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A Sufficient Condition for Existence of the Maximal Nonpositive Solution for a Certain Quadratic Matrix Equation*

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Abstract. *This paper is concerned with a quadratic matrix equation that arises in the analysis of mass-spring systems. By the use of a basic fixed-point iteration, we propose a sufficient condition for the existence of the maximal nonpositive solution to this equation. We also show that Bernoulli's method and Newton's method converge to the maximal nonpositive solution under the proposed sufficient condition. Moreover, we discuss some properties of the maximal nonpositive solution and provide a numerical result to validate the proposed condition.*

Key words : Quadratic Matrix Equation, M-matrix, Fixed-Point Iteration, Newton's Method, Bernoulli's Method.

AMS Subject Classifications : 65F30, 65H10, 15A24

1. Introduction

We consider the quadratic matrix equation (QME)

$$Q(X) = AX^2 + BX + C = 0, \tag{1}$$

with $X \in \mathbb{R}^{n\times n}$, an unknown matrix, A, B and $C \in \mathbb{R}^{n\times n}$ are the known coefficient matrices. Throughout this paper, we shall always assume that A is a diagonal matrix with positive diagonal elements, B is a nonsingular M-matrix and C is an M-matrix such that $B^{-1}C \geq 0$. The concept of the M-matrix and its properties will be introduced in the latter part of this section. The above assumptions are motivated by a quadratic eigenvalue problem (QEP)

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$$Q(\lambda)x = (\lambda^2 A + \lambda B + C)x = 0, \tag{2}$$

arising from a damped mass-spring system [11,12] with

$$A = \mathrm{diag}(m_1, \cdots, m_n),\ B = \tau\mathrm{tridiag}(-1,3,-1)\ \text{ and }\ C = \kappa\mathrm{tridiag}(-1,3,-1), \tag{3}$$

where, for each $i = 1,2,\cdots,n$, m_i is the weight of the i-th mass and $\tau(\kappa)$ is some damping (stiffness) constant. It is not difficult to see that the matrices A, B and C in (3) have simple structures and satisfy the conditions supposed in (1).

To solve the quadratic eigenvalue problem (2), Higham and Kim [10] indicated that the quadratic matrix polynomial $Q(\lambda)$ can be factored as

$$Q(\lambda) = (\lambda A + AX + B)(\lambda I - X).$$

Therefore, if the solution X of the corresponding QME (1) is formed, the eigenpairs of QEP (2) can be constructed computionally from the solution X. Such an approach shall be called the solution method and depends heavily on finding the solution efficiently in QME (1).

Newton's method and Fixed-point methods are usual iterative schemes for computing the solution of QME (1). It is well known that theses methods may have local convergence, if they are well defined and not incorporating any global strategy such as linear search. There are some basic problems pertaining to the existence of solutions to QME (1) and to determination of which solution the iterative methods will converge to. In this paper, we give a sufficient condition for the existence of a maximal nonpositive solution in QME (1), which is only tied to the coefficient matrices and can easily be verified. We also show that , under our proposed condition, Newton's method and Bernoulli's method (a special fixed-point method) with the initial zero matrix converge to the maximal nonpositive solution. As a byproduct, we further demonstrate some properties of the maximal nonpositive solution.

The concept of an M-matrix and some of its properties are stated as follows and can be found in [13]. For matrices $A, B \in \mathbb{R}^{n\times n}$, we write $A \geq B(A > B)$ if $a_{ij} \geq b_{ij}(a_{ij} > b_{ij})$ for all i, j. A real square matrix A is called a Z-matrix if all its off-diagonal elements are nonpositive. It is clear that any Z-matrix A can be written as $sI - B$ with $B \geq 0$. A Z-matrix $A = sI - B$ with $B \geq 0$ is called an M-matrix if $s \geq \rho(B)$, where $\rho(\cdot)$ denotes the spectral radius. It is called a singular M-matrix if $s = \rho(B)$ and a nonsingular M-matrix if $s > \rho(B)$.

Lemma 1.1. *Given that A is a Z-matrix , A is an M-matrix if and only if there exists a nonzero vector $v \geq 0$ such that $Av \geq 0$.*

Lemma 1.2. *For a Z-matrix A, the following statements are equivalent:*
(a) A is a nonsingular M-matrix.
(b) A is nonsingular and satisfies $A^{-1} \geq 0$.
(c) $Av > 0$ for some vector $v > 0$.
(d) All eigenvalues of A have positive real parts.

It is not difficult to see that the next result is a direct consequence of (a) and (c) in Lemma 1.2.

Lemma 1.3. *Let A be a nonsingular M-matrix. If $B \geq A$ is a Z-matrix, then B is also a nonsingular M-matrix.*

In the sections that follow, for a square matrix $A \in \mathbb{R}^{n\times n}$, we will denote by $\lambda(A)$ the set of eigenvalues of A, and $\|A\|$ the Frobenius norm of A. The operator vec(A) represents a column vector of size $n^2 \times 1$ whose entries come from A by stacking up columns of A. The rest of this paper is organized as follows. We review Newton's method and Bernoulli's method for finding a nonpositive solution of QME (1) in the next section. In Section 3, we construct a basic fixed-point iteration and to subsequently propose a sufficient condition of the existence of the maximal nonpositive solution. In Section 4, we establish the convergence of Newton's method and Bernoulli's method under the proposed sufficient condition. Some properties of the maximal nonpositive solution are also discussed. At last, Section 5 is devoted to a numerical experiment that validates the proposed sufficient condition.

2. Newton's and Bernoulli's Methods

While reviewing Newton's and Bernoulli's methods for finding a solution of QME (1), we first show that QME (1) can be transformed to a simpler form. In fact, by multiplying (1) by A^{-1} from the left, we have

$$\bar{Q}(X) = X^2 + A^{-1}BX + A^{-1}C = 0. \tag{4}$$

Notice further that $A^{-1}B$ is a Z-matrix and $(A^{-1}B)^{-1} = B^{-1}A \geq 0$, then, by Lemma 1.2, $A^{-1}B$ is a nonsingular M-matrix. Since C is a M-matrix, there exists a vector $v \geq 0$ such that $Cv \geq 0$. It then follows, by Lemma 1.1, from the inequality $A^{-1}Cv \geq 0$ that $A^{-1}C$ is also an M-matrix Furthermore, we have $(A^{-1}B)^{-1}A^{-1}C = B^{-1}C \geq 0$, which indicates that the QME (4) is equivalent to

$$Q(X) = X^2 + BX + C = 0, \tag{5}$$

with B as a nonsingular M-matrix and C as an M-matrix satisfying $B^{-1}C \geq 0$.
Here and after, we shall consider only iterative methods for solving QME (5). Bernoulli's method, as considered in [10], is of the following form

$$(X_k + B)X_{k+1} = -C, \quad k = 0, 1, \cdots\cdots, \tag{6}$$

and is essentially a special fixed-point iteration. The convergence result, under assumptions we have made in the Introduction, is summarized in what follows. Note that an additional condition is necessary to guarantee that the sequence $\{X_k\}$ converges to a nonpositive solution.

Theorem 2.1. *For Bernoulli's method, with initial matrix $X_0 = 0$, if there is a negative matrix X such that $Q(X) \leq 0$, then QME (5) has a nonpositive solution X^* and the sequence $\{X_k\}$ produced by (6) converges to X^* with $X^* \geq X$. Moreover, the matrix X^* is such that $X^* + B$ is an M-matrix.*

Proof. We prove this theorem by induction on k. In view of (6), we have

$$X_1 = -B^{-1}C \leq X_0 \quad \text{and} \quad X_0 > X,$$

where X is a negative matrix such that $Q(X) \leq 0$. Moreover, it is clear that $X_k + B$ is a nonsingular M-matrix. Then the following

$$X_k > X, \quad X_k + B \text{ is a nonsingular M-matrix and } X_{k+1} \le X_k, \tag{7}$$

should hold true for $k = 0$.
Suppose next that (7) is true for all $k = i \ge 0$. We are going to show that it is true for $i+1$ too. By the use of (6) and the induction assumption, we have

$$(X_i + B)(X_{i+1} - X) = (X_i + B)X_{i+1} - BX - X_iX$$
$$= X^2 - X_iX = (X - X_i)X > 0. \tag{8}$$

Since $I \otimes (B + X_i)$ is a nonsingular M-matrix, then by Lemma 1.2, it follows that $X_{i+1} > X$. Moreover, in view of (6), we have

$$(X_{i+1} + B)(X_{i+1} - X) = ((X_{i+1} - X) + (X + B))(X_{i+1} - X)$$
$$= (X_{i+1} - X)^2 + (X - X_i)X_{i+1} - Q(X) > 0, \tag{9}$$

which shows, by Lemma 1.2, that $I \otimes (B + X_{i+1})$ is a nonsingular M-matrix. Finally, the inequality

$$(X_{i+1} + B)(X_{i+2} - X_{i+1}) = -C - (X_{i+1} + B)X_{i+1}$$
$$= -C - X_{i+1}^2 - BX_{i+1} = (X - X_i)X_{i+1} \le 0, \tag{10}$$

indicates, by Lemma 1.2, that $X_{i+2} \le X_{i+1}$. So we have shown by induction that (7) is true for all $k \ge 0$. Bernoulli's method is therefore well defined and the sequence $\{X_k\}$ in (6) is monotonically non-increasing and bounded from below by X. Let $lim_{k\to\infty} X_k = X^*$. It is obvious then that X^* is a nonpositive solution to (5). By the definition of M-matrix, $B + X_k$ can be written as $sI - N_k$ with $N_k \ge 0$ and $s > \rho(N_k)$ for all $k \ge 0$. Therefore, $B + X^* = sI - N$, with $s \ge \rho(N)$, where $N = lim_{k\to\infty} N_k$. So, $B + X^*$ is a M-matrix. ■

We turn now to Newton's method. Since the Fréchet derivative of Q at X is a linear map $Q_X^{'}: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by

$$Q_X^{'}(E) = EX + (X + B)E,$$

the basic Newton iterative format

$$X_{k+1} = X_k - (Q_{X_k}^{'})^{-1} Q(X_k), \quad k = 0, 1, \cdots\cdots, \tag{11}$$

(if $(Q_{X_k}^{'})^{-1}$ is nonsingular at X_k) can then be rewritten as

$$X_{k+1}X_k + (X_k + B)X_{k+1} = X_k^2 - C, \quad k = 0, 1, \cdots\cdots. \tag{12}$$

The convergence of Newton's method, under the assumptions of the Introduction, may be listed as follows.

Theorem 2.2. *For Newton's method with initial matrix $X_0 = 0$. If there is a negative matrix X such that $Q(X) \le 0$, then QME (5) has a nonpositive solution X^* and the sequence $\{X_k\}$ produced by (12) converges to X^* with $X^* \ge X$. Moreover, the matrix X^* is such that*

$$M_{X^*} = (X^*)^T \otimes I + I \otimes (X^* + B)$$

is an M-matrix, where $\otimes$ is the Kronecher product.

Proof. Let $M_{X_k} = (X_k)^T \otimes I + I \otimes (X_k + B)$. The proof is similar to that of Theorem 2.1 and is carried out by induction on the statement

$$X_{k+1} \le X_k,\ \ X_k > X \ \text{ and } \ M_{X_k} \ \text{ is a nonsingular M-matrix} \tag{13}$$

for each $k = 1, 2. \dots$ ■

Remark 2.1. We will see later that X^* in Theorem 2.1 and Theorem 2.2 is the maximal nonpositive solution of QME (5).

3. A Sufficient Condition for the Existence of the Maximal Nonpositive Solution

Theorems 2.1 and 2.2 illustrate that the negative matrix X plays an important role in their proof. However, such a negative matrix X may not exist and in this case, searching for such a negative matrix is definitely fruitless.

In this section, we propose a sufficient condition for the existence of the nonpositive solution to QME (5) which may be based on another fixed-point iteration

$$X_{k+1} = -B^{-1}X_k^2 - B^{-1}C, \quad k = 0, 1, \cdots\cdots, \tag{14}$$

with $X_0 = 0$. This condition is only tied with the coefficient matrices of QME (5) and can easily be verified.

Theorem 3.1. *If $B - C - I$ is a nonsingular M-matrix, then QME (5) has a nonpositive solution X^* with $\rho(X^*) \le 1$. Also $B + X^*$ and $B + X^* - C$ are both nonsingular M-matrices. In particular, the matrix X^* is the maximal nonpositive solution of (5).*

Proof. It follows from (14) that $X_1 = -B^{-1}C \le X_0$. We now assume that

$$X_k \le X_{k-1} \le 0 \tag{15}$$

for $k = i \ge 0$, then the inequality

$$X_{i+1} - X_i = -B^{-1}[X_i(X_i - X_{i-1}) + (X_i - X_{i-1})X_{i-1}] \le 0$$

shows that (15) is true for $k = i + 1$. Therefore the sequence $\{X_k\}$ is monotonically non-increasing by induction.

If $B - C - I$ is nonsingular M –matrix, then there exists a vector $v > 0$ such that

$$(B - C - I)v > 0.$$

Since $B^{-1} \ge 0$ with no zero row, we, by lemma 1.2, have

$$v > B^{-1}v + B^{-1}C\,v\,. \tag{16}$$

Furthermore, we will show, by induction, that $B + X_k$ are nonsingular M –matrices and $-X_k v < v$ for all $k \ge 0.$ The above statement holds obviously for $k = 0$. Suppose then that $B + X_k$ is a nonsingular M-matrix and $-X_k\, v < v$ is true for $k = i$. Moreover, according to (14) and (16) we have

$$-X_{i+1}\,v = B^{-1}X_{i+1}^{2}\,v + B^{-1}C\,v < B^{-1}v + B^{-1}C\,v < v. \qquad (17)$$

This results with

$$(B+X_{i+1})v > B\,v - v > C\,v\,. \qquad (18)$$

Since $B^{-1} \geq 0$ we have $(I+B^{-1}X_{i+1})v > B^{-1}C\ v \geq 0$. Therefore, by virtue of Lemma1.2, $I+B^{-1}X_{i+1}$is a nonsingular M-matrix. Hence $B+X_{i+1}$ is nonsingular with

$$(B+X_{i+1})^{-1} = (I+B^{-1}X_{i+1})^{-1}B^{-1} \geq 0\,.$$

By lemma 1.2 and (17), $B+X_k$ are all nonsingular M-matrices and $-X_k\,v < v$ for all $k \geq 0$. Now the sequence $\{X_k\}$ is monotonically non-increasing and bounded from below, and has a limit X^* with $(I+X^*)\ v \geq 0$, i.e. $\rho(X^*) \leq 1$. We also know from the earlier proof that the sequence $\{(B+X_k)^{-1}\}$ is monotonically non-decreasing and bounded from above. Hence the limit $B+X^*$ is a nonsingular M-matrix. Using moreover the limit in (18) yields

$$(B+X^*)\ v \geq B\,v - v > C\,v\,.$$

Since $B+X^*$is a Z –matrix, then $B+X^* - C$ is a nonsingular M –matrix.

If there is another nonpositive solution X^{**} of (5), it is clear that $X_0 \geq X^{**}$. Assume then that $X_k \geq X^{**}$ holds for $k = i$. by induction and the inequality

$$X_{i+1} - X^{**} = -B^{-1}[X_i(X_i - X^{**}) + (X_i - X^{**})X^{**}] \geq 0,$$

we deduce that $X^* \geq X^{**}$. Hence X^* is the maximal nonpositive solution of (5). ■

We can also identify additional properties of the maximal nonpositive solution of QME (5). Before that however, we state the following lemma. Its proof is straightforward.

Lemma 3.1. *Let* $W = \begin{bmatrix} 0 & -I \\ C & B \end{bmatrix}$ *and X be a solution of QME (5), then*

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}^{-1} W \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} -X & -I \\ 0 & B+X \end{bmatrix}.$$

Moreover, the eigenvalues of $B+X$ *and* $-X$ *are just the eigenvalues of W.*

Proposition 3.1. *Let X be a solution, and* X^**be the maximal nonpositive solution, of QME(5), then the following holds.*
(a) If C is nonsingular, then $B+X$ *and X are both nonsingular matrices.*
(b) If $B-C-I$ *is a nonsingular M –matrix while C is a singular one, then* X^* *is a singular matrix with the same multiplicities of the zero eigenvalue of C.*

Proof. It follows from

$$\mathrm{rank}(C) = \mathrm{rank}((B+X)X) \leq \min\{\mathrm{rank}(X), \mathrm{rank}(X+B)\}$$

that (a) is true. If $B-C-I$ is a nonsingular M-matrix, then $B+X^*$ is nonsingular by Theorem 3.1. Moreover, from Lemma 3.1 and since C is singular it follows also that X^* and C must have the same multiplicities of the zero eigenvalue. ■

The sufficient condition of Theorem 3.1 can however further be appropriately relaxed as shown in the result that follows.

Theorem 3.2. *If the matrix $B - C - I$ is an irreducible singular M-matrix, then QME (5) has the maximal nonpositive solution X^* with $\rho(X^*) \le 1$. $B + X^*$ is a nonsingular M-matrix and $B + X^* - C$ is an irreducible M-matrix.*

Proof. We know from the proof of Theorem 3.1 that the sequence $\{X_k\}$ produced by (14) is monotonically non-increasing. Hence if $B - C - I$ is an irreducible singular M-matrix, then (by the Perron-Frobenius theory) there is a vector $v > 0$ such that

$$(B - C - I)v = 0.$$

As in the previous proof of Theorem 3.1, the iterative matrices $\{X_k\}$ satisfy $-X_k v < v$, and $\{B + X_k\}$ are all nonsingular M-matrices for all $k \ge 0$. By the same assertion in Theorem 3.1, it is not difficult to show that the limit X^* of $\{X_k\}$ with $\rho(X^*) \le 1$ is such that $B + X^*$ is a nonsingular M-matrix. Also, X^* is the maximal nonpositive solution of (5) and

$$(B + X^* - C)v \ge (B - C - I)v = 0.$$

So $B + X^* - C$ is, by Lemma 1.1, an M-matrix. Finally, the irreducibility of M-matrix $B - C - I$ requires that $B - C$ is still irreducible with nonpositive off-diagonal elements. Moreover, since X^* is a nonpositive matrix, $B + X^* - C$ is an irreducible matrix ; and here the proof completes. ■

4. Convergence of Iterative Methods Under the Sufficient Condition

In this section, we illustrate the convergence of Bernoulli's and Newton's methods under the proposed sufficient condition of last section.

Theorem 4.1. I*f $B - C - I$ is a nonsingular M-matrix, then the Bernoulli's sequence $\{X_k\}$ with $X_0 = 0$ is monotonically non-increasing and is convergent to the maximal nonpositive solution X^*. Moreover, if C is a nonsingular M-matrix such that $C^{-1}B$ is a Z-matrix, X^* is nonsingular and $(-X^*)^{-1}$ is an M-matrix.*

Proof. Let v be the positive vector such that $(B - C - I)v > 0$. It follows from (6) and (16) that $-X_1 v = B^{-1}Cv < v$ and $X_1 \le X_0$. Also, $B + X_0$ is a nonsingular M-matrix. Therefore, the statement

$$B + X_{k-1} \text{ is a nonsingular M-matrix, } -X_k v < v \text{ and } X_k \le X_{k-1} \le 0, \tag{19}$$

holds for $k = 1$.

Now, we assume that (19) is true for $k = i$. By the same assertion in proof of Theorem 3.1, $B + X_i$ is also a nonsingular M-matrix. It follows from the assumption $-X_i v < v$ that $Cv < Cv + v + X_i v < (B + X_i)v$. Here since there is no zero row in $(B + X_i)^{-1}$, it is clear that $-X_{i+1}v = (X_i + B)^{-1}Cv < v$, and in view of (6), we have

$$(X_i + B)(X_{i+1} - X_i) = -C - BX_i - X_i^2$$

$= (X_{i-1} - X_i)X_i \leq 0.$

A left hand side multiplication of the above inequality by the nonnegative matrix $(X_i + B)^{-1}$ yields $X_{i+1} - X_i \leq 0$. We have thus proven (19) for $k = i+1$. Therefore, the Bernoulli's sequence $\{X_k\}$ is monotonically non-increasing and bounded from below. As before, let $X^* = lim_{k\to\infty} X_k$, and it is obvious that X^* is a nonpositive solution.

If there is another solution $X^{**} \leq 0$, the equality

$$(X_{k-1} + B)(X_k - X^{**}) = (X^{**} - X_{k-1})X^{**},$$

together with induction yields $X_k - X^{**} \geq 0$ for $k \geq 0$. Hence, the limit of $\{X_k\}$ satisfies $X^* \geq X^{**}$. So X^* is the maximal nonpositive solution of (5). If C is nonsingular, X^* is nonsingular by Proposition 3.1. Also, $-X_1^{-1}v = C^{-1}Bv > v + C^{-1}v > v$. Now we assume that $-X_k^{-1}v > v$ for $k = i$. Then it follows from the nonsingularity of $B + X_i$ that $X_{i+1} = -(X_i + B)^{-1}C$ is nonsingular and

$$-X_{i+1}^{-1}v = C^{-1}Bv + C^{-1}X_i v > C^{-1}(B - I)v > v. \tag{20}$$

Note that $-X_{i+1}^{-1} = C^{-1}B + C^{-1}X_i$ is also a Z-matrix if $C^{-1}B$ is a Z-matrix. Hence, (20) and the boundedness of $\{X_k\}$ imply that $(-X^*)^{-1}v \geq v > 0$. So,by Lemma 1.2, $(-X^*)^{-1}$ is a nonsingular M- matrix. ■

Remark 4.1. The above theorem indicates particularly that Bernoulli's iteration is well defined under the sufficient condition proposed in Section 3, and the singularity of the maximal nonpositive solution X^* depends only on the singularity of C. Therefore, all eigenvalues of X^* are negative with $\rho(X^*) \leq 1$ when C is nonsingular and $C^{-1}B$ is a Z-matrix.

As for the convergence rate of Bernoulli's method, we have the following result whose proof is analogous to that in [8]. For that reason it is not reported.

Theorem 4.2. *Let X^* be the maximal nonpositive solution of (5). For Bernoulli's method (6), we have*

$$\limsup_{k\to\infty} \sqrt[k]{\|X^* - X_k\|} \leq \rho((-X^*)^T \otimes (B + X^*)^{-1}). \tag{21}$$

Moreover, the equality holds if X^* is negative.

Let $M_{X^*} = (X^*)^T \otimes I + I \otimes (X^* + B)$ be defined as in Theorem 2.2. The next proposition specifies the spectral radius in (21).

Proposition 4.1. *Let X^* be the maximal nonpositive solution of (5).*
a) M_{X^*} is a nonsingular M-matrix if and only if $\rho((-X^*)^T \otimes (B + X^*)^{-1}) < 1$.
b) If M_{X^*} is an irreducible singular M-matrix, $\rho((-X^*)^T \otimes (B + X^*)^{-1}) = 1$.

Proof. Since $M_{X^*} = I \otimes (B + X^*) - (-X^*)^T \otimes I$ is a regular splitting of M_{X^*}, then assertion a) is a direct conclusion of Theorem 3.29 of [13]. Moreover, if M_{X^*} is a singular M-matrix, then there is a vector $v \neq 0$ such that $M_{X^*} \cdot v = 0$, i.e.,

$$[I \otimes (B + X^*)]\, v = ((-X^*)^T \otimes I)\, v.$$

A left hand side multiplication of the above inequality by $[I \otimes (B + X^*)]^{-1}$ yields assertion b). ■

For the convergence of Newton's method under the proposed sufficient condition, we only have a rather rough result, that we report next.

Theorem 4.3. *Let X^* be the maximal nonpositive solution of (5). If $B - C - I$ is a nonsingular M-matrix and $\rho((B + X^*)^{-1}) < 1/\rho(X^*)$, then the Newton's sequence $\{X_k\}$ produced by (12) with $X_0 = 0$ is monotonically non-increasing and converges to X^*.*

Proof. The proof is similar to that of Theorem 2.2. In fact, we also aim at showing that

$$X_{k+1} \le X_k, \quad X_k \ge X^*, \quad M_{X_k} \text{ is a nonsingular } M-\text{matrix}, \tag{22}$$

holds for $k \ge 0$. Obviously (22) is true for $k = 0$. Suppose then that (22) holds for $k = i$. Replacing X by X^* in (8) yields $X_{i+1} \ge X^*$. Since $\rho((B + X^*)^{-1}) < 1/\rho(X^*)$ and by properties of Kronecker product, we have $\rho((-X^*)^T \otimes (B + X^*)^{-1}) < 1$. It follows from Proposition 4.1 then that M_{X^*} is a nonsingular M-matrix. In view of Lemma 1.3, $M_{X_{i+1}}$ is a nonsingular M-matrix. Finally, the inequality

$$(X_{i+2} - X_{i+1})X_{i+1} + (X_{i+1} + B)(X_{i+2} - X_{i+1})$$

$$= -X_{i+1}^2 - BX_{i+1} - C = -(X_{i+1} - X_i)^2 \le 0 \tag{23}$$

shows that $X_{i+2} \ge X_{i+1}$. So (22) is true for $k = i + 1$. The rest follows along the same steps as in the proof of Theorem 2.2. ■

For the convergence rate of Newton′s method, we refer the reader to [3,4,5,6,8] for more details.

5. A Numerical Result

In this section, we verify the proposed sufficient condition for QME (5). Our experiment were done using MATLAB 7.1 on a PC with 2.13GHz AMD processor, which has unit roundoff $u = 2^{-53} \approx 1.1 \times 10^{-16}$. The stopping criterion in Bernoulli's method and Newton's method, as in [10], is

$$\frac{\|X_{k+1} - X_k\|_1}{\|X_k\|_1} \le nu,$$

where n is the dimension of the problem.

Example 5.1. Consider quadratic matrix equation (1) with $A = I_3$; $B(1,1) = B(3,3) = 20$, $B = \text{tridiag}(-10, 30, -10)$; $C = \text{tridiag}(-5, 15, -5)$. Since $B - C - I_3 = \begin{bmatrix} 5 & -5 & 0 \\ -5 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix}$ is a nonsingular M-matrix, there is a maximal nonpositive solution

$$X^* = \begin{bmatrix} -0.8679 & -0.0075 & -0.0875 \\ -0.1596 & -0.5139 & -0.1596 \\ -0.0875 & -0.0075 & -0.8679 \end{bmatrix}.$$

Newton's method with $X_0 = 0$ converges to X^* after 4 iterations and $\lambda(M_{X^*}) = \{38.4902, 38.6705, 38.9424, 8.1192, 8.2994, 8.5713, 18.2588, 18.4391, \text{st}\}$ is a nonsingular M-matrix. Bernoulli's method converges to X^* after 17 iterations and $\lambda(X^* + B) = \{39.4509, 19.2195, 9.0799\}$ and $B + X^*$ is a nonsingular M-matrix.

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