable $C(\omega)$ such that the solution of equation (2) satisfies*
$x^*(t) \leq C(\omega)\log\lambda(t)^{(\theta\vee\mu+l+\Theta\eta+r\alpha_1^{-1})/2},$
where

$$\Theta = -2\nu + 2\beta + \frac{2c_1\|x_\infty\|_V}{\sqrt{\alpha_1}}.$$

Proof. By Itô formula [2], taking $V(t,x(t)) = \|x(t)-x_\infty\|_H^2\xi^{-1}(t)$, we get

$$\|x(t)-x_\infty\|_H^2\xi^{-1}(t) = \|x(0)-x_\infty\|_H^2\xi^{-1}(0)$$

$$+\int_0^t \frac{d\xi^{-1}(s)}{ds}\|x(s)-x_\infty\|_H^2 ds - 2\int_0^t \xi^{-1}(s)\langle \nu Ax(s), x(s)-x_\infty\rangle ds$$

$$-2\int_0^t \xi^{-1}(s)\langle B(x(s)), x(s)-x_\infty\rangle ds + 2\int_0^t \xi^{-1}(s)\langle F(x(s)), x(s)-x_\infty\rangle ds$$

$$+\int_0^t \xi^{-1}(s)\|G(s,x(s))\|_{L_2^0}^2 ds + 2\int_0^t (\xi^{-1}(s)\|x(s)-x_\infty\|_H, G(s,x(s))dW(s)))$$

Consider (3), and

$$\int_0^t \langle \nu Ax_\infty, x(s)-x_\infty\rangle ds + \int_0^t \langle B(x_\infty), x(s)-x_\infty\rangle ds$$

$$= \int_0^t \langle F(x_\infty), x(s)-x_\infty\rangle ds,$$

to write

$$\|x(t)-x_\infty\|_H^2\xi^{-1}(t) = \|x(0)-x_\infty\|_H^2\xi^{-1}(0) + \int_0^t \frac{d\xi^{-1}(s)}{ds}\|x(s)-x_\infty\|_H^2 ds$$

$$-\int_0^t \nu\xi^{-1}(s)\langle Ax(s)-Ax_\infty, x(s)-x_\infty)\rangle ds + \int_0^t \xi^{-1}(s)\langle F(x(s)) - F(x_\infty), x(s)-x_\infty\rangle ds$$

$$-2\int_0^t \xi^{-1}(s)\langle B(x(s)) - B(x_\infty), x(s)-x_\infty\rangle ds + \int_0^t \xi^{-1}(s)\|G(s,x(s))\|_{L_2^0}^2 ds$$

$$+2\int_0^t (\xi^{-1}(t)\|x(s)-x_\infty\|_H, G(s,x(s))dW(s)))$$

Here we used that,

$$\langle B(x(s)) - B(x_\infty), x(s)-x_\infty\rangle = b(x(s)-x_\infty, x_\infty, x(s)-x_\infty)$$

$$\leq c_1\|x(s)-x_\infty\|_H^{\frac{1}{2}}\|x(s)-x_\infty\|_V^{\frac{1}{2}}\|x_\infty\|_V\|x(s)-x_\infty\|_H^{\frac{1}{2}}\|x(s)-x_\infty\|_V^{\frac{1}{2}}$$

$$\leq \frac{c_1}{\sqrt{\alpha_1}}\|x_\infty\|_V\|x(s)-x_\infty\|_V^2$$

Therefore,

$$\|x(t)-x_\infty\|_H^2\xi^{-1}(t) \leq \|x(0)-x_\infty\|_H^2\xi^{-1}(0)+\int_0^t\left(-2\nu+2\beta+\frac{2c_1\|x_\infty\|_V}{\sqrt{\alpha_1}}\right)$$

$$\times\|x(s)-x_\infty\|_V^2\xi^{-1}(s)ds \quad + \quad \int_0^t\frac{d\xi^{-1}(s)}{ds}\|x(s)-x_\infty\|_H^2ds$$

$$+\int_0^t\xi^{-1}(s)\|G(s,x(s)\|_{L_2^0}^2ds+2\int_0^t(\xi^{-1}(t)\|x(s)-x_\infty\|_H,G(s,x(s)dW(s)).$$

Utilizing Lemma 2. 2 leads to

$$P\left\{\omega : \sup_{k0\leq t\leq w}\left[\int_0^t(\xi^{-1}(s)\|x(s)-x_\infty\|_H,G(s,x(s)dW(s)))\right.\right.$$

$$\left.\left.-\frac{\alpha}{2}\int_0^t Q\|x(s)-x_\infty\|_H^2\xi^{-1}(s)ds > \beta\right]\right\} \leq e^{-\alpha\beta}.$$

After taking

$$\alpha = 2\xi(k), \quad \beta = \xi^{-1}(k)log\lambda(k), \quad w = k,$$

we apply the well known Borel–Cantelli Lemma to obtain the fact that there exists an integer $k_0(\omega)$ for almost all $\omega \in \Omega$ such that

$$\|x(t)-x_\infty\|_H^2\xi^{-1}(t) \leq \int_0^t(\Theta\xi^{-1}(s)+\alpha_1^{-1}\frac{d\xi^{-1}(s)}{ds})\|x(s)-x_\infty\|_V^2ds$$

$$+\int_0^t\xi^{-1}(s)\|G(s,x(s)\|_{L_2^0}^2ds+\xi^{-1}(k)\log\lambda(k)+\int_0^t Q\|x(s)-x_\infty\|_H^2\xi^{-1}(s)ds$$

Therefore,

$$\|x(t)-x_\infty\|_H^2 \leq \xi(t)\int_0^t(\Theta\xi^{-1}(s)+ \ \alpha_1^{-1}\frac{d\xi^{-1}(s)}{ds})\|x(s)-x_\infty\|_V^2ds$$

$$+\xi(t)\int_0^t\xi^{-1}(s)\varphi_1(s)\|x(s)-x_\infty\|_H^2ds+\xi(t)\xi^{-1}(k)\log\lambda(k)+\xi(t)\int_0^t\varphi_2(s)ds.$$

Applying Bellmann–Gronwall's inequality, we get

$$\|x(t)-x_\infty\|_H^2 \leq \left[\xi(t)\xi^{-1}(k)\log\lambda(k)+\xi(t)\int_0^t\varphi_2(s)ds\right]$$

$$\times\exp\left[\xi(t)\int_0^t\xi^{-1}(s)\varphi_1(s)ds+\xi(t)\int_0^t(\Theta\xi^{-1}(s)+\alpha_1^{-1}\frac{d\xi^{-1}(s)}{ds})ds\right].$$

By conditions (L1) and (L2), for any $\varepsilon > 0$ there exists a random integer $k_1 = k_1(w)$ such that if $k-1 \leq t \leq k$, $k \geq k_1 \vee k_0$ we have

$$\log\|x_t - x_\infty\|_H^2 \leq \log[\log\lambda^{\theta+\varepsilon}(t)(M+\varepsilon) + \log\lambda^{\mu+\varepsilon}(t)]$$

$$+ [(l+\varepsilon) + \Theta(\eta+\varepsilon) + (r\alpha_1^{-1}+\varepsilon)] \log\log\lambda(t),$$

which implies immediately that

$$\lim_{t\to\infty} \frac{\log\|x(t)-x_\infty\|_H^2}{\log\log\lambda(t)} \leq (\theta+\varepsilon) \vee (\mu+\varepsilon) + l + \Theta\eta + r\alpha_1^{-1} + \varepsilon.$$

Letting $\varepsilon \to 0$ ends up with

$$\lim_{t\to\infty} \frac{\log\|(x(t)-x_\infty)\|_H^2}{\log\log\lambda(t)} \leq \theta \vee \mu + l + \Theta\eta + r\alpha_1^{-1}. \quad \text{a.s.}$$

Finally, invoke lemma 6.3, of [5], to realize the fact that there exists a random variable $C(\omega)$ such that

$$x^*(t) \leq C(\omega)\left[\log\lambda(t)^{(\theta\vee\mu+l+\Theta\eta+r\alpha_1^{-1})/2}\right], \quad \text{a.s.},$$

and here the proof ends. ■

4. Exponential Stability Behavior

Here we shall prove the almost sure exponential stability behavior of weak solutions to stochastic Navier–Stokes equations (2). For this purpose, we utilize the following conditions:

(G1) Let $\varphi(t)$ be a continuous non negative function such that:
$\|G(t,x(t))\|_{L_2^0}^2 \leq \|x(t)-x_\infty\|_H^2\varphi(t),$

(G2) Let $\rho(t)$ be a continuous non negative function and $\mu \in \mathbb{R}$ such that:

$$\lim_{t\to\infty} \frac{(\rho(k)-2)\int_0^t \varphi(s)ds}{t} \leq \mu.$$

Theorem 4. 1. *Let $x_\infty \in V$. Suppose that the hypothesis of Theorem 3.2 and conditions (G1)–(G2) are satisfied. Then, any weak solution $x(t)$ to (2) satisfies*

$$\limsup_{t\to\infty} \frac{\log\|x(t)-x_\infty\|_H}{t} \leq -2\nu + 2\beta + \frac{2c_1}{\sqrt{\alpha_1}}\|x_\infty\|_V + 2\mu,$$

provided that

$$2\nu > 2\beta + \frac{2c_1\|x_\infty\|_H}{\sqrt{\alpha_1}} + 2\mu.$$

Proof. By Itô formula,

$$\log\|x(t)-x_\infty\|_H^2 = \log\|x(0)-x_\infty\|_H^2 - 2\int_0^t \frac{\langle \nu Ax(s), x(s)-x_\infty\rangle}{\|x(s)-x_\infty\|_H^2}ds$$

$$-2\int_0^t \frac{\langle B(x(s)), x(s)-x_\infty\rangle}{\|x(s)-x_\infty\|_H^2}ds + 2\int_0^t \frac{\langle F(x(s)), x(s)-x_\infty\rangle}{\|x(s)-x_\infty\|_H^2}ds$$

$$-2\int_0^t \frac{\|G(s,x(s))\|_{L_2^0}^2}{\|x(s)-x_\infty\|_H^2}ds + 2\int_0^t \frac{(x(s)-x_\infty, G(s,x(s))dW(s))}{\|x(s)-x_\infty\|_H^2}.$$

Further assumption of

$$\log\|x(t)-x_\infty\|_H^2 \le \log\|x(0)-x_\infty\|_H^2 - 2\int_0^t \varphi(s)ds$$

$$-2\int_0^t\left(\nu - \beta - \frac{c_1}{\sqrt{\alpha_1}}\|x_\infty\|_V\right)ds + 2\int_0^t \frac{(x(s)-x_\infty, G(s,x(s))dW(s))}{\|x(s)-x_\infty\|_H^2}.$$

with Lemma 2. 2 implies that

$$P\left\{\omega : \sup_{0\le t\le \omega}\left[\int_0^t \frac{(x(s)-x_\infty, G(s,x(s))dW(s))}{\|x(s)-x_\infty\|_H^2} - \frac{u}{2}\int_0^t \frac{2\|G(s,x(s)\|_{L_2^0}^2}{\|x(s)-x_\infty\|_H^2}ds \ge v\right]\right\} \le e^{-uv}.$$

Then by assuming $\varepsilon > 0$ to be arbitrarily small, and taking

$$u = \rho(\varepsilon k), \quad v = \rho(\varepsilon k)^{-1}\log k, \quad w = k\varepsilon, \quad k = 1,2,\ldots,$$

together with the Borel–Cantelli lemma, we realize that for almost all $\omega \in \Omega$, there exists an integer $k_0(\varepsilon,\omega) > 0$ such that

$$\int_0^t \frac{(x(s)-x_\infty, G(s,x(s)dW(s))}{\|x(s)-x_\infty\|_H^2} \le \rho(k\varepsilon)^{-1}\log k + \rho(k\varepsilon)\int_0^t \frac{\|G(s,x(s))\|_{L_2^0}^2}{\|x(s)-x_\infty\|_H^2}ds.$$

Therefore,

$$\log\|x(t)-x_\infty\|_H^2 \le \log\|x(0)-x_\infty\|_H^2 - 2\int_0^t\left(\nu - \beta - \frac{c_1}{\sqrt{\alpha_1}}\|x_\infty\|_V\right)ds$$

$$-2\int_0^t \varphi(s)ds + 2\rho(k\varepsilon)^{-1}\log k + \rho(k\varepsilon)\int_0^t \varphi(s)ds$$

Hence,

$$\log\|x(t)-x_\infty\|_H^2 \le \log\|x(0)-x_\infty\|_H^2 - 2\int_0^t\left(\nu - \beta - \frac{c_1}{\sqrt{\alpha_1}}\|x_\infty\|_V\right)ds$$

$$+2\rho(k\varepsilon)^{-1}\log k + [\rho(k\varepsilon) - 2]\int_0^t \varphi(s)ds,$$

for all $(k-1)\varepsilon \le t \le k\varepsilon$, $k \ge k_0(\varepsilon,\omega) \vee k_1(\varepsilon)$; from which it immediately follows that

$$\limsup_{t\to\infty} \frac{\log\|x(t)-x_\infty\|_H}{t} \le -2\nu + 2\beta + \frac{2C_1}{\sqrt{\alpha_1}}\|x_\infty\|_V + 2\mu, \quad \text{a.s.} \qquad \blacksquare$$

Acknowledgements
The author wishes to thank the referee for his/her very helpful comments and suggestions. He would also like to thank Professor T. E. Govindan for interesting remarks.

References

[1] A. Bensoussan, Stochastic Navier–Stokes equations, *Acta Applicandae Mathematicae* **38**, (1995), 367–304.

[2] A. Bensoussan, and R. Temam, Stochastic Navier–Stokes, *Journal of Functional Analysis* **13**, (1973), 195–222.

[3] T. Caraballo, J. A. Langa, and T. Taniguchi, The Exponential Behavior and Stabilizability of Stochastic 2*D* Navier–Stokes Equations, *Journal of Differential Equations* **179** (2), (2002), 714–737.

[4] M. Capinski, and N. J. Cutland, Existence of global stochastic flow and attractors for Navier–Stokes equations, *Probability Theory and Related Fields* **115**, (1999), 121–151.

[5] S. Chadrasekhar, *Stochastic*, *Statistical and Hydromagnetic Problems in Physics and Astronomy*, *Selected Papers* Vol. 3, University of Chicago Press, Chicago, 1989.

[6] N. J. Cutland, and K. Grzesiak, Optimal control for two–dimensional stochastic Navier–Stochastic equation, *Applied Mathematics and Optimization* **55**, (2007), 61–91.

[7] G. Da Prato, and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[8] B. Ferrario, Pathwise regularity of non–linear Itô equations, *Journal of Mathematical Analysis and Applications* **19**, (2001), 135–150.

[9] B. Ferrario, Stochastic Navier–Stokes equations: Analysis of the noise to have a unique inmeasure, *Annali di Matematica Pura ed Applicata (IV)* **CLXXVII**, (1999), 331–347.

[10] K. Liu, Large time behaviour for a class of turbulence models–stochastic Burgers equations, *Journal of Physis A: Mathematical and General* **30**, (1997), 3991–4004.

[11] X. Mao, *Exponential Stability of Stochastic Differential Equations*, Dekker, NY, 1994.

[12] X. Mao, and K. Liu, Exponential stability of non–linear stochastic evolution equations, *Stochastic Processes and their Applications* **78**, (1998), 173–193.

[13] E. Pardoux, *Équations aux Dérivées Partielles Stochastiques Nonlinéaires Monotones*, Thesis, Université Paris XI, 1975.

[14] R. Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1995.

[15] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer–Verlag, NY/Berlin, 1988.

Article history: Submitted March, 04, 2011; Revised May,15, 2011; Accepted June, 02, 2011.