Journal of Numerical Mathematics and Stochastics, **3** (1) : 13-19, 2011 © ***JNM@S***
http://www.jnmas.org/jnmas3-2.pdf
Euclidean Press, LLC
Online: ISSN 2151-2302

A Third-Order Family of Newton-Like Iteration Methods for Solving Nonlinear Equations*

P. WANG

Institute of Mathematics, Jilin University, Changchun 130012, P. R. China,
E-mail: pwang@jlu.edu.cn

Abstract. *In this paper, we present a third-order family of Newton-like iteration methods for solving nonlinear equations. Methods that avoid computation of second-order derivatives and turn out to require, per iteration, one evaluation of the function and two evaluations involving its first derivative only. Analysis of this family of methods demonstrates that its iterative solution is cubically convergent. Numerical examples are given to illustrate the efficiency and good performance of these rather novel methods.*

Key words : Newton's Method, Order of Convergence, Newton-Like Iteration Methods, Nonlinear Equations, Functional Equations.

AMS Subject Classifications : 65H11, 65K05

1. Introduction

Quite often in scientific and engineering practices a need arises to solve nonlinear equations of the form,

$$f(x) = 0, \tag{1}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function and D is an open interval. In this paper, we consider iterative methods for finding a simple root, α, for this equation, i.e. $f(\alpha) = 0$, while $f'(\alpha) \neq 0$, that may use f and f' but not the higher-order derivatives of f.

Equation (1) is well-known to be solvable iteratively by Newton's method and a range of its variants [13] as well as by other techniques. The solution by Newton's method, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0, \tag{2}$$

*Work has been supported partially by the Natural Science Foundation of China (Grant # 10926158), the Doctoral Fund of the Ministry of Education of China (Grant # 20090061120038) and the Basic Science Research Foundation of Jilin University (Grant # 2009032287).

happens to converge quadratically in some neighborhood of α [11].

Some Newton-type methods with third-order convergence that do not require the computation of second-order derivatives have been developed in [2, 3, 7, 9, 12, 14]. Other classes of those iterative methods invoke the Adomian decomposition method as in [1], He's homotopy perturbation method [5] or Liao's homotopy analysis method [2]. One class of those methods have been derived based on quadrature formulas for the computation of the integral

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt, \tag{3}$$

arising from Newton's theorem. Weerakoon and Fernando rederived in [14] a Newton's method by the rectangular rule to compute the integral of (3) and advanced, based on the trapezoidal rule, the following modified Newton's iteration

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - f(x_n)/f'(x_n))}. \tag{4}$$

exhibiting a third-order convergence.

The midpoint rule for the integral of (3) was shown in [3, 12] to yield

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/(2f'(x_n)))}. \tag{5}$$

The method (5) has also been independently derived by Homeier in [7] . A further multivariate version of this method has been discussed in [4, 6].

By applying Newton's theorem to the inverse function $x = f(y)$ instead $y = f(x)$, Homeier derived in [7]the following cubically convergent iteration scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left[\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/(f'(x_n)))}\right]. \tag{6}$$

The method leading to (6) has also been independently derived by Özban in [12].

Finally, Kou, Li and Wang considered in [9] Newton's theorem on a new interval of integration and arrived at the following cubically convergent iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n + f(x_n)/f'(x_n))}{f'(x_n)}. \tag{7}$$

Any of the aforementioned methods happens to require three functional evaluations of the given function and for its first-order derivative, but no evaluations of the second- or higher-order derivatives. Iterative methods with a third-order convergence rate, not requiring the computation of second-order derivatives, is both important and interesting from the practical point of view and is an area of current active research.

In this paper, we present a rather novel family of these Newton-like third-ordr methods and prove that their pertaining iterative solution to be cubically convergent. Their efficiency is demonstrated by numerical examples.

2. The Family of Iterative Methods

Let us consider the one point Newton-Cotes formula [8]

$$\int_{x_n}^{x} f'(t)dt \simeq (x - x_n)\, f'\left(\frac{x_n + x}{2}\right).$$

Applying this formula with

$$x = x_n - \frac{f(x_n)}{f'(x_n)},$$

to (3), leads obviously to the midpoint rule (5).

In this way, by using the following rule

$$\int_{x_n}^{x} f'(t)dt \simeq (x - x_n)[(1-\beta)f'(x_n) + \beta f'(x_n - f(x_n)/(2\beta f'(x_n)))],$$

to approximate the right integral of (3) and looking for $f(x) = 0$, we obtain a family of new iterative methods

$$x_{n+1} = x_n - \frac{f(x_n)}{(1-\beta)f'(x_n) + \beta f'(x_n - f(x_n)/(2\beta f'(x_n)))}, \qquad (8)$$

where $\beta \neq 0$. A similar approximation can be found in [10].

Remark 2.1. Obviously, when $\beta = 1$, formula (8) becomes the midpoint rule (5). Moreover, it becomes the trapezoidal rule (4) when $\beta = 1/2$. Otherwise, the family (8) is more general.

3. Convergence Analysis

In this section, we study the convergence of the family (8) of Newton-like methods.

Theorem 3.1. *Let $\alpha \in D$, an open interval, be a simple zero of sufficiently differentiable function $f : D \subset R \to R$, and let $e_n = x_n - \alpha$ with $c_k = f^{(k)}(\alpha)/k!$. If x_0 is sufficiently close to α, then (i) the order of convergence of the solution by the methods defined in (8) is three, and (ii) this solution satisfies the error equation*

$$e_{n+1} = -\frac{2}{c_1^3}\left[(3 + \frac{3}{4\beta})c_1^2 c_3 + 6c_1 c_2^2 + 2c_2^2\right]e_n^3 + O(e_n^4).$$

Proof. Let α be a simple zero of f. Consider the iteration function F defined by

$$F(x) = x - \frac{f(x)}{(1-\beta)f'(x) + \beta f'(x - f(x)/(2\beta f'(x)))},$$

where $\beta \neq 0$.

From the Taylor expansion of $F(x_n)$ around $x = \alpha$, we obtain

$$x_{n+1} = F(x_n) = F(\alpha) + C_1 e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4), \qquad (9)$$

where $e_n = x_n - \alpha$ and $C_k = F^{(k)}(\alpha)/k!$.

Taking into consideration that $f(\alpha) = 0$, we have

$$F(\alpha) = \alpha,\ F'^{(2)}(\alpha) = 0 \tag{10}$$

and

$$F^{(3)}(\alpha) = -\left(1 + \frac{1}{4\beta}\right)\frac{f^{(3)}(\alpha)}{f'(\alpha)} - \frac{3\,f^{(2)}(\alpha)^2}{f'^2} - \frac{f^{(2)}(\alpha)^2}{f'^3}. \tag{11}$$

Substituting (10) and (11) into (9), on the assumption that $c_k = f^{(k)}(\alpha)/k!$, leads to

$$x_{n+1} = \alpha - \frac{1}{c_1^3}\left(6\,(1 + \tfrac{1}{4\beta})c_1^2 c_3 + 12\,c_1 c_2^2 + 4\,c_2^2\right)e_n^3 + O(e_n^4).$$

Therefore, we have

$$e_{n+1} = -\frac{2}{c_1^3}\left((3 + \tfrac{3}{4\beta})c_1^2 c_3 + 6\,c_1 c_2^2 + 2\,c_2^2\right)e_n^3 + O(e_n^4)\ ;$$

implying an explicit third-order convergence. Here the proof completes. ■

Remark 3.1. When $\beta = -1/4$ and $f^{(2)}(\alpha) = 0$, the method (8) provides an iterative solution with a quartic convergence.

It is worth noting finally that the family of methods (8) includes, as a particular case, when $\beta = 3/4$, the new third-order iterative solution procedure

$$x_{n+1} = x_n - \frac{4\,f(x_n)}{f'(x_n) + 3\,f'(x_n - 2\,f(x_n)/(3\,f'(x_n)))}. \tag{12}$$

4. Numerical Examples

Computational tests for the reported family of methods were done by means of MAPLE using 64 digit floating point arithmetics (Digits := 64). The level of approximation of the solution (root) is directly tied to the precision ϵ of the computer, where. This has been set at $\epsilon = 10^{-27}$ for all the present computations. The adopted stopping criteria for the computer programs were: (i) $|x_{n+1} - x_n| < \epsilon$, and (ii) $|f(x_{n+1})| < \epsilon$. So, when the stopping criteria are satisfied, computations are terminated and the emerging x_n is the corresponding approximate root . The test nonlinear functions used are the same functions previously entertained by Weerakoon and Fernando [14], and by Chun [2]. These are namely:

$$f_1(x) = \sin^2 x - x^2 + 1,$$
$$f_2(x) = x^2 - e^x - 3\,x + 2,$$
$$f_3(x) = x\,e^{x^2} - \sin^2 x + 3\cos x + 5,$$
$$f_4(x) = e^{x^2 + 7x - 30} - 1.$$

The numerical test results for various cubically convergent iterative schemes are summarized in Table 1. In addition to results of the method (12), (WM), advanced in this paper, the table reports on comparative results by the Newton method (NM), the method of Weerakoon and Fernando (4) (WF), the method derived from midpoint rule (5) (MP), the method of Homeier (6) (HM), the method of Kou et al. (7) (KLW), the Abbasbandy method [1] (AM), defined by

Table 1. Comparison of solutions by various iterative methods

Method	IT	NFE	x_n	$\lvert f(x_{n+1})\rvert$
f_1 ; $x_0 = 1$				
NM	7	14		
WF	5	15		
MP	5	15		
AM	5	15		
HM	4	12	$1.4044916482153412260350868178$;	$-3.26e-29$
KLW	5	15		
CM1	5	20		
CM2	5	15		
WM	4	12		
f_2 ; $x_0 = 2$				
NM	6	12		
WF	5	15		
MP	4	12	$0.25753028543986076045536730494$;	$-1.04e-29$
AM	5	15		
HM	5	15		
KLW	4	12	$0.25753028543986076045536730499$;	$-1.99e-29$
CM1	4	16		
CM2	4	12	$0.25753028543986076045536730494$;	$-1.04e-29$
WM	4	12		
f_3 ; $x_0 = -2$				
NM	9	18		
WF	7	21		
MP	6	18		
AM	6	18		
HM	6	18	$-1.2076478271309189270094167584$;	$-8.92e-28$
KLW	4	12		
CM1	6	24		
CM2	6	18		
WM	4	12		
f_4 ; $x_0 = 3.5$				
NM	13	26		
WF	9	27		
MP	8	24		
AM	7	21		
HM	8	24	$3.0000000000000000000000000000$;	$0.00e-01$
KLW	8	24		
CM1	9	36		
CM2	9	27		
WM	8	24		

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2 f^{(2)}(x_n)}{2f'(x_n)^3} - \frac{f(x_n)^5 f^{(2)}(x_n)^2}{2f'(x_n)^5},$$

and Chun method [2] (CM1), defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + h\,\frac{2f(x^*_{n+1})}{f'(x_n)+f'(x^*_{n+1})},$$

where $x^*_{n+1} = x_n - f(x_n)/f'(x_n)$ and $h = -1$. The results exhibited in Table 1 relate also to another Chun method [2], (CM2), defined viz

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + h\,\frac{f(x_n)\,f(x^*_{n+1})}{[f(x_n)+f(x^*_{n+1})]\,f'(x_n)}.$$

For each of the test functions listed above we sought an approximation x_n of the root α of equation $f(x) = 0$ after n iterations. Table 1 reports moreover on the absolute values, $|f(x_n)|$, for the corresponding functions. Also displayed are the number of iterations,(IT), to approximate the zero and the number of function evaluations (NFE). This is counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the first derivative. The computational results show that the cubically convergent methods in general, and the new method pertaining to(12) in particular, can be quite competitive with the standard Newton method.

5. Conclusions

In this paper, we presented a new family of modified Newton-like methods which includes, as two particular cases, the midpoint rule and the trapezoidal rule. The methods require, per iteration, one evaluation of the function and two evaluations of its first-order derivative. We have demonstrated that each family member yields a cubically convergent solution, and observed from numerical examples that the proposed methods show at least the same performance as that of other known methods of the same order.

Acknowledgments
The author would like to take this opportunity to thank Professor Yong Li for his guidance and continual encouragement. He is also grateful to the referee for a number of useful remarks.

References

[1] S. Abbasbandy, Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method, *Applied Mathematics and Computation* **145**, (2003), 887-893.

[2] C. Chun, Construction of Newton-like iteration methods for solving nonlinear equations, *Numerical Mathematics* **104**, (2006), 297-315.

[3] M. Frontini, and E. Sormani, Some variant of Newton's method with third-order convergence, *Applied Mathematics and Computation* **140**, (2003), 419-426.

[4] M. Frontini, and E. Sormani, Third-order methods from quadrature formulae for solving systems of nonlinear equations, *Applied Mathematics and Computation* **149**, (2004), 771-782.

[5] J. He, Newton-like iteration method for solving algebraic equations, *Communications on Nonlinear Science and Numerical Simulation* **3**, (1998) 106-109.

[6] H. H. H. Homeier, A modified Newton method with cubic convergence: the multivariate case, *Journal of Computational and Applied Mathematics* **169**, (2004), 161-169.

[7] H. H. H. Homeier, On Newton-type methods with cubic convergence, *Journal of Computational and Applied Mathematics* **176**, (2005), 425-432.

[8] D. E. Kincaid, and E. W. Cheney, *Numerical Analysis*, Brooks/Cole Publishing Company, Pacific Grove, CA, 1990.

[9] J. Kou, Y.Li, and X. Wang, A modification of Newton method with third-order convergence, *Applied Mathematics and Computation* **181**, (2006), 1106-1111.

[10] M. A. Noor, and M. Waseem, Some iterative methods for solving a system of nonlinear equations, *Computers and Mathematics with Applications* **57**, (2009), 101-106.

[11] A. M. Ostrowski, *Solution of Equations in Euclidean and Banach Space*, Academic Press, New York, 1973.

[12] A. Y. Ōzban, Some new variants of Newton's method, *Applied Mathematics Letters* **17**, (2004), 677-682.

[13] J. F. Traub, *Iterative Methods for the Solution of Equations*, Chelsea Publishing company, New York, 1982.

[14] S. Weerakoon, and G. I. Fernando, A variant of Newton's method with accelerated third-order convergence, *Applied Mathematics Letters* **17**, (2000), 87-93.

Article history : Submitted November, 16, 2010; Received in revised form January, 17, 2011; Accepted January, 25, 2011.