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Numerical Evaluation of the Stochastic Integral Using Mechanical Quadrature Rules

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Abstract. *Many authors have investigated numerical methods for solving stochastic differential equations and their boundary value problems. The situation with numerical stochastic integration, as a subject of its own, has so far been different. It has not been studied with the same vigor as of investigating ordinary numerical integration. Consequently, a timely question appears to arise here on whether usual numerical quadrature methods can be employed to evaluate the stochastic integral. This paper is an attempt to answer this question by presenting a modified mechanical quadrature rule for this purpose. Convergence of this numerical integration method is also investigated and a pertaining result is reported.*

Key words : Itô Calculus, Stochastic Integral, Brownian Motion, Orthogonal Polynomials.

AMS Subject Classifications : 41A10, 41A30, 65C05

1. Introduction

Stochastic calculus, which is almost as old as the ordinary calculus, see, e.g., [5], naturally appears when modeling real life physical systems, where purely deterministic partial differential equations (PDEs) fail to predict the real trajectories of the associated dynamics. In many such situations it turns out to be quite reasonable to modify the PDEs just by embedding the possibility of random effects into the dynamics of these systems. Such modifications may result with an explicit distinction between the stochastic and the deterministic approaches, see [4], to modeling. Generally speaking, when separate deterministic and stochastic terms (or parts) are incorporated in a model, there is a likelihood for a need to evaluate two types of integrals: ordinary and stochastic. Interestingly, despite the ongoing increasing research in numerical methods for stochastic differential equations (SDEs), still there is apparently less reported work on numerical methods for stochastic integrals, see [1,7], in comparison with the existing huge numerical literature on ordinary integrals. Consequently, a timely question appears to arise here on whether usual numerical quadrature methods can be employed to evaluate a stochastic integral $\int_0^T G\, dW,$ in which G is an ordinary function and W is a random variable. This paper is an attempt to answer this question by presenting a modified mechanical quadrature rule for this Itô stochastic integral.

The rest of this paper consists of two sections. In section 2, we define the previous stochastic integral and establish its notations with relevant preliminaries. In section 3, we present the numerical quadrature method that approximates the stochastic integral. Convergence of this numerical integration method is also investigated in this section and a pertaining result is reported.

2. Definitions and Preliminaries

Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ be the collection of all outcomes, $\omega_i, i = 1,2,3,..$ of a random experiment and let $A \subset \Omega$. Define then a measure $P : A \to [0,1]$ with $P(\Omega) = 1$. The triplet $(\Omega,\ A,\ P)$ is the usual probability space, which is measurable. A mapping X from Ω into R is a random variable of this experiment, that is measurable with respect to A, if and only if $\{\omega \in \Omega : X(\omega) \in B\} \in A$ holds for all Borel-sets, B.

A stochastic process can be defined by means of this X if it is additionally parameterized as follows.

Definition 2.1. A collection $\{X(t,.)\ : t \geq 0\}$ of random variables $X : \Omega \times T \to R$, $T = [0,\infty]$ is called a stochastic process. Here for each point $\omega \in \Omega$, the mapping $t \to X(t,\omega)$ is a realization, sample path or trajectory of the stochastic process.

Definition 2.2. A real-valued stochastic process that depends continuously on $t \in [0,T]$ $W = \{W(t), t \geq 0\},\ W : T \times \Omega \to R$ is called a Brownian motion (BM) or a Wiener process if

1. $W(0) = 0$ a.s.,
2. $W(t) - W(s)$ is $N(0, t-s)$, Gaussian distribution with mean 0 and variance $s - t$, for all $t > s > 0$,
3. for all times $0 < t_1 < t_2 <\ldots< t_n$, the random variables $W(t_1), W(t_2) - W(t_1), \ldots,$ $W(t_n) - W(t_{n-1})$ are independent.

Also, the expected values of $W(t)$ and $W^2(t)$ are given respectively by $E(W(t))\ = 0,$ $E(W^2(t))\ = t$ for each time $t > 0$.

Definition 2.3. Let $0 < t_1 < t_2 <\ldots< t_n$ be a partition of the interval $[0,T]$ and let $h(W(t),t) = h(t)$ be a continuous function on $[0,T]$. The stochastic integral $\int_0^T h(t)dW(t)$, which satisfies $E\left(\int_0^T h(t)dW(t)\right) = 0,$ is defined according to Itô as

$$I(h) = \int_0^T h(t)dW(t) = \lim_{n\to\infty} \sum_{j=0}^{n} h(t_j)(W(t_{j+1}) - W(t_j)). \tag{1}$$

Among the many other definitions of $\int_0^T h(t)dW(t)$ is the Stratonovich integral, conceived as

$$\int_0^T h(t)dW(t) = \lim_{n\to\infty} \sum_{j=0}^{n} h\left(\frac{t_j + t_{j+1}}{2}\right)(W(t_{j+1}) - W(t_j)).$$

Definition 2.4. (Convergence) If $I(h)$ is the exact value of a stochastic integral and $\tilde{I}_n(h)$ is its approximate value, then we have

1. Strong convergence when

$$E(|I(h) - \tilde{I}_n(h)|) \to 0, \; n \to \infty.$$

2. Weak convergence when

$$|E(I(h)) - E(\tilde{I}_n(h))| \to 0, \; n \to \infty.$$

3. Mean square convergence when

$$E(|I(h) - \tilde{I}_n(h)|^2) \to 0, \; n \to \infty.$$

Examples: According to Itô's definition one has

1. $\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T,$
2. $\int_0^T h_{m+1}(W,t)dW(t) = h_m(W,t),$

where $h_m(t)$ is the Hermite polynomial of degree m given by

$$h_m(x,t) = \frac{(-t)^m}{m!} e^{\frac{x^2}{2t}} \frac{d^m}{dt^m}(e^{\frac{-x^2}{2t}}). \qquad (2)$$

3. Main Results

In this section, we will show that the stochastic integral (1) could be approximated by a Gaussian quadrature. To arrive at our main result (Theorem 3.1), we need first to state the following lemmata.

Lemma 3. 1. [2] *If a random variable X is distributed as* $N(\mu,\sigma^2)$*,a Gaussian distribution, then the random variable* $Y = a + bX, \; b \neq 0$ *is distributed as* $N(a + b\mu, b^2\sigma^2)$.

Lemma 3. 2. *If* $W(t)$ *is a Wiener process, and* $V(t) = \frac{2}{T}W(\frac{T}{2}(t+1)) - t$*, then* $V(t) \sim N(-t, \frac{2(1+t)}{T})$.

Proof. The proof of this lemms is trivial; by assuming

$$a = -t, \quad b = \frac{2}{T}.$$

in Lemma 3.1.

Theorem 3.1. *Let* $\xi_1 < \xi_2 < \xi_3 < \ldots < \xi_N$ *be the roots of a Legendre polynomial of degree N and let* $\varsigma_k, \; k = 1,2,\ldots,N$ *be the corresponding weights. A quadrature approximation to the stochastic integral (1) is*

$$\int_0^T h(t)dW(t) = \sum_{k=1}^{N} \varrho_k\, g(\xi_k) + R(f), \tag{3}$$

where $R(f)$ is a remainder term of this formula and $\rho_k \sim N(0, \frac{4(\xi_{k+1}-\xi_k}{T^2})$.

Proof. Use the linear transformation $\xi = \frac{T(1+t)}{2}$, to write

$\int_0^T h(t)dW(t) = \int_{-1}^1 g(\xi)dW^*(\xi),$

where $g(t) = h(\frac{T}{2}(1+t))$ and $W^*(t) = W(\frac{T(1+t)}{2})$. Setting $W^*(t) = \frac{T}{2}\ (t + V(t))$ leads to

$\int_0^T h(t)dW(t) = \frac{T}{2}\left\{\int_{-1}^1 g(t)dt + \int_{-1}^1 g(t)dV(t)\right\} = \frac{T}{2}\{I_1 + I_2\}.$

Since the function $g(t)$ is assumed to be continuous, then a quadrature formula could be used to represent I_1 by

$I_1 = \sum_{k=1}^N \varsigma_k\, g(\xi_k) + R_1(f).$

According to Itô's definition of the stochastic integral, the second integral, I_2, is equal to

$$I_2 = \sum_{k=1}^{N} g(\xi_k)(V(\xi_{k+1}) - V(\xi_k)) + R_2(f) = \sum_{k=1}^{N} g(\xi_k)\, \zeta_k + R_2(f)$$

where

$\zeta_k = V(\xi_{k+1}) - V(\xi_k).$

Hence,

$$\int_0^T h(t)dW(t) = \frac{T}{2} \sum_{k=1}^{N} \varrho_k\, g(\xi_k) + R(f), \tag{4}$$

where

$\varrho_k = \varsigma_k + \zeta_k, \quad R(f) = \frac{T}{2}\ \{R_1(f) + R_2(f)\}.$

Apply finally Lemma 3.2 to Itô's definition of the stochastic integral to conclude that

$$\varrho_k \sim N(0, \frac{4(\xi_{k+1} - \xi_k}{T^2}), \tag{5}$$

and to end the proof. ■

The ϱ_k's of (5) are random numbers. Moreover, for the construction of quadrature formulae with minimal error, the ϱ_k's should be nonnegative numbers. It should be noted however that the error $R_2(f)$ has no closed form; unlike the error $R_1(f)$ which has a closed form that depends on the differentiability of the integrand $h(t)$, see [3].

3.1. **Convergence**

Theorem 3. 2. *If the stochastic integral*

$I(h) = \int_0^T h(t)dW(t),$

exists and is approximated by

$$\tilde{I}(h) = \tfrac{T}{2} \sum_{k=1}^{N} \varrho_k \, g(\xi_k),$$

then

$$(i)\ E(I(h) - \tilde{I}(h)) = 0,$$

$$(ii)\ E([I(h) - \tilde{I}(h)]^2) \le B \left(\tfrac{\log N}{N} \right)$$

where B is a constant.

Proof. It is straightforward to see that

$$0 \le |E(I(h) - \tilde{I}(h))| \le |E(I(h)| + |E(\tilde{I}(h))| = 0 + \left| E(\tfrac{T}{2} \sum_{k=1}^{N} \varrho_k \, g(\xi_k) \right|$$

$$\le \tfrac{T}{2} \sum_{k=1}^{N} |E(\varrho_k)| \, |E(g(\xi_k))| = 0,$$

which implies the strong convergence of (i). To prove the second part (ii), let us obtain an estimate for $E([I(h)]^2)$ as follows.

$$E([I(h)]^2) = E\left(\left[\int_0^T h(t) dW(t) \right]^2 \right) = E\left(\left[\sum_{k=1}^{N} h_k \, (W_{k+1} - W_k) \right]^2 \right)$$

$$= \sum_{k=1}^{N} E([h_k \, (W_{k+1} - W_k)]^2) = \sum_{k=1}^{N} E([h_k]^2) \, E([W_{k+1} - W_k)]^2)$$

$$= \sum_{k=1}^{N} E([h_k]^2) \, (t_{k+1} - t_k) = T \sum_{k=1}^{N} E([h_k]^2)$$

Since $g(t) = h(\tfrac{T}{2}(1 + t))$, then

$$E([I(h)]^2) \le A \sum_{k=1}^{N} E([g_k]^2), \tag{6}$$

always holds for some constant A. Subsequently

$$E([I(h) - \tilde{I}(h)]^2) = E\left(\left[I(h) - \tfrac{T}{2} \sum_{k=1}^{N} \varrho_k \, g(\xi_k) \right]^2 \right)$$

$$= E\left(I^2(h) - 2I(h) \sum_{k=1}^{N} \varrho_k \, g(\xi_k) + \left[\sum_{k=1}^{N} \varrho_k \, g(\xi_k) \right]^2 \right)$$

$$= E(I^2(h)) + \sum_{k=1}^{N} E((\varrho_k\, g(\xi_k))^2) + 2 \sum_{k=1}^{N} \sum_{k=1}^{N} E(\varrho_k\, \varrho_j\, g(\xi_k))\, g(\xi_j))$$

$$= E(I^2(h)) + \sum_{k=1}^{N} E(\varrho_k\, E(g(\xi_k))^2)$$

$$\leq A \sum_{k=1}^{N} E([g_k]^2) + \frac{4}{T^2} \sum_{k=1}^{N} E(g(\xi_k)^2)\,(\xi_{k+1} - \xi_k).$$

As roots of an orthogonal polynomial satisfy [6],

$$\xi_{k+1} - \xi_k \leq C\, \frac{\log N}{N},$$

then there exists a constant B such that

$$E([I(h) - \tilde{I}(h)]^2) \leq B\left(\frac{\log N}{N}\right),$$

and here the proof ends. ■

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