Journal of Numerical Mathematics and Stochastics,* 3** (1) : 37-45, 2011 © ***JNM@S
http://www.jnmas.org/jnmas3-5.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

Parametric Estimation for SDEs With Additive Sub-Fractional Brownian Motion*

A. DIEDHIOU, C. MANGA, and I. MENDY

Département de Mathématiques, Université de Ziguinchor, B. P. 523 Ziguinchor, Sénégal,
E-mail: asana@usad.sn

Abstract. *We study the maximum likelihood estimator for stochastic differential equations with additive sub-fractional Brownian motion. The study applies Girsanov transform to the sub-fractional Brownian motion and employs the theory of regularity and supremum estimation for Gaussian process.*

Key words : Sub-fractional Brownian Motion, Maximum Likelihood Estimator, Girsanov Transform.

AMS Subject Classifications : 60G15, 60G35, 62M09, 62M40

1. Introduction

Recent developments in stochastic calculus to engulf fractional Brownian motion have led to many applications to the statistical inference. As a result many authors are currently interested in the study of parameter estimation problems for diffusion type processes satisfying stochastic differential equations driven by the fractional Brownian motion.

Among these we may cite the work of Prakasa Rao [13, 12], considered pioneering for this method of estimation. Other authors have studied these aspects too, see e.g. Comte [2] Norros et al. [11], Le Breton [8], Decreusefond and Üstünel [3], Kleptsyna et al. ([7],[6]), Tudor et al.[16]. The case of parameter estimation for stochastic equations with an additive fractional Brownian sheet was studied by Tudor et al.[14]. An obvious extension of this work is to study the sub-fraction Brownian motion case. Elements of stochastic calculus covering sub-fractional Brownian motion have recently been considered by Tudor [15], and stochastic differential

*This work was partially supported by grants from FIRST (Fonds d'Impulsion pour la Recherche Scientific et Technique.)

equations driven by sub-fractional Brownian motion has also been considered by Mendy [9].

The aim of this work is to construct the maximum likelihood estimator (MLE) for the parameter θ in the equation

$$X_t = \theta \int_0^t b(X_s)ds + S_t^H, \; 0 \leq t \leq T, \tag{1}$$

where θ is unknown constant drift, S^H is a sub-fractional Brownian motion with Hurst parameter $H \in (0,1)$ and b is a function which satisfies a linear growth condition. Our construction of the estimator is based on the Girsanov transform and uses the relation between the sub-fractional Brownian motion and standard Brownian motion and the theory of regularity and supremum estimation for Gaussian process. In this respect, recently Mendy [10] has studied parameter estimation problems for sub-fractional Ornstein-Ulhenbeck process by using the Girsanov transform, the relation between sub-fractional Brownian motion, Malliavin calculus and Gaussian regularity theory.

This paper is organized as follows. Section 2 contains some preliminaries on the sub-fractional Brownian motion. In section 3, we give estimate of the solution of (1). Section 4 contains the proof of the existence of the MLE for the parameter θ. Finally, in section 5, we present two different expressions of the MLE.

2. Sub-fractional Brownian Motion

Let $S^H = \{S_t^H, t \in [0,T]\}$ be a sub-fractional Brownian motion with Hurst parameter $H \in (0,1)$, in a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$. S^H is a centered Gaussian process with covariance function C_H given by :

$$\mathbb{E}[S_s^H S_t^H] = s^H + t^H - \tfrac{1}{2}[(s+t)^H + |s-t|^H].$$

This process was introduced by Bojdecky et al. [1] as an intermediate process between standard Brownian and fractional Brownian motions.

In fact sub-fractional Brownian motion S^H has in the past been represented by a standard Brownian motion W that is constructed from it. For details and references on this construction see Tudor [15] and Dzhaparidze et al. [4]. Consider now the kernels n_H and ψ_H introduced in [4] and in [15] viz.

$$\begin{aligned} n_H(t,s) &= \frac{\sqrt{\pi}}{2^H \Gamma(H+\frac{1}{2})} s^{\frac{1}{2}-H} \frac{d}{ds}\left(\int_s^t (x^2-s^2)^{H-\frac{1}{2}} dx\right) 1_{(0,t)}(s) \\ &= \frac{\sqrt{\pi}}{2^H \Gamma(H+\frac{1}{2})} s^{\frac{3}{2}-H}\left(\frac{(x^2-s^2)^{H-\frac{1}{2}}}{t} + \int_s^t \frac{(x^2-s^2)^{H-\frac{1}{2}}}{x^2} dx\right) 1_{(0,t)}(s) \end{aligned}$$

and

$$\begin{aligned} \psi_H(t,s) = \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2}-H)} \Big[& t^{H-\frac{3}{2}} (t^2-s^2)^{\frac{1}{2}-H} - \\ & (H-\frac{3}{2}) \int_s^t (x^2-s^2)^{\frac{1}{2}-H} x^{H-\frac{3}{2}} dx \Big] \; 1_{(0,t)}(s) \end{aligned} \tag{2}$$

The following lemma due to Mendy [9] gives an estimate of ψ which we use throughout this paper.

Lemma 2. 1. *(i) For every $H < \frac{1}{2}$ and $T > 0$,*

$|\psi_H(t,s)| \leq C(H)s^{2H-2}, \ 0 \leq s < t \leq T,$

(ii) For every $H > \frac{1}{2}$ and $T > 0$,

$|\psi_H(t,s)| \leq C(H)s^{H-3/2}(t-s)^{1/2-H} + C'(H)s^{H-3/2}, \ 0 \leq s < t \leq T,$

where $C(H)$ and $C'(H)$ are two generic positive constants depending only on H.

According to Dzhaparidze et al. [4] and Tudor [15], we have the relations between the sub-fractional Brownian motion and the Brownian motion constructed by it, established by the following result.

Theorem 2. 1. *The process*

$$W_t = \int_0^t \psi_H(t,s)dS_s^H. \tag{3}$$

is the unique Brownian motion such that

$$S_t^H = c(H)\int_0^t n_H(t,s)dW_s, \tag{4}$$

where

$$c^2(H) = \frac{\Gamma(1+2H)\sin\pi H}{\pi}.$$

Moreover S^H and W generate the same filtration.

In what follows we shall denote by n_H also the operator on $L^2([0,T])$ induced by the kernel

$$n_H(f)(t) = \int_0^t n_H(t,s)f(s)ds,$$

and similarly for ψ_H. Note that the operator ψ_H is indeed the inverse of the operator n_H.

3. On the Solution

Consider equation (1) driven by the sub-fractional Brownian motion S^H with Hurst parameter $H \in (0,1)$. Suppose that $X_0 = 0$ and $\theta > 0$. Equation (1) has, by the way, been considered by Mendy [9] for Hurst parameter $H \in (0,1)$ in the more general context $b(x) = b(s,x)$ with $s \in [0,T]$. It has been proved in [9] that, if b satisfies a linear growth condition

$$\sup_{s\in[0,T]}\sup_{x\in\mathbb{R}}|b(s,x)| \leq M(1+|x|), \tag{5}$$

then Equation (1) has a unique weak solution; which will be assumed throughout this paper.

Since our main objective is the construction of a maximum likelihood estimator from the observation of the trajectory of the process X that satisfies (1), we will need some estimates on the supremum of this processes.

Lemma 3. 1. *For every* $s,t \in [0,T]$,

$$\sup_{s\leq t}|X_s| \leq \left(Ct + \sup_{s\leq t}|S_s^H|\right)e^{Kt}. \tag{6}$$

Proof. Consideration of (5) and Gronwall's lemma in

$|X_s| \leq \theta\int_0^s |b(X_u)|du + |S_s^H|,$

leads to

$$\begin{aligned}|X_s| &\leq \theta\int_0^s C(1+|X_u|)du + \sup_{u\leq s}|S_s^H| \\ &\leq \left(Ct + \sup_{u\leq s}|S_s^H|\right)e^{Cs},\ s\in[0,T].\end{aligned}$$

■

4. Maximum Likelihood Estimator

Our construction is based on the following observation (see Mendy [9]). Given an adapted process with integrable trajectories $u = \{u_t, t\in[0,T]\}$, consider the transform

$$\tilde{S}_t^H = S_t^H + \int_0^t u_s ds, \tag{7}$$

to write

$$\begin{aligned}\tilde{S}_t^H &= S_t^H + \int_0^t u_s ds = c(H)\int_0^t n_H(t,s)dW_s + \int_0^t u_s ds \\ &= c(H)\int_0^t n_H(t,s)d\tilde{W}_s,\end{aligned} \tag{8}$$

where

$$\begin{aligned}\tilde{W}_t &= W_t + \int_0^t\left(\int_0^s\left(\psi_H(s,r)\int_0^r u_z dz\right)dr\right)ds \\ &= W_t + \int_0^t v_s ds,\end{aligned} \tag{9}$$

with

$$v_s = \int_0^s\left(\psi_H(s,r)\int_0^r u_z dz\right)dr. \tag{10}$$

As a consequence we may deduce the following version of Girsanov's theorem for the sub-fractional Brownian motion.

Theorem 4. 1. *Let* S^H *be a sub-fractional Brownian motion and let* $u = \{u_s; s\in[0,T]\}$ *be a process adapted to the filtration generated by* S^H. *Then let* W *be a standard Brownian motion constructed from* S^H *by (3) and* v_s *be defined by (10).*

Assume further that
(i) $v_s \in L^2(\Omega \times [0,T])$,
(ii) $\mathbb{E}(V_T) = 1$,
where
$V_T = \exp(-\int_0^T v_s dW_r - \frac{1}{2}\int_0^T v_s^2 dr)$.
Then under the new probability $\tilde{P}$, *with* $\frac{d\tilde{P}}{dP} = V_T$, *the process* $\tilde{W}$ *given by (9) is a standard Brownian motion and the process* $\tilde{S}^H$ *given by (8) is a sub-fractional Brownian motion.*

The rest of this section is devoted to construction of a maximum likelihood estimator for the parameter θ in (1) by using the Girsanov theorem (4.1).

Proposition 4. 1. *Let*
$Q_t = \int_0^t \psi_H(t,r)\left(\int_0^r b(X_s)ds\right)dr$,
then, given an observation over $[0,t]$, *the MLE for* θ *in (1) is*

$$\theta_t = -\frac{\int_0^t Q_u\, dW_u}{\int_0^t Q_u^2\, du}. \tag{11}$$

Proof. let us denote by P_θ the law of the process X_t that is unique for (1). Then the MLE is obtained by taking $\sup_\theta F_\theta$, where $F_\theta = \frac{dP_\theta}{dP_0}$. The conclusion (11) follows then by Girsanov theorem (4.1) if we show that V_T is well-defined and $\mathbb{E}(V_T) = 1$. The pertaining Q_t would, in view of (5) and Lemma 2.1, satisfy

$$|Q_t| \leq \int_0^t |\psi_H(t,r)| \int_0^r |b(X_u)|\, dudr$$

$$\leq \int_0^t |\psi_H(t,r)| \int_0^r C(1+|(X_u)|)\, dudr \tag{12}$$

$$\leq \int_0^t r\,|\psi_H(t,r)| \int_0^r C\left(1+\sup_{u\leq s} |(X_u)|\right) ds$$

$$\leq C\int_0^t r\,|\psi_H(t,r)|\, ds \left(1+\sup_{u\leq s} |(X_u)|\right)$$

$$\leq C(H,T)\left(1+\sup_{u\leq s}|(X_u)|\right). \tag{13}$$

Notice that Q_t is an adapted process and taking into account that X_t has the same regularity properties as the sub-fractional Brownian motion, we deduce that $Q \in L^2([0,T])$ almost surely and V_t is well defined.

To prove that $\mathbb{E}(V_T) = 1$, it suffices to invoke Theorem 1.1, page 152 in Frieman [5] and to note, by (6) and (13), that there exists an $a > 0$, such that

$$\sup_{0\leq u\leq t} \mathbb{E}(\exp[a\, Q_u^2]) \leq \infty. \qquad \blacksquare$$

5. Alternative Form of the Estimator

In this section, we will give other forms for the maximum likelihood estimator. By (1), and via integrating the quantity $\psi_H(t,s)$ with respect s between 0 and t, we can write

$$\int_0^t \psi_H(t,s)dX_s = \theta \int_0^t \psi_H(t,s)b(X_s)ds + W_t. \tag{14}$$

On the other hand, according to (1), we have

$$X_t = c(H)\int_0^t n_H(t,s)d\tilde{W}_s, \tag{15}$$

where $\tilde{W}$ is given by

$$\tilde{W}_t = W_t + \int_0^t \left(\int_0^s \psi_H(s,r)\left(\int_0^r b(X_s)ds\right)dr\right)ds. \tag{16}$$

Therefore, the following equality

$$\tilde{W}_t = \int_0^t \psi_H(t,s)dX_s. \tag{17}$$

holds, Then (14)-(17) we obtain

$$\int_0^t \left(\int_0^s \psi_H(s,r)\left(\int_0^r b(X_s)ds\right)dr\right)ds = \int_0^t \psi_H(t,s)b(X_s)ds.$$

Thus the function $t \mapsto \int_0^t \psi_H(t,s)X_s ds$ is absolutely continuous with respect the Lebesgue measure and

$$Q_t = \frac{d}{dt}\int_0^t \psi_H(t,s)b(X_s)ds. \tag{18}$$

By (16), the MLE is obtain by taking the $\sup_\theta F_\theta$, where

$$F_\theta = \log\left(\frac{dP_\theta}{dP_0}\right) = -\theta\int_0^t Q_s d\tilde{W}_s + \frac{1}{2}\theta^2\int_0^t Q_s^2 ds.$$

As a consequence, the maximum likelihood estimator θ_t has the equivalent form

$$\theta_t = \frac{\int_0^t Q_s d\tilde{W}_s}{\int_0^t Q_s^2 ds}. \tag{19}$$

The last formula shows explicitly that the estimator θ_t is observable if it is possible to observe the whole trajectory of the solution X.

Now we derive a form for the MLE by using the sub-fractional fundamental martingale given by Tudor [15].

Denote

$d_H = \frac{2^{H-1/2}}{C(H)\Gamma(3/2-H)\sqrt{\pi}}$,

in the process

$$M_t^H = d_H \int_0^t s^{1/2-H} dW_s, \tag{20}$$

where W is a standard Brownian motion, which is called the sub-fractional fundamental martingale. Since

$W_t = \int_0^t \psi_H(t,s) dS_s^H$,

we have

$$M_t^H = \int_0^t k_H(t,s) dS_s^H, \tag{21}$$

and $\omega_H(t) =< M >_t = \lambda_H t^{2-2H}$, where $k_H(t,s) = d_H s^{1/2-H} \psi_H(t,s)$ and $\lambda_H = \frac{d_H^2}{2-2H}$. The integral in (21) can be defined in a Wiener sense with respect to the sub-fractional Brownian motion. The filtration generated by M coincides with the one generated by S^H.

Let us integrate the deterministic kernel $k_H(t,s)$ with respect to both sides of (1) and get

$$Z_t = \int_0^t k_H(t,s) dX_s = \theta \int_0^t k_H(t,s) b(X_s) ds + M_t. \tag{22}$$

We deduce that

$X_t = \int_0^t K_H(t,s) dZ_s$

where $K_H(t,s) = \frac{C(H)}{d_H} s^{H-1/2} n_H(t,s)$. The sample paths of the process $\{X_t, t \geq 0\}$ are smooth enough so that the process R_t represented by

$$R_t = \frac{d}{d\omega_H(t)} \int_0^t k_H(t,s) b(X_s) ds, \; t \in [0,T]. \tag{23}$$

is well-defined where the derivative is understood in the sense of absolute continuity with respect to the measure generated by ω_H. Moreover the sample paths of the process Q_t belong to $L^2([0,T])$ a.s. Finally from (22) and (23) we obtain that

$$Z_t = \theta \int_0^t R_s d\omega_H(s) + M_t. \tag{24}$$

and then the MLE for the parameter θ in (1) can be written as

$$\theta_t = \frac{\int_0^t R_s dM_s}{\int_0^t R_s^2 d\omega_H(s)}. \tag{25}$$

Remark 5. 1. If θ_0 is the true parameter, it can be shown that

$$\frac{dP_{\theta}^{T}}{dP_{\theta_0}^{T}} = \exp[(\theta_0 - \theta)\int_0^T R_s dM_s] - \frac{1}{2}(\theta_0 - \theta)^2 \int_0^T R_s^2 d\omega_H(s).$$

Consequently,

$$\hat{\theta}^T - \theta_0 = \frac{\int_0^t R_s dM_s}{\int_0^t R_s^2 d\omega_H(s)}.$$

To show that $\hat{\theta}^T$ is strongly consistent, that is $\lim_{T\to+\infty} \hat{\theta}^T - \theta_0 = 0$, one can use theorems 3.1 and 3.2 of [13].

References

[1] T. L. G. Bojdecki, L. G. Gorostiza, and A. Talarczyk, Sub-fractional Brownian motion and its relation to occupation times, *Statistics and Probability Letters* **69**, (2004), 405-419.

[2] F. Comte, Simulation and estimation of long memory continuous-time models, *Journal of Time Series Analysis* **17**(1), (1996), 19-36.

[3] L. Decreusefond, and A. S. Üstünel, Stochstic analysis of the fractional Brownian motion, *Potential Analysis* **10**, (1999), 177-214.

[4] K. Dzhaparidze, and H. Van Zanten, A series expansion of fractional Brownian motion, *Probability Theory and Related Fields* **130**, (2004), 39-55.

[5] A. Frieman, *Stochastic Differential Equations and Applications*, Academic Press, NY (1975).

[6] M. L. Kleptsyna, and A. Le Breton, Some explicit statistical results about elementary fractional type models, *Nonlinear Analysis: Theory Methods and Applications* **47**, (2001), 4783-4794.

[7] M. L. Kleptsyna, A. Le Breton, and M. C. Roubaud, Parameter estimation and optimal filtering for fractional type stochastic systems, *Statistical Inference for Stochastic Processes* **3**, (2000), 173-182.

[8] A. Le Breton, Filtering and parameter estimation in a sample linear model driven by a fractional Brownian motion, *Statistics and Probability Letters* **38**(3), (1998), 263-274.

[9] I. Mendy, A Stochastic differential equation driven by a sub-fractional Brownian motion, (2010), Preprint.

[10] I. Mendy, and A. F. Yode, Parameter estimation of the sub-fractional Ornstein-Uhlenbeck type process, (2010), Preprint.

[11] I. Norros, E. Valkeila, and J. Virtamo, An elementary approach to a Girsanov formula and other analytic results on fractional Brownian motions, *Bernoulli* **5**(4), (1999), 571-587.

[12] B. L. S. Prakasa Rao, Parametric estimation for linear stochastic differential equations driven by fractional Brownian motion, *Random Operators and Stochastic Equations* **11**(3), (2003), 229-242.

[13] B. L. S. Prakasa Rao, *Statistical Inference for Diffusion Type Processes*, Arnold and Oxford University Press, NY (1999).

[14] T. Söttinen, and C. A. Tudor, parameter estimation for stochastic equations with additive fractional Brownian sheet, *Statistical Inference for Stochastic Processes* **11**, (2008), 221-236.

[15] C. A. Tudor, On the Wiener integral with respect to a sub-fractional Brownian motion on interval, *Journal of Mathematical Analysis and Applications* **351**, (2007), 456-468.

[16] C. A. Tudor, and F. Viens, Statistic Aspect of the fractional stochastic calculus, *The Annals of Statistics* **35** (3), (2007), 1183-1212.

Article history : Submitted February, 18, 2011; Accepted May, 10, 2011.