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On the Existence of Solutions to Fully Coupled RFBSDEs With Monotone Coefficients*

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Abstract. *In this work we prove the existence of a solution of a class of Reflected Forward Backward Stochastic Differential Equations (RFBSDEs) by weakening the usual Lipschitz conditions on the generator of the backward equation and the drift of the forward equation. These coefficients are monotonic but can be discontinuous and the diffusion term can be degenerate.*

Key words : Reflected Forward Backward Stochastic Differential Equations, Comparison Theorem, Increasing Process.

AMS Subject Classifications : 60H10, 35B51

1. Introduction

The aim of this work consists in finding a solution to a class of RFBSDEs under monotonic hypotheses on the generator of the backward equation and the drift of the forward equation. More precisely, we consider the coupled system of SDEs

$$\left.\begin{aligned} X_t &= x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s)\, dW_s , \\ Y_t &= \Gamma + \int_t^T f(s, X_s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s\, dW_s , \\ Y_t &\geq L_t , \int_0^T (Y_t - L_t) dK_t = 0. \end{aligned}\right\} \qquad (1)$$

Fully coupled FBSDEs can be encountered in various problems such as the probabilistic representation of viscosity solutions of quasilinear partial differential equations (PDEs), (see [5]), and stochastic optimal control, among others. In 1999, fully coupled forward-backward

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stochastic differential equations and their connection with PDEs were studied intensively by Pardoux and Tang (see [16]). In 2006, Antonelli and Hamadène (see [1]) gave one existence result for coupled FBSDEs under a non-Lipschitzian assumption. Unfortunately, most existence or uniqueness results on solutions of forward-backward stochastic differential equations need regularity assumptions. The coefficients are required to be at least continuous which is somehow too strong for some applications.

In 2008, inspired by [1], Ouknine and Ndiaye (see [14]) gave the first result which proves existence of a solution of a forward-backward stochastic differential equation with discontinuous coefficients and a degenerate diffusion coefficient where the terminal condition is not necessarily bounded. However, there are few results about reflected forward-backward stochastic differential equation in which the solution of the BSDE stays above a given bound. This work can be seen as an extension of ([14]) with the obstacle constraint.

2. Assumptions and Notations

Let $[0,T]$ be a fixed time interval. We will always take s to be in $[0,T]$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and W a d-dimensional Brownian motion defined on this space. We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration of W. We suppose also that the stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ satisfy the usual conditions.

The processes to be analysed in this work are assumed to take place in the following spaces :

(a_1) $\mathcal{S}^2$, the set of adapted and continuous processes $U=(U_t)_{0\leq t\leq T}$ such that

$$\|U\|_{\mathcal{S}^2}^2=\mathbb{E}\left(\sup_{0\leq t\leq T}|U_t|^2\right)<\infty,$$

(a_2) $\mathcal{H}^2$, the set of $\mathcal{F}_t$-progressively measurable processes Z, such that

$$\|Z\|_{\mathcal{H}^2}^2=\mathbb{E}[\int_0^T|Z_s|^2ds]<\infty,$$

(a_3) $\mathcal{S}_i^2$, the subset of $\mathcal{S}^2$ which contains non-decreasing processes $K=(K_t)_{0\leq t\leq T}$ with $K_0=0$.

3. Main Result

The main result of this work deals with $b:[0,T]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$, a measurable and bounded, by M, function, such that for all $s\in[0,T]$, $b(s,.,.)$ is increasing and left continuous. The following approximating lemma happens to play an important role in the proof of this result.

Lemma 3. 1. [14] *For every b, $\exists$ a family of measurable functions $(b_n(s,x,y),n\geq 1,s\in[0,T],$ $x,y\in\mathbb{R})$ such that:*

(l_1) *for all sequence $(x_n,y_n)\uparrow(x,y)$, $(x,y)\in\mathbb{R}^2$ we have*

$$\lim_{n\to\infty} b_n(s,x_n,y_n)=b(s,x,y),$$

(l_2) $(x,y)\mapsto b_n(s,x,y)$ *is increasing, for all $n\geq 1$, $s\in[0,T]$,*

(l_3) *$n \mapsto b_n(s,x,y)$ is increasing, for all $x \in \mathbb{R},\ y \in \mathbb{R},\ s \in [0,T]$,*
(l_4) *$|b_n(s,x,y) - b_n(s,x',y')| \leq 2nM(|x-x'|+|y-y'|)$ for all $n \geq 1,\ s \in [0,T],\ M \in \mathbb{R}_+^*$,*
(l_5) *$\sup_{n\geq 1} \sup_{s\in[0,T]} \sup_{x,y\in\mathbb{R}} |b_n(s,x,y)| \leq M$ for all $n \geq 1,\ s \in [0,T],\ x,y \in \mathbb{R}$.*

Theorem 3. 1. *Assume the following conditions to hold.*
(i) *$f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, a measurable and bounded function such that for all $s \in [0,T],\ z \in \mathbb{R},\ f(s,.,.,z)$ is increasing, left continuous and Lipschitzian with respect to z uniformly in x, y and s i.e. $\exists\Lambda \in \mathbb{R}_+^*$ such that*

$$|f(s,x,y,z) - f(s,x,y,z')| \leq \Lambda|z-z'|,\ s \in [0,T],\ x,y,z,z' \in \mathbb{R}.$$

(ii) *$\sigma : [0,T] \times R \to R$, a continuous function satisfying the following conditions:*

$$|\sigma(s,x)| \leq \Lambda(1+|x|),$$

and

$$|\sigma(s,x) - \sigma(s,x')| \leq \Lambda|x-x'|, s \in [0,T],\ x,x' \in \mathbb{R}.$$

(iii) Γ is a random variable F_T- measurable and square integrable.
(iv) L is a continuous obstacle which is progressively measurable, real valued such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} (L_t^+)^2\right] < \infty \quad \text{and} \quad L_T \leq \Gamma \quad a.s.$$

Then the fully coupled reflected forward-backward system of stochastic differential equations (1) has at least one solution $(X,Y,Z,K) \in S^2 \otimes S^2 \otimes H^2 \otimes S_i^2$.

Proof. Consider the following RBSDE:

$$\left.\begin{aligned} &Y_t^0 = \Gamma + M\int_t^T ds + K_T^0 - K_t^0 - \int_t^T Z_s^0\, dW_s\ , \\ &Y_t^0 \geq L_t\ ,\ \int_0^T (Y_t^0 - L_t)dK_t^0 = 0, \end{aligned}\right\} \tag{2}$$

which has a unique solution satisfying $\|Y_t^0\|_{S^2} < \infty$.
Let us also define S as the unique solution of the SDE

$$S_t = x + \int_0^t M\, ds + \int_0^t \sigma(s,S_s)\, dW_s.$$

Step1: Focuses on illustrating the existence of two increasing processes $(Y^k)_{k\geq 1}$ and $(X^k)_{k\geq 1}$ satisfying:

$$\left.\begin{aligned} &X_t^k = x + \int_0^t b(s,\ X_s^k,\ Y_s^k)ds + \int_0^t \sigma(s,\ X_s^k)\, dW_s\ , \\ &Y_t^k = \Gamma + \int_t^T f(s,\ X_s^{k-1},\ Y_s^k,\ Z_s^k)ds + K_T^k - K_t^k - \int_t^T Z_s^k\, dW_s\ , \\ &Y_t^k \geq L_t\ ,\ \int_0^T (Y_t^k - L_t)dK_t^k = 0. \end{aligned}\right\} \tag{3}$$

For $n \geq 1$, (b_n) is the sequence defined in lemma 3.1. Consider then the SDE:

$$X_t^{0,n} = x + \int_0^t b_n(s, X_s^{0,n}, Y_s^0)ds + \int_0^t \sigma(s, X_s^{0,n})dW_s. \tag{4}$$

According to properties (l_4) and (l_5), this equation has a unique strong solution; and $(l_3) \Rightarrow b_n(s,x,Y_s^0) \leq b_{n+1}(s,x,Y_s^0)$. We deduce from the comparison theorem of SDEs that the sequence $(X^{0,n})_{n\geq 1}$ is increasing. Moreover, since $b_n(s,x,Y_s^0) \leq M$, then this comparison theorem implies again that $\forall t \leq T,\ X_t^{0,n} \leq S_t$ a.s. Therefore $X^{0,n} \nearrow X^0$.
We will show moreover that X^0 is a solution of the SDE(4). Indeed since $X_s^{0,n} \nearrow X_s^0$, (l_1) implies that

$$\lim_{n\to\infty} b_n(s, X_s^{0,n}, Y_s^0) = b(s, X_s^0, Y_s^0).$$

The functions $b_n(s,.,.)$ are measurable and bounded. The dominated convergence theorem happens to give

$$\int_0^t b_n(s, X_s^{0,n}, Y_s^0)ds \longrightarrow \int_0^t b(s, X_s^0, Y_s^0)ds.$$

On the other hand,

$$\mathbb{E}\Big[\int_0^t [\sigma(s, X_s^{0,n}) - \sigma(s, X_s^0)]^2 ds\Big] \leq K^2 \mathbb{E}\Big[\int_0^t \mid X_s^{0,n} - X_s^0 \mid^2 ds\Big] \to 0,$$

when $n \to \infty$. Hence we may invoke then Doob's inequality to deduce that

$$\int_0^{\cdot} \sigma(s, X_s^n)dW_s \longrightarrow \int_0^{\cdot} \sigma(s, X_s^0)dW_s,$$

(where the limit is taken in the sense of ucp's convergence).
Therefore:

$$X_t^0 = x + \int_0^t b(s, X_s^0, Y_s^0)ds + \int_0^t \sigma(s, X_s^0)dW_s.$$

Thus the couple of processes $(X_s^0, Y_s^0)_{s\in[0,T]}$ is well defined.

Let us define the random function f^1 viz

$$f^1(s,y,z) := f(s, X_s^0(\omega), y, z),$$

where by hypothesis, the function f is measurable, bounded, increasing and left continuous in the y variable. Then we can construct the following sequence of functions

$$f_n^1(s,y,z) = n\int_{y-\frac{1}{n}}^{y} f(s, X_s^0(\omega), u, z)du.$$

The (l_1), (l_4) and Lipschitz's condition with respect to z uniformly in x and y provide for the existence of a unique triplet of processes $(Y^{1,n}, Z^{1,n}, K^{1,n}) \in \mathcal{S}^2 \otimes \mathcal{H}^2 \otimes \mathcal{S}_i^2$ satisfying

$$\left.\begin{array}{l} Y_t^{1,n} = \Gamma + \int_t^T f_n^1(s, Y_s^{1,n},\ Z_s^{1,n})ds + K_T^{1,n} - K_t^{1,n} - \int_t^T Z_s^{1,n}\ dW_s\ , \\ Y_t^{1,n} \geq L_t\ ,\ \int_0^T (Y_t^{1,n} - L_t)\ dK_t^{1,n} = 0. \end{array}\right\} \tag{5}$$

Since the terminal value of the RBSDE (5) is independent on n and the function $n \mapsto f_n^1(s,.,.)$ is increasing, the comparison theorem for RBSDEs allows for,

$$\forall t \leq T, \quad Y_t^0 \leq Y_t^{1,n} \leq Y_t^{1,n+1}, \quad K_t^0 \geq K_t^{1,n} \geq K_t^{1,n+1}.$$

Now, let us prove the convergence of the sequences $(Y_t^{1,n})_{n\geq 0}$ and $(Z_t^{1,n})_{n\geq 0}$.
Since $\int_0^T (Y_t^{1,n} - L_t)dK_t^{1,n} = 0$, then it follows from Itô's formula that

$$\mathbb{E}\Big[|Y_t^{1,n}|^2 + \int_t^T |Z_s^{1,n}|^2 ds\Big] = \mathbb{E}\Big[|\Gamma|^2 + 2\int_t^T Y_s^{1,n} f_n^1(s, Y_s^{1,n}, Z_s^{1,n})ds + 2\int_t^T L_s dK_s^{1,n}\Big]$$
$$\leq \mathbb{E}|\Gamma|^2 + \epsilon^2 \mathbb{E}\Big[\int_t^T |f_n^1(s, Y_s^{1,n}, Z_s^{1,n})|^2 ds\Big] + \frac{1}{\epsilon^2}\mathbb{E}\Big[\int_t^T |Y_s^{1,n}|^2 ds\Big] + 2\mathbb{E}\Big[\int_t^T L_s dK_s^{1,n}\Big]$$
$$\leq \mathbb{E}(|\Gamma|^2) + \epsilon^2 (M.T) + \frac{1}{\epsilon^2}\mathbb{E}\Big[\int_t^T |Y_s^{1,n}|^2 ds\Big] + \frac{1}{\delta}\mathbb{E}[\sup_{t\leq s\leq T}(L_s^+)^2] + \delta \mathbb{E}\Big[(K_T^{1,n} - K_s^{1,n})^2\Big]$$
$$\leq C\Big(1 + \mathbb{E}(|\Gamma|^2) + \mathbb{E}\Big[\int_t^T |Y_s^{1,n}|^2 ds\Big]\Big) + \frac{1}{\delta}\mathbb{E}[\sup_{t\leq s\leq T}(L_s^+)^2] + \delta \mathbb{E}\Big[(K_T^{1,n} - K_s^{1,n})^2\Big],$$

where ϵ and δ are universal non-negative real constants while C is a constant depending only on M and T.
On the other hand, since

$$K_T^{1,n} - K_t^{1,n} = Y_t^{1,n} - \Gamma - \int_t^T f_n^1(s, Y_s^{1,n}, Z_s^{1,n})ds + \int_t^T Z_s^{1,n} dW_s,$$

we have

$$\mathbb{E}\Big[(K_T^{1,n} - K_t^{1,n})^2\Big] \leq \widetilde{C}\mathbb{E}\Big[|\Gamma|^2 + |Y_t^{1,n}|^2 + \Big(\int_t^T f_n^1(s, Y_s^{1,n}, Z_s^{1,n})ds\Big)^2 + \Big(\int_t^T Z_s^{1,n} dW_s\Big)^2\Big]$$
$$\leq \widetilde{C}\mathbb{E}\Big[1 + |\Gamma|^2 + |Y_t^{1,n}|^2 + \int_t^T |Z_s^{1,n}|^2 ds\Big],$$

where $\widetilde{C}$ is a constant.
Injection of this inequality in the previous one leads to

$$\mathbb{E}\Big[|Y_t^{1,n}|^2 + \int_t^T |Z_s^{1,n}|^2 ds\Big] \leq \overline{C}\Big[1 + \mathbb{E}\Big(\int_t^T |Y_s^{1,n}|^2 ds\Big)\Big],$$

where $\overline{C}$ is an appropriate real constant. Finally Gronwall's inequality gives $\mathbb{E}[|Y_t^{1,n}|^2] < \infty$.
From Itô's formula and Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t^{1,n}|^2\Big] < \infty.$$

Hence $\|Y_t^{1,n}\|_{\mathcal{S}^2} < \infty.$ Furthermore, the sequence $(Y_t^{1,n})_{n\geq 0}$ is increasing and bounded from above in $\mathcal{S}^2$. Then it converges in $\mathcal{S}^2$ to a process denoted by Y^1.

For the sequence $(Z_t^{1,n})_{n\geq 0}$, let us apply the Itô's formula to the function $x \mapsto |x|^2$ and the difference of processes $Y_s^{1,k} - Y_s^{1,h}$ between s and T . So

$$\left.\begin{aligned} |Y_t^{1,k} - Y_t^{1,h}|^2 &= 2\int_t^T (Y_s^{1,k} - Y_s^{1,h})\Big[f_1^k(s, Y_s^{1,k}, Z_s^{1,k}) - f_1^h(s, Y_s^{1,h}, Z_s^{1,h})\Big]ds \\ &\quad - 2\int_t^T (Y_s^{1,k} - Y_s^{1,h})(Z_s^{1,k} - Z_s^{1,h})dW_s - \int_t^T |Z_s^{1,k} - Z_s^{1,h}|^2 ds \\ &\quad + 2\int_t^T (Y_s^{1,k} - Y_s^{1,h})dK_s^{1,k} + 2\int_t^T (Y_s^{1,h} - Y_s^{1,k})dK_s^{1,h}. \end{aligned}\right\} \quad (6)$$

Let us introduce the local martingale $M_t = \int_0^t Y_s^{1,k} Z_s^{1,k} dW_s$, and show that it is uniformly integrable.

$$\begin{aligned}\mathbb{E}\sqrt{< M >_T} &= \mathbb{E}\sqrt{\int_0^T |Y_s^{1,k}|^2.|Z_s^{1,k}|^2 ds}\\ &\le \mathbb{E}\sqrt{\sup_{0\le t\le T}|Y_t^{1,k}|^2}\,.\sqrt{\int_0^T |Z_s^{1,k}|^2 ds}\\ &\le \tfrac{1}{2}\mathbb{E}[\sup_{0\le t\le T}|Y_t^{1,k}|^2] + \tfrac{1}{2}\mathbb{E}\Big[\int_0^T |Z_s^{1,k}|^2 ds\Big] < \infty\end{aligned}$$

On the other hand the Burkholder-Davis-Gundy inequality implies that

$$\mathbb{E}\Big[\sup_{0\le t\le T}|M_s|\Big] \le \widehat{C}\,\mathbb{E}[\sqrt{< M >_T}\,] < \infty$$

where $\widehat{C}$ is a universal constant .

Therefore (M_t) is a local and uniformly integrable martingale. So it is a martingale. The same holds for $\int_0^t (Y_s^{1,k} - Y_s^{1,h})(Z_s^{1,k} - Z_s^{1,h})dW_s$.

Taking the expectation of the left-hand side of (6)leads to

$$\begin{aligned}\mathbb{E}(|Y_t^{1,k} - Y_t^{1,h}|^2) &= 2\mathbb{E}\Big(\int_t^T (Y_s^{1,k} - Y_s^{1,h})[f_1^k(s, Y_s^{1,k}, Z_s^{1,k}) - f_1^h(s, Y_s^{1,h}, Z_s^{1,h})]ds\Big)\\ &\quad - \mathbb{E}\Big(\int_t^T |Z_s^{1,k} - Z_s^{1,h}|^2 ds\Big) + 2\mathbb{E}\Big(\int_t^T (Y_s^{1,k} - Y_s^{1,h})dK_s^{1,k}\Big)\\ &\quad + 2\mathbb{E}\int_t^T (Y_s^{1,h} - Y_s^{1,k})dK_s^{1,h}.\end{aligned}$$

Utilizing the fact that $Y_t^{1,i} \ge L_t$, $\int_0^T (Y_t^{1,i} - L_t)dK_t^{1,i} = 0,\ i \in \{h, k\}$ and Hölder's inequality, allows writing

$$\begin{aligned}\mathbb{E}(\int_0^T |Z_s^{1,k} - Z_s^{1,h}|^2 ds) &\le \mathbb{E}\Big[|Y_s^{1,k} - Y_s^{1,h}|^2 + \int_0^T |Z_s^{1,k} - Z_s^{1,h}|^2 ds\Big]\\ &\le K_1\Big[\mathbb{E}(\int_0^T [f_1^k(s, Y_s^{1,k}, Z_s^{1,k}) - f_1^h(s, Y_s^{1,h}, Z_s^{1,h})]^2 ds)\Big]^{\frac{1}{2}}\\ &\quad \times \Big[\mathbb{E}(\int_0^T (Y_s^{1,k} - Y_s^{1,h})^2 ds)\Big]^{\frac{1}{2}} \longrightarrow 0\end{aligned}$$

which is true due to the boundedness of the functions f_1^k and convergence of the the sequence $(Y_t^{1,k})_{k\ge1}$.

Clearly then $(Z_t^{1,n})_{n\ge0}$ is a Cauchy sequence in $\mathcal{H}^2$. Thus it converges to a limit $Z \in \mathcal{H}^2$.

Since

$$K_t^{1,n} = Y_0^{1,n} - Y_t^{1,n} - \int_0^t f_n^1(s, Y_s^{1,n}, Z_s^{1,n})ds + \int_0^t Z_s^{1,n} dW_s,$$

we have also $\mathbb{E}\Big[\sup_{0\le s\le T}|K_s^{1,n} - K_s^{1,p}|^2\Big] \to 0$ as $n, p \to \infty$. Hence there exists an $\mathcal{F}_t$-adapted nondecreasing and continuous process $(K_t)_{t\le T}$ (with $K_0 = 0$) such that

$$\mathbb{E}\Big[\sup_{0\le s\le T}|K_s^{1,n} - K_s^1|^2\Big] \to 0 \text{ as } n \to \infty.$$

Furthermore, since $(Y_t^{1,n}, K_t^{1,n})_{0\le t\le T}$ tends to $(Y_t^1, K_t^1)_{0\le t\le T}$ uniformly in t, in probability, then the measure $dK_t^{1,n}$ converges to dK_t^1 weakly in probability, and consequently

$\int_0^T (Y_t^{1,n} - L_t) dK_t^{1,n} \to \int_0^T (Y_t^1 - L_t) dK_t^1$ in probability as $n \to \infty$. Moreover

$$0 \le \int_0^T (Y_t^1 - L_t) dK_t^1 = \int_0^T (Y_t^1 - Y_t^{1,n}) dK_t^1 + \int_0^T (Y_t^{1,n} - L_t)(dK_t^1 - dK_t^{1,n}) + \int_0^T (Y_t^{1,n} - L_t) dK_t^{1,n}.$$

It follows then that $\int_0^T (Y_t^1 - L_t) dK_t^1 = 0$.

On the other hand, $(Y_s^{1,n}, Z_s^{1,n}) \longrightarrow (Y_s^1, Z_s^1)$ and because of (l_1) we have

$f_n^1(s, Y_s^{1,n}, Z_s^{1,n}) \longrightarrow f^1(s, Y_s^1, Z_s^1) = f(s, X_s^0, Y_s^1, Z_s^1)$.

Since the functions f_n^1 are measurable and bounded, the dominated convergence theorem implies

$\lim_{n \to \infty} \int_t^T f_n^1(s, X_s^0, Y_s^1, Z_s^1) ds = \int_t^T f(s, X_s^0, Y_s^1, Z_s^1) ds$.

We also have

$\int_t^T Z_s^{1,n} dW_s \longrightarrow \int_t^T Z_s^1 dW_s$,

with $K_T^{1,n} \to K_T^1$ in the sense that $\mathbb{E}\left[\sup_{0 \le s \le T} |K_s^{1,n} - K_s^1|^2 \right] \to 0$ as $n \to \infty$.

Finally we obtain a triplet $(Y_t^1, Z_t^1, K_t^1)_{0 \le t \le T}$ satisfying the following equation:

$Y_t^1 = \Gamma + \int_t^T f(s, X_s^0, Y_s^1, Z_s^1) ds + K_T^1 - K_t^1 - \int_t^T Z_s^1 dW_s$,

$Y_t^1 \ge L_t$, $\int_0^T (Y_t^1 - L_t) dK_t^1 = 0$.

As for the limit, we also have $\forall t \le T, \quad Y_t^0 \le Y_t^1$ and $\| Y_t^1 \|_{\mathcal{S}^2} < \infty$.

Next, Consider the forward component linked with Y^1,

$$X_t^{1,n} = x + \int_0^t b_n(s, X_s^{1,n}, Y_s^1) ds + \int_0^t \sigma(s, X_s^{1,n}) dW_s. \tag{7}$$

Since $Y^0 \le Y^1$ and $(b_n(s,.,.))_{n \ge 0}$ is increasing in space and with respect to n, we have

$b_n(s, x, Y_s^0) \le b_n(s, x, Y_s^1) \le b_{n+1}(s, x, Y_s^1)$.

we also have through the comparison theorem for SDE's that

$$\forall t \le T, \quad X_t^{0,n} \le X_t^{1,n} \le X_t^{1,n+1}. \tag{8}$$

Repeating what we have done on the construction of X^0, we can show the existence of a process X^1 in $\mathcal{S}^2$ which is an increasing limit of the sequence $(X^{1,n})_{n \ge 0}$ and such that,

$\forall t \le T, \quad X_t^1 = x + \int_0^t b(s, X_s^1, Y_s^1) ds + \int_0^t \sigma(s, X_s^1) dW_s$.

Taking the limit in(8), one gets $\forall t \le T, \quad X_t^0 \le X_t^1$.

Having found a solution $(X^1, Y^1, Z^1, K^1) \in \mathcal{S}^2 \otimes \mathcal{S}^2 \otimes \mathcal{H}^2 \otimes \mathcal{S}_i^2$ of (1), we can proceed by induction to find the anticipated solution.

Step2: Let us suppose that we have built the sequence of solutions (X^i, Y^i, Z^i, K^i) for all $i \le k-1$, i.e. for all $i = 1, \cdots, k-1$ and $t \le T$, for

$$\begin{cases} X_t^i = x + \int_0^t b(s,\ X_s^i,\ Y_s^i)ds + \int_0^t \sigma(s,\ X_s^i)\ dW_s\ , \\ Y_t^i = \Gamma + \int_t^T f\,(s,\ X_s^{i-1},\ Y_s^i,\ Z_s^i)ds + K_T^i - K_t^i - \int_t^T Z_s^i\ dW_s\ , \\ Y_t^i \geq L_t\ ,\ \int_0^T (Y_t^i - L_t)dK_t^i = 0. \end{cases}$$

where $t \leq T, \quad X_t^{i-1} \leq X_t^i, \quad Y_t^{i-1} \leq Y_t^i$ and $\| Y_t^i \|_{\mathcal{S}^2} < \infty$.
Define then the random function

$$f^k(s,y,z) := f(s,X_s^{k-1}(\omega),y,z).$$

By hypothesis $f^k(s,y,z)$ is measurable and bounded. Then we can build the sequence of functions f_n^k to satisfy (l_1),(l_2),(l_3),(l_4) and (l_5), and consider the following RBSDE

$$\begin{cases} Y_t^{k,n} = \Gamma + \int_t^T f\,(s,\ Y_s^{k,n},\ Z_s^{k,n})ds + K_T^{k,n} - K_t^{k,n} - \int_t^T Z_s^{k,n}\ dW_s\ , \\ Y_t^{k,n} \geq L_t\ ,\ \int_0^T (Y_t^{k,n} - L_t)\ dK_t^{k,n} = 0. \end{cases}$$

Since $f^k(s,y,z) = f(s,X_s^{k-1}(\omega),y,z)$, $f^{k-1}(s,y,z) = f(s,X_s^{k-2}(\omega),y,z)$ and $X_s^{k-1} \leq X_s^{k-2}$, the increase of the function f in x implies that $f^{k-1}(s,y,z) \leq f^k(s,y,z)$. This allows us to state that $f_n^{k-1}(s,y,z) \leq f_n^k(s,y,z), \forall n \geq 0.$ Thus the comparison theorem for RBSDEs gives us

$$\forall t \leq T, \quad Y_t^{k-1,n} \leq Y_t^{k,n}. \tag{9}$$

The same calculations done with $Y_t^{1,n}$ show that

$$\sup_{n,k} \| Y^{k,n} \|_{\mathcal{S}^2} < \infty. \tag{10}$$

From this we deduce that the sequence $(Y^{k,n})_{n\geq 0}$ is convergent in $\mathcal{S}^2$ to a process denoted as Y^k. Similar calculations made with $Z_t^{1,n}$ allows to state that the sequence $(Z^{k,n})_{n\geq 0}$ is convergent in $\mathcal{S}^2$ to a process denoted Z^k. Then $(Y_s^{k,n},Z_s^{k,n}) \longrightarrow (Y_s^k,Z_s^k)$ and $Y_s^{k,n} \nearrow Y_s^k$, and by virtue of (l_1) we have

$$\lim_{n\to\infty} f_n^k(s,Y_s^{k,n},Z_s^{k,n}) = f^k(s,Y_s^k,Z_s^k) = f(s,X_s^{k-1},Y_s^k,Z_s^k).$$

Since the functions f_n^k are measurable and bounded, the dominated convergence theorem implies, on one hand, that

$$\lim_{n\to\infty} \int_t^T f_n^k(s,Y_s^{k,n},Z_s^{k,n})ds = \int_t^T f(s,X_s^{k-1},Y_s^k,Z_s^k)ds.$$

On the other hand, $X_s^{k,n} \longrightarrow X_s^k$ and $K_T^{k,n} \to K_T^k$, in the sense that $\mathbb{E}\left[\sup_{0\leq s\leq T} |K_s^{k,n} - K_s^k|^2\right] \to 0$ as $n \to \infty$.
As in the previous step, we obtain a triplet $(Y_t^k,Z_t^k,K_t^k)_{0\leq t\leq T}$ satisfying the following equation:

$$\begin{cases} Y_t^k = \Gamma + \int_t^T f\,(s,X_s^{k-1},Y_s^k,Z_s^k)ds + K_T^k - K_t^k - \int_t^T Z_s^k\ dW_s\ , \\ Y_t^k \geq L_t\ ,\ \int_0^T (Y_t^k - L_t)\ dK_t^k = 0. \end{cases}$$

Taking the limit in (9) and (10), together with $\forall t \leq T, \quad Y_t^0 \leq Y_t^1$ leads to $\| Y_t^1 \|_{\mathcal{S}^2} < \infty$.

Clearly, the sequence $(Y^k)_k$ is increasing and bounded, it converges on one process which we shall denote by Y_t. We need to show now that $(Z^k)_k$ is a Cauchy sequence.

Applying the Itô's formula to the function $x \mapsto |x|^2$ and to the process $Y_.^k - Y_.^h$ between t

and T, we obtain:

$$(Y_t^k - Y_t^h)^2 = 2\int_t^T (Y_s^k - Y_s^h)[f(s,X_s^{k-1},Y_s^k,Z_s^k) - f(s,X_s^{h-1},Y_s^h,Z_s^h)]ds$$
$$- 2\int_t^T (Z_s^k - Z_s^h)(Y_s^k - Y_s^h)dW_s - \int_t^T |Z_s^k - Z_s^h|^2 ds$$
$$+2\int_t^T (Y_s^k - Y_s^h)d(K_s^k - K_s^h).$$

But

$$\int_t^T (Y_s^k - Y_s^h)d(K_s^k - K_s^h) = \int_t^T (Y_s^k - Y_s^h)dK_s^k + \int_t^T (Y_s^h - Y_s^k)dK_s^h$$
$$\leq \int_0^T (Y_s^k - L_s)dK_s^k + \int_0^T (Y_s^h - L_s)dK_s^h = 0.$$

As previously, taking the expectation in each term and using the Hölder's inequality leads to

$$\mathbb{E}[\int_0^T |Z_s^k - Z_s^h|^2 ds] \leq \mathbb{E}\Big[(Y_0^k - Y_0^h)^2 + \int_0^T |Z_s^k - Z_s^h|^2 ds\Big]$$
$$\leq 2[\mathbb{E}(\int_0^T [f(s,X_s^{k-1},Y_s^k,Z_s^k) - f(s,X_s^{h-1},Y_s^h,Z_s^h)]^2 ds)]^{\frac{1}{2}}.[\mathbb{E}(\int_0^T (Y_s^k - Y_s^h)^2 ds)]^{\frac{1}{2}}.$$

But because $f(s,.,.,.)$ is bounded and $Y^k - Y^h \longrightarrow 0$ we have $\mathbb{E}[\int_0^T |Z_s^k - Z_s^h|^2 ds] \longrightarrow 0.$ Then $(Z^k)_{k\geq 0}$ is a Cauchy sequence in $\mathcal{H}^2$ with $Z = \lim_{k\to\infty} Z^k$.

Let us return now to the forward component and consider the SDE

$$X_t^{k,n} = x + \int_0^t b_n(s,X_s^{k,n},Y_s^k)ds + \int_0^t \sigma(s,X_s^{k,n})dW_s.$$

By repeating the same work made with X^1, i.e. by changing 1 to k, we obtain the same conclusion about X^k in satisfying

$$X_t^{k-1,n} \leq X_t^{k,n} \leq S_t, \quad X_t^{k,n} \longrightarrow X_t^k,$$
$$X_t^k = x + \int_0^t b_n(s,\ X_s^k,\ Y_s^k)\ ds + \int_0^t \sigma(s,X_s^k)\ dW_s,$$
$$X_t^{k-1} \leq X_t^k \leq S_t.$$

The sequence $(X^k)_k$ is increasing and bounded from above, then it converges in $\mathcal{H}^2$ to a process denoted as X.

By the left continuity of b we have: $b(s,X_s^k,Y_s^k) \longrightarrow b(s,X_s,Y_s)$ when $k \longrightarrow \infty$.

Since the function $b(s,.,.)$ is measurable and bounded, the dominated convergence theorem implies, on one-hand, that

$$\int_0^t b(s,\ X_s^k,\ Y_s^k)ds \longrightarrow \int_0^t b(s,\ X_s,\ Y_s)ds.$$

On the other hand

$$\mathbb{E}\Big[\int_0^t [\sigma(s,X_s^k) - \sigma(s,X_s)]^2 ds\Big] \leq K^2\ \mathbb{E}\Big[\int_0^t |X_s^k - X_s|^2 ds\Big] \to 0,$$

when $n \to \infty$, since $X_s^k \to X_s$. Then $\int_0^{\cdot} \sigma(s,X_s^k)\ dW_s \longrightarrow \int_0^{\cdot} \sigma(s,X_s)\ dW_s$.

So,

$$X_t = x + \int_0^t b(s,X_s,Y_s)\ ds + \int_0^t \sigma(s,X_s)\ dW_s.$$

Let us show now that, $\forall t \leq T,\ (X_t, Y_t, Z_t, K_t)_{t\leq T}$ satisfies,

$$\left.\begin{array}{l} Y_t = \Gamma + \int_t^T f(s,\ X_s,\ Y_s,\ Z_s)ds + K_T - K_t - \int_t^T Z_s\, dW_s\ , \\ Y_t \geq L_t\ ,\ \int_0^T (Y_t - L_t)dK_t = 0. \end{array}\right\} \qquad (11)$$

Since $\lim_{k\to\infty} X_t^k = X$, $\lim_{k\to\infty} Y_t^k = Y$, $\lim_{k\to\infty} Z_t^k = Z$ and f is left continuous in y and lipschitzian in z, then

$$f(X_s^k, Y_s^k, Z_s^k) \longrightarrow f(X_s, Y_s, Z_s).$$

Moreover f is measurable and bounded, then the dominated convergence theorem implies that

$$\int_t^T f(s, X_s^k, Y_s^k, Z_s^k)ds \longrightarrow \int_t^T f(s, X_s, Y_s, Z_s)ds.$$

On the other hand, $Z_s^k \longrightarrow Z_s$. So $\mathbb{E}\int_0^T |Z_s^k - Z_s|^2 ds \to 0$, leading to

$$\int_0^T Z_s^k dW_s \longrightarrow \int_0^T Z_s dW_s.$$

Then by the Burkholder-David-Gundy inequality, it follows that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t^k - Y_t^h|^2\right] \to 0 \quad and \quad \mathbb{E}\left[\sup_{0\leq t\leq T} |K_t^k - K_t^h|^2\right] \to 0.$$

Therefore

$$\int_0^T (Y_t^k - L_t)dK_t^k \to \int_0^T (Y_t - L_t)dK_t,$$

which implies that

$$\int_0^T (Y_t - L_t)dK_t = 0.$$

Finally, $\forall t \leq T,\ (X_t, Y_t, Z_t, K_t)_{t\leq T}$ clearly satisfies (11), and here the proof ends. ■

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