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Sample Path Exponential Stability of Stochastic Neutral Partial Functional Differential Equations

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Abstract. *In this paper, we study the almost sure moment exponential stability of mild solutions of stochastic neutral partial functional differential equations in real separable Hilbert spaces using local Lipschitz conditions. Even in the special case, when the neutral term is zero, the results obtained here appear to be new and complement the study in [Taniguchi, et al, J. Differential Eqns. 181 (2002), 72-91].*

Key words : Stochastic Neutral Partial Functional Differential Equations, Mild Solutions, Local Lipschitz Conditions, Almost Sure Exponential Stability.

AMS Subject Classifications : 60H20, 34K50, 93E15

1. Introduction

In this paper, we consider a semilinear neutral stochastic differential equation in a real separable Hilbert space of the form:

$$d[x(t)+f(t,x_t)] = [A\,x(t)+a(t,x_t)]dt+b(t,x_t)dw(t), \qquad t>0; \tag{1}$$

$$x(t)=\varphi(t), \quad t\in[-r,0] \quad (0\le r<\infty); \tag{2}$$

where $x_t(s)=x(t+s), -r\le s\le 0,$ and the equation to be made precise later (see Section 2). Equation (1) when $f\equiv 0$ has been well studied, see [4,8] and the references cited therein.

A study of such class of equations (1) was initiated recently in Govindan [2]. Using a global Lipschitz condition on the nonlinear terms $f(t,u)$, $a(t,u)$ and $b(t,u)$, existence and stability problems were addressed in [2]. Subsequently, in Govindan [3], the existence and uniqueness of a mild solution was considered by assuming only a local Lipschitz condition. Even in the deterministic case (when $b\equiv 0$), very little is known on equation (1) though this class of equations models problems of stabilization of lumped control systems, see Hernandez et al [6]. See, also [5].

In this paper, our goal is to study the exponential stability of the quadratic moments of a mild solution of equation (1) exploiting local Lipschitz conditions on the nonlinear terms; and using the latter to deduce the almost sure exponential stability of the second moment of the sample paths of a mild solution. Taniguchi, et al, [9] considered such problems in the special case (when $f = 0$) for the p^{th} – moment ($p > 2$) of a mild solution. So, when $p = 2$, the results established here appear to be new even in this special case.

The format of the rest of the paper is as follows. In Section 2, we give the preliminaries containing several definitions from Taniguchi [8] and lemmas. We state all our assumptions and also an existence and uniqueness result of a mild solution in Section 3. Section 4 is devoted to the main result on almost sure exponential stability of the quadratic moments of a mild solution of equation (1). In Section 5, an example is given to illustrate the theory.

2. Preliminaries

Let X, Y be real separable Hilbert spaces and $L(Y,X)$ be the space of bounded linear operators mapping Y into X. For convenience, we shall use the same notation $|\cdot|$ to denote the norms in X, Y and $L(Y,X)$ and use $(\cdot,\cdot)$ to denote inner–product of X and Y without any confusion. Let $(\Omega, B, P, \{B_t\}_{t\geq 0})$ be a complete probability space with an increasing right continuous family $\{B_t\}_{t\geq 0}$ of complete sub–σ–algebras of B. Let $\beta_n(t)(n = 1,2,3,\dots)$ be a sequence of real-valued standard Brownian motions mutually independent defined on this probability space. Set

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \beta_n(t) e_n, \quad t \geq 0,$$

where $\lambda_n \geq 0 (n = 1,2,3,\dots)$ are nonnegative real numbers and $\{e_n\}(n = 1,2,3,\dots)$ is a complete orthonormal basis in Y. Let $Q \in L(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$. The above Y–valued stochastic process $w(t)$ is called a Q–Wiener process. Now, we define a real–valued stochastic integral of Y–valued B_t–adapted predictable process $h(t)$ with respect to the Q–Wiener process $w(t)$.

Definition 2.1. Let h(t) be a Y–valued B_t–adapted predictable process such that $E\int_0^t |h(t)|^2 dt < \infty$ for any $t \in R^+$, where $R^+ = [0,\infty)$. Then, we define the real–valued stochastic integral $\int_0^t \langle h(s), dw(s)\rangle$ by

$$\int_0^t \langle h(s), dw(s)\rangle = \sum_{n=1}^{\infty} \int_0^t (h(s), e_n) dw(s) e_n,$$

where $w(s)e_n = (w(s), e_n) = \sqrt{\lambda_n}\, \beta_n(s)$.

Definition 2.2. Let h(t) be an $L(Y,X)$–valued function and let λ be a sequence $\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots\}$. Then we define

$$|h(t)|_\lambda = \left\{ \sum_{n=1}^{\infty} |\sqrt{\lambda_n}\, h(t) e_n|^2 \right\}^{1/2}.$$

If $|h(t)|_\lambda^2 < \infty$, then h(t) is called λ–Hilbert–Schmidt operator and let $\sigma(\lambda)(Y,X)$ denote the space of all λ–Hilbert–Schmidt operators from Y to X.

Next, we define the X–valued stochastic integral with respect to the Y–valued Q–Wiener process $w(t)$.

Definition 2.3. Let $\Phi : R^+ \to \sigma(\lambda)(Y,X)$ be a predictable, B_t–adapted process. Then, for any

Φ satisfying $\int_0^t E|\Phi(s)|_\lambda^2 ds < \infty$ we define the X–valued stochastic integral $\int_0^t \Phi(s)dw(s) \in X$ with respect to w(t) by

$$\left(\int_0^t \Phi(s)dw(s), h\right) = \int_0^t \langle \Phi^*(s)h, dw(s)\rangle, \quad h \in X,$$

where Φ^* is the adjoint operator of Φ.

A semigroup $\{S(t), t \geq 0\}$ is said to be exponentially stable if there exist positive constants M and a such that $\|S(t)\| \leq M \exp(-at)$, $t \geq 0$, where $\|\bullet\|$ denotes the operator norm in $L(X,X)$. If $M = 1$, the semigroup is said to be a contraction. If $\{S(t), t \geq 0\}$ is an analytic semigroup, see Pazy, [7, p.60] with infinitesimal generator A such that $0 \in \rho(A)$ (the resolvent set of A) then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$ as a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha))$ is dense in X and the expression

$$\|x\|_\alpha = |(-A)^\alpha x|, \quad x \in D((-A)^\alpha),$$

defines a norm on $X_\alpha = D((-A)^\alpha)$.

Let C be the space of continuous functions $x : [-r, 0] \to X$ with the norm $\|x\|_C = \sup_{-r \leq s \leq 0} |x(s)|$.

We now make the equation (1) precise: let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{S(t), t \geq 0\}$ defined on X. Let the functions $a(t,u), f(t,u)$ and $b(t,u)$ be defined as follows: $a : R^+ \times C \to X$, $f : R^+ \times C \to X_\alpha$ and $b : R^+ \times C \to L(Y,X)$ are Borel measurable; and for each (t,u) are measurable with respect to the σ–algebra B_t. Let $MC(t)$ denote the space of all B_t- measurable functions which belong to $L^2(\Omega, C)$, that is, $MC(t)$ is the space of all B_t- measurable $C-$ valued functions $\eta : \Omega \to C$ with the norm $E\|\eta\|_C^2 = E \sup_{-r \leq s \leq 0} |\eta(s)|^2 < \infty$, see [9]. The past process $\varphi \in MC(t)$.

Next, we introduce the concept of a mild solution of equation (1).

Definition 2.4. An X – valued stochastic process $\{x(t), t \in [-r, T]\} (0 < T < \infty)$ is called a mild solution of equation (1) if

(i) x(t) is B_t–adapted with $\int_0^T |x(t)|^2 dt < \infty$, a.s.,

(ii) $x(t) = \varphi(t)$, $t \in [-r, 0]$ a.s., and

(iii) x(t) satisfies the integral equation

$$x(t) = S(t)[\varphi(0) + f(0,\varphi)] - f(t, x_t) - \int_0^t A\, S(t-s) f(s, x_s) ds$$
$$+ \int_0^t S(t-s) a(s, x_s) ds + \int_0^t S(t-s) b(s, x_s) dw(s), \quad \text{a.s.}, \quad t \in [0, T].$$

We will need the following results in the sequel.

Theorem 2.1. [7, p.74] *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$. If $0 \in \rho(A)$ then,*

(a) $S(t) : X \to X_\alpha$ *for every* $t > 0$ *and* $\alpha \geq 0$.

(b) *For every* $x \in X_\alpha$ *we have*

$$S(t) A^\alpha x = A^\alpha S(t) x.$$

(c) *For every* $t > 0$ *the operator* $A^\alpha S(t)$ *is bounded and*

$$\|A^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-at}, \quad a > 0.$$

(d) *Let* $0 < \alpha \leq 1$ *and* $x \in D(A^{\alpha})$ *then*

$$\|S(t)x - x\| \leq C_{\alpha} t^{\alpha} \|A^{\alpha} x\|.$$

Lemma 2.1. [5] *Let* $-A$ *be the infinitesimal generator of an analytic semigroup of bounded linear operators* $\{S(t), t \geq 0\}$ *in X. Then, for any stochastic process* $F : [0,\infty) \to X$ *which is strongly measurable with* $\int_0^T E|(-A)^{\alpha}F(t)|^p dt < \infty$, $p \geq 2$ *and* $0 < T \leq \infty$, *the following inequality holds for* $0 < t \leq T$:

$$E\left|\int_0^t (-A)S(t-s)F(s)ds\right|^p \leq k(p,a,\alpha)\int_0^t E|(-A)^{\alpha}F(s)|^p ds,$$

provided $1/p < \alpha < 1$, *where*

$$k(p,a,\alpha) = M_{1-\alpha}^p \frac{(p-1)^{p\alpha-1}[\Gamma((p\alpha-1)/(p-1))]^{p-1}}{(pa)^{p\alpha-1}},$$

and $\Gamma(\bullet)$ *is the Gamma function.*

3. An Existence and Uniqueness Result

In this section, we consider the existence and uniqueness of a mild solution of equation (1) using local Lipschitz conditions.

Let the following assumptions hold a.s.:

(H1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X and that the semigroup is a contraction,

(H2) The functions $a(t,u)$ and $b(t,u)$ are continuous and that there exist positive constants $C_i = C_i(T)$, $i = 1,2$ such that

$$|a(t,u) - a(t,v)| \leq C_1\|u - v\|_C,$$

$$|b(t,u) - b(t,v)|_{\lambda} \leq C_2\|u - v\|_C,$$

for all $t \in [0,T]$ and $u,v \in C$.

Under this assumption, we may suppose that there exists a positive constant $C_3 = C_3(T)$ such that

$$|a(t,u)|^2 + |b(t,u)|_{\lambda}^2 \leq C_3^2(1 + \|u\|_C^2).$$

(H3) The function $f(t,u)$ is continuous and that there exists a positive constant $C_4 = C_4(T)$ such that

$$\|f(t,u) - f(t,v)\|_{\alpha} \leq C_4\|u - v\|_C,$$

for all $t \in [0,T]$ and $u,v \in C$.

Under this assumption, we may suppose that there exists a positive constant $C_5 = C_5(T)$ such that

$$\|f(t,u)\|_{\alpha} \leq C_5(1 + \|u\|_C).$$

(H4) $f(t,u)$ is continuous in the quadratic mean sense:

$$\lim_{t\to s} E\|f(t,x_t) - f(s,x_s)\|_{\alpha}^2 \to 0.$$

In the rest of the paper, we shall restrict α to the interval $1/2 < \alpha < 1$.

Theorem 3.1. [3] *Suppose that the hypotheses (H1)–(H4) are satisfied. Then, the problem*

(1)–(2) has a unique mild solution. Further, if $t_m < \infty$, then $\lim_{t\uparrow t_m} E\|x(t)\|_\alpha^2 = \infty$.

To prove this theorem, assume $T > 0$ is a fixed time. Let Γ_T be the subspace of all continuous processes x which belong to the space $C([-r,T],L^2(\Omega,X))$ with the norm $\|x\|_{\Gamma_T} < \infty$, where $\|x\|_{\Gamma_T} := \sup_{0\le t\le T}(E\|x_t\|_C^2)^{1/2}$. See [9].

Define a map G on Γ_T:

$$(Gx)(t) = S(t)[\varphi(0) + f(0,\varphi)] - f(t,x_t) - \int_0^t AS(t-s)f(s,x_s)ds + \int_0^t S(t-s)a(s,x_s)ds + \int_0^t S(t-s)b(s,x_s)dw(s), \quad t > 0, \tag{3}$$

$$(Gx)(t) = \varphi(t), \quad t \in [-r,0]. \tag{4}$$

3.1. **Sketch of the proof of Theorem 3.1**

First, show the continuity of the map G defined on $[0,T]$ and taking values in $L^2(\Omega,X)$ thereby showing that it is a well-defined map in the space $C([-r,T],L^2(\Omega,X))$. Second, show that G maps Γ_T into itself.

Let $x,y \in \Gamma_T$. Then for any fixed $t \in [0,T]$, we have

$$\begin{aligned}
&E\|(Gx)_t - (Gy)_t\|_C^2 = E\sup_{-r\le\theta\le0}|(Gx)(t+\theta) - (Gy)(t+\theta)|^2 \\
&\le 4\Big\{E\sup_{-r\le\theta\le0}|f(t+\theta,x_{t+\theta}) - f(t+\theta,y_{t+\theta})|^2 \\
&+E\sup_{-r\le\theta\le0}\left|\int_0^{t+\theta}(-A)S(t+\theta-s)[f(s,x_s) - f(s,y_s)]ds\right|^2 \\
&+E\sup_{-r\le\theta\le0}\left|\int_0^{t+\theta}S(t+\theta-s)[a(s,x_s) - a(s,y_s)]ds\right|^2 \\
&+E\sup_{-r\le\theta\le0}\left|\int_0^{t+\theta}S(t+\theta-s)[b(s,x_s) - b(s,y_s)]dw(s)\right|^2\Big\} \\
&\le 4C_4^2\|(-A)^{-\alpha}\|^2\sup_{0\le t\le T}E\|x_t - y_t\|_C^2 \\
&+ 4\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}TC_4^2\sup_{0\le t\le T}E\|x_t - y_t\|_C^2 \\
&+ 4TC_1^2\sup_{0\le t\le T}E\|x_t - y_t\|_C^2 \\
&+ 4\kappa TC_2^2\sup_{0\le t\le T}E\|x_t - y_t\|_C^2.
\end{aligned}$$

Now choosing $T > 0$ sufficiently small, we can find a positive number $K(T) \in (0,1)$ such that

$$\|Gx - Gy\|_{\Gamma_T} \le K(T)\|x-y\|_{\Gamma_T},$$

for any $x,y \in \Gamma_T$. Hence, by the Banach fixed point theorem, G has a unique fixed point $x \in \Gamma_T$ and this fixed point is the unique mild solution of equation (1) on $[0,T]$. Next, we continue the solution for $t \ge T$, see Govindan [3] and the references therein.

4. Almost Sure Exponential Stability

In this section, we consider the exponential stability of the second moment of a trivial solution of equation (1). For this we need a further assumption. See [9].

(H5) There exist nonnegative real numbers $Q_1, Q_2 \geq 0$, $0 \leq Q_3 < 1$ and continuous functions $\xi_i : R^+ \to R^+, j = 1,2,3$ such that

$$E|a(t,u)|^2 \leq Q_1 E\|u\|_C^2 + \xi_1(t),$$

$$E|b(t,u)|_\lambda^2 \leq Q_2 E\|u\|_C^2 + \xi_2(t),$$

and

$$\|f(t,u)\|_\alpha \leq Q_3\|u\|_C + \xi_3(t), \quad \text{a.s.}$$

for $t \geq 0$, and there exist nonnegative real numbers $P_1, P_2, P_3 \geq 0$ and $\delta > a > 0$ such that

$$|\xi_i(t)| \leq P_j e^{-\delta t}, \quad j = 1,2,3; \quad t \geq 0.$$

Assume from now on that $a(t,0) = b(t,0) = f(t,0) \equiv 0$ a.e. t so that equation (1) admits a trivial solution.

The following lemma is needed to consider the main result.

Lemma 4.1. [5] *Let $-A$ be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$. Then, for any stochastic process $F : R^+ \to X$ which is strongly measurable with $\int_0^T E|(-A)^\alpha F(t)|^2 dt < \infty$, $0 < T < \infty$, the following inequality holds:*

$$E|\int_0^t (-A)S(t-s)F(s)ds|^2 \leq \frac{M_{1-\alpha}^2 \Gamma(2\alpha-1)}{a^{2\alpha-1}} \int_0^t e^{-a(t-s)} E|(-A)^\alpha F(s)|^2 ds,$$

provided $1/2 < \alpha < 1$.

Theorem 4.1. *Let the hypotheses (H1)–(H5) hold. Suppose that the semigroup $\{S(t), t \geq 0\}$ is exponentially stable. Then, the mild solution of equation (1) satisfies*

$$E\|x_t\|_C^2 \leq K e^{-\theta t}, \quad t \geq 0; \quad K, \theta > 0,$$

provided

$$\frac{4}{(1-Q_3)^2}\left[\frac{2M_{1-\alpha}^2\ \Gamma(2\alpha-1)\ Q_3^2}{a^{2\alpha-1}} + \frac{Q_1}{a} + 4Q_2\right] < a.$$

Proof. Consider the mild solution

$$x(s) = S(s)[\varphi(0) + f(0,\varphi)] - f(s,x_s) - \int_0^s AS(s-\tau)f(\tau,x_\tau)d\tau$$
$$+ \int_0^s S(s-\tau)a(\tau,x_\tau)d\tau + \int_0^s S(s-\tau)b(\tau,x_\tau)dw(\tau).$$

By assumption (H5) and Theorem 2.1, we obtain

$$|x(s)| \leq |S(s)\varphi(0)| + |S(s)f(0,\varphi)| + |f(s,x_s)| + \left|\int_0^s AS(s-\tau)f(\tau,x_\tau)d\tau\right|$$
$$+ \left|\int_0^s S(s-\tau)a(\tau,x_\tau)d\tau\right| + \left|\int_0^s S(s-\tau)b(\tau,x_\tau)dw(\tau)\right|$$
$$\leq \|S(s)\|\|\varphi\|_C + \|S(s)\|\{Q_3\|\varphi\|_C + \xi_3(s)\}$$

$$+ \{Q_3\|x_s\|_C + \xi_3(s)\} + \left|\int_0^s AS(s-\tau)f(\tau,x_\tau)d\tau\right|$$
$$+ \left|\int_0^s S(s-\tau)a(\tau,x_\tau)d\tau\right| + \left|\int_0^s S(s-\tau)b(\tau,x_\tau)dw(\tau)\right|.$$

Applying first Lemma 7.2 [1, p.182] and then Lemma 4.1 and Lemmas 4.1–4.2 [8], we have

$$(1-Q_3)^2E\|x_t\|_C^2 \le 4\Big\{E[Me^{-at}(1+Q_3)\|\varphi\|_C + 2\xi_3(t)]^2$$
$$+ \tfrac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{a^{2\alpha-1}}\int_0^t e^{-a(t-s)}E|(-A)^\alpha f(s,x_s)|^2ds,$$
$$+ \tfrac{1}{a}\int_0^t e^{-a(t-s)}E|a(s,x_s)|^2ds$$
$$+ 4\int_0^t e^{-a(t-s)}E|b(s,x_s)|_\lambda^2 ds\Big\}.$$

Assumption (H5) then yields

$$e^{at}E\|x_t\|_C^2 \le \tfrac{4}{(1-Q_3)^2}\{E[(1+Q_3)\|\varphi\|_C + 2P_3]^2$$
$$+ \tfrac{2M_{1-\alpha}^2\Gamma(2\alpha-1)}{a^{2\alpha-1}}\int_0^t e^{as}[Q_3^2E\|x_s\|_C^2 + \xi_3^2(s)]ds,$$
$$+ \tfrac{1}{a}\int_0^t e^{as}[Q_1E\|x_s\|_C^2 + \xi_1(s)]ds$$
$$+ 4\int_0^t e^{as}[Q_2E\|x_s\|_C^2 + \xi_2(s)]ds\Big\}$$
$$\le \tfrac{4}{(1-Q_3)^2}\Big\{2(1+Q_3)^2E\|\varphi\|_C^2 + P_3^2 + (\tfrac{P_1}{a} + P_2)$$
$$+ \Big[\tfrac{2M_{1-\alpha}^2\Gamma(2\alpha-1)Q_3^2}{a^{2\alpha-1}} + \tfrac{Q_1}{a} + 4Q_2\Big]\int_0^t e^{as}E\|x_s\|_C^2ds\Big\}.$$

Letting

$$K = \tfrac{4}{(1-Q_3)^2}\{2(1+Q_3)^2E\|\varphi\|_C^2 + P_3^2 + (\tfrac{P_1}{a} + P_2)\}$$

and

$$\gamma = \tfrac{4}{(1-Q_3)^2}\Big[\tfrac{2M_{1-\alpha}^2\Gamma(2\alpha-1)Q_3^2}{a^{2\alpha-1}} + \tfrac{Q_1}{a} + 4Q_2\Big],$$

we get

$$e^{at}E\|x_t\|_C^2 \le K + \gamma\int_0^t e^{as}E\|x_s\|_C^2ds.$$

Invoking Gronwall's lemma, we have

$$E\|x_t\|_C^2 \le Ke^{-\theta t}, \quad t \ge 0,$$

where $\theta = a - \gamma$. ■

Theorem 4.2. Suppose that all the conditions of Theorem 4.1 hold. Then the mild solution of equation (1) satisfies

$$\lim_{t\to\infty}\sup \tfrac{1}{t}\log|x(t)| \le -\tfrac{\theta}{4}, \quad \text{a. s.}$$

Proof. Let N be a sufficiently large positive integer. Let $N \le t \le N+1$. Then,

$$x(t) = S(t-N)[x(N)+f(N,x(N)] - f(t,x_t) - \int_N^t AS(t-s)f(s,x_s)ds$$
$$+\int_N^t S(t-s)a(s,x_s)ds + \int_N^t S(t-s)b(s,x_s)dw(s).$$

Letting $L = 1-Q_3$, we have

$$|x(t)| \leq \tfrac{1}{L}|S(t-N)[x(N)+f(N,x(N))| + \tfrac{1}{L}\left|\int_N^t AS(t-s)f(s,x_s)ds\right|$$
$$+\tfrac{1}{L}\left|\int_N^t S(t-s)a(s,x_s)ds\right| + \tfrac{1}{L}\left|\int_N^t S(t-s)b(s,x_s)dw(s)\right|.$$

Thus, for any $\varepsilon_N > 0$, we obtain

$$P\left\{\sup_{N\le t\le N+1}|x(t)| > \varepsilon_N\right\} \leq P\left\{\sup_{N\le t\le N+1}\tfrac{1}{L}|S(t-N)[x(N)+f(N,x(N))]| > \tfrac{\varepsilon_N}{4}\right\}$$
$$+P\left\{\sup_{N\le t\le N+1}\tfrac{1}{L}\left|\int_N^t AS(t-s)f(s,x_s)ds\right| > \tfrac{\varepsilon_N}{4}\right\}$$
$$+P\left\{\sup_{N\le t\le N+1}\tfrac{1}{L}\left|\int_N^t S(t-s)a(s,x_s)ds\right| > \tfrac{\varepsilon_N}{4}\right\}$$
$$+P\left\{\sup_{N\le t\le N+1}\tfrac{1}{L}\left|\int_N^t S(t-s)b(s,x_s)dw(s)\right| > \tfrac{\varepsilon_N}{4}\right\}$$
$$\leq (\tfrac{4}{\varepsilon_N})^2 E\left[\sup_{N\le t\le N+1}\tfrac{1}{L^2}|S(t-N)[x(N)+f(N,x(N))]|^2\right]$$
$$+(\tfrac{4}{\varepsilon_N})^2 E\left[\sup_{N\le t\le N+1}\tfrac{1}{L^2}\left|\int_N^t AS(t-s)\,f(s,x_s)ds\right|^2\right]$$
$$+(\tfrac{4}{\varepsilon_N})^2 E\left[\sup_{N\le t\le N+1}\tfrac{1}{L^2}\left|\int_N^t S(t-s)a(s,x_s)ds\right|^2\right]$$
$$+(\tfrac{4}{\varepsilon_N})^2 E\left[\sup_{N\le t\le N+1}\tfrac{1}{L^2}\left|\int_N^t S(t-s)b(s,x_s)dw(s)\right|^2\right]$$
$$= \sum_{i=1}^4 I_i, \quad \text{say.}$$

In view of assumption (H3) and Theorem 4.1, we have

$$I_1 \leq \left(\tfrac{4}{L\varepsilon_N}\right)^2[2E|x(N)|^2 + 2Q_3^2E\|x_t\|_C^2 + 2P_3^2e^{-2\delta N}]$$
$$\leq \left(\tfrac{4}{L\varepsilon_N}\right)^2[(1+2Q_3^2)E\|x_t\|_C^2 + 2P_3^2e^{-2\delta N}]$$
$$\leq \left(\tfrac{4}{L\varepsilon_N}\right)^2[(1+2Q_3^2)Ke^{-\theta N} + 2P_3^2e^{-2\delta N}].$$

Hence, one can find a constant $L_1 > 0$ such that

$$I_1 \leq \left(\tfrac{4}{L\varepsilon_N}\right)^2 L_1e^{-\theta N/2}.$$

Next, by assumption (H2), Lemma 2.1 and Theorem 4.1, we have

$$I_2 \le \left(\frac{4}{L\varepsilon_N}\right)^2 E\left|\int_N^{N+1} AS(t-s)f(s,x_s)ds\right|^2$$
$$\le \left(\frac{4}{L\varepsilon_N}\right)^2 \frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}} \int_N^{N+1} E|(-A)^\alpha f(s,x_s)|^2 ds$$
$$\le \left(\frac{4}{L\varepsilon_N}\right)^2 \frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}} 2\int_N^{N+1} [Q_3^2 E\|x_s\|_C^2 + \xi_3(s)]ds$$
$$\le \left(\frac{4}{L\varepsilon_N}\right)^2 L_2 e^{-\theta N},$$

for some constant $L_2 > 0$, and

$$I_3 \le \left(\frac{4}{L\varepsilon_N}\right)^2 E\left|\int_N^{N+1} S(t-s)a(s,x_s)ds\right|^2$$
$$\le \left(\frac{4}{L\varepsilon_N}\right)^2 \int_N^{N+1} E|a(s,x_s)|^2 ds$$
$$\le \left(\frac{4}{L\varepsilon_N}\right)^2 \int_N^{N+1} [Q_1 E\|x_s\|_C^2 + \xi_1(s)]ds$$
$$\le \left(\frac{4}{L\varepsilon_N}\right)^2 L_3 e^{-\theta N},$$

for some constant $L_3 > 0$.
Finally, by using Lemma 5.1 [2], Lemma 7.7 [1] together with assumption (H2) and Theorem 4.1:

$$I_4 \le \left(\frac{4}{L\varepsilon_N}\right)^2 4\int_N^{N+1} [Q_2 E\|x_s\|_C^2 + \xi_2(s)]ds$$
$$\le \left(\frac{4}{L\varepsilon_N}\right)^2 L_4\, e^{-\theta N}, \quad L_4 > 0.$$

Hence, there exists a constant $\eta > 0$ such that

$$P(\sup_{N\le t\le N+1}|x(t)|> \varepsilon_N) \le \frac{\eta}{\varepsilon_N^2} e^{-\theta N/2}.$$

Therefore, the conclusion follows from the Borel–Cantelli lemma. ■

5. An Example

Consider the stochastic partial neutral functional differential equation with finite delays r_1, r_2 and $r_3(\infty > r > r_i \ge 0, i = 1,2,3)$:

$$d\left[z(t,x) + \frac{\alpha_3}{\|(-A)^\alpha\|}\int_{-r_3}^0 f_1(t,z(t+u,x))du\right] = \left[\frac{\partial^2}{\partial x^2}z(t,x) + \alpha_1 a_1(t,z(t-r_1 x))\right]dt$$
$$+ \alpha_2 b_1(t,z(t-r_2,x))d\beta(t), \quad t > 0; \tag{5}$$
$$\alpha_i \ge 0, \quad i = 1,2,3; \quad z(t,0) = z(t,\pi) = 0, \quad t > 0;$$
$$z(s,x) = \varphi(s,x), \quad \varphi(s,\bullet) \in L^2[0,\pi], \quad -r \le s \le 0, \quad 0 \le x \le \pi;$$

where $\beta(t)$ is a standard one–dimensional Wiener process and $E\|\varphi\|_{\alpha,0}^2 < \infty$. Note that when $\alpha_3 = 0$, equation (5) reduces to the stochastic heat equation, see [4,9].

Let the functions $a_1(t,u), b_1(t,u)$ and $f_1(t,u)$ be defined as follows: $a_1 : R^+ \times R \to R$, $f_1 : R^+ \times R \to R$ and $b_1 : R^+ \times R \to R$ are continuous with respect to the second argument. Moreover, let the following assumptions hold a.s.:
(A1) The functions $a_1(t,u)$ and $b_1(t,u)$ satisfy the local Lipschitz conditions:

$|a_1(t,u_1) - a_1(t,u_2)| \leq C_1(t)|u_1 - u_2|, \quad C_1(t) > 0;$

$|b_1(t,u_1) - b_1(t,u_2)| \leq C_2(t)|u_1 - u_2|, \quad C_2(t) > 0;$

for all $u_1, u_2 \in R$.
(A2) The function $f_1(t,u)$ is continuous in t and satisfies:

$|f_1(t,u_1) - f_1(t,u_2)| \leq C_3(t)|u_1 - u_2|, \quad C_3(t) > 0;$

for all $u_1, u_2 \in R$.
(A3) The function $f_1(t,u)$ is continuous in the sense that

$\lim_{t\to s}|f_1(t,u_t) - f_1(s,u_s)| \to 0.$

(A4) There exist constants Q_1, Q_2 and $Q_3 \geq 0$ and continuous functions $\xi_1, \xi_2, \xi_3 : R^+ \to R^+$ as in assumption (H5) such that

$|a_1(t,u)|^2 \leq Q_1|u|^2 + \xi_1(t),$

$|b_1(t,u)|^2 \leq Q_2|u|^2 + \xi_2(t),$

and

$|f_1(t,u)| \leq Q_3|u| + \xi_3(t),$

for all $u \in R$ and $t \geq 0$.

Assume further that $a_1(t,0) = b_1(t,0) = f_1(t,0) \equiv 0$ a.e. t so that equation (5) admits a trivial solution.
Take $X = L^2[0,\pi]$, $Y = R = (-\infty,\infty)$. Define $A : X \to X$ by $A = \partial^2/\partial x^2$ with domain $D(A) = \{w \in X : w, \partial/\partial x$ are absolutely continuous, $\partial^2 w/\partial x^2 \in X, w(0) = w(\pi) = 0\}$. Then

$Aw = \sum_{n=1}^{\infty} n^2(w,w_n)w_n, \quad w \in D(A),$

where $w_n(x) = \sqrt{2/\pi}\ \sin nx$, $n = 1,2,3,\ldots$, is the orthonormal set of eigenvectors of A. It is well–known that $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$ in X, and is given by

$S(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}(w,w_n)w_n, \quad w \in X,$

that statisfies $\|S(t)\| \leq e^{-\pi^2 t}$, $t \geq 0$. We define A^α (actually $|A|^\alpha$) for self–adjoint operator A by the classical spectral theorem

$|A|^\alpha e^{-At}w = \sum_{n=1}^{\infty}(n^2)^\alpha e^{-n^2 t}(w,w_n)w_n,$

see Taniguchi, et al [9].

Define now

$$f(t,z_t) = \frac{\alpha_3}{\|(-A)^\alpha\|}\int_{-r_3}^{0} z(t+u,x)du,$$

$$a(t,z_t) = \alpha_1 a_1(t,z(t-r_1,x)),$$

and

$$b(t,z_t) = \alpha_2 b_1(t,z(t-r_2,x)).$$

Next,

$$\|f(t,z_t)\|_\alpha = \left|\frac{\alpha_3}{\|(-A)^\alpha\|}\int_{-r_3}^{0}(-A)^\alpha f_1(t,z(t+u,x))du\right|$$
$$\leq \frac{\alpha_3}{\|(-A)^\alpha\|}C_3(t)\int_{-r_3}^{0}|(-A)^\alpha z(t+u,x)|du$$

$$\leq \alpha_3 r_3 C_3(t) \|z_t\|_C, \quad \text{a.s.}$$

This shows that $f : R^+ \times C \to X_\alpha$ and it follows that $f(t,u)$ satisfies a local Lipschitz condition with constant $\alpha_3 \gamma_3 C_3(t)$. Similarly, one can show that $a : R^+ \times C \to X$ and $b : R^+ \times C \to L(Y,X)$. Hence, equation (5) can be expressed as equation (1).
By assumption (A1), we have

$$|a(t,z_t^1) - a(t,z_t^2)|^2 = \int_0^\pi |a_1(t,z^1(t-r_1,x)) - a_1(t,z^2(t-r_1,x))|^2 dx$$

$$\leq C_1^2(t) \int_0^\pi |z^1(t-r_1,x) - z^2(t-r_1,x)|^2 dx$$

$$\leq \pi C_1^2(t) \|z_t^1 - z_t^2\|_C,$$

demonstrating that $a(t,u)$ satisfies a local Lipschitz condition. It can be verified similarly for $b(t,u)$.
Further, by assumption (A4):

$$|a(t,z_t)|^2 = \int_0^\pi |a_1(t,z(t-r_1,x))|^2 dx$$

$$\leq \int_0^\pi \left[Q_1 |z(t-r_1,x)|^2 + \xi_1(t) \right] dx$$

$$\leq Q_1 \pi \|z_t\|_C^2 + \pi \xi_1(t), \quad t \geq 0.$$

The remaining conditions can be verified similarly. Thus, all the assumptions of Theorem 4.2 are fulfilled. Therefore, the almost sure exponential stability of a solution of equation (5) follows provided

$$Q_3 < 1/3r_3,$$

and for $0 < m < 1$,

$$\frac{4}{(1-Q_3)^2} \left[\frac{2[(1-\alpha)/(1-m)]^{2(1-\alpha)} \Gamma(2\alpha-1) Q_3^2}{\pi^{2(2\alpha-1)}} + \frac{Q_1}{\pi^2} + 4\, Q_2 \right] < \pi^2.$$

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