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A Streamline Diffusion Method for the Mass-Spring System

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Abstract. *We apply in this article the streamline diffusion method to solve a second order hyperbolic boundary-value problem (BVP) with a special boundary condition. We also prove an a priori error estimate of this solution for a mass-spring model problem of this BVP. The theory is illustrated by a numerical example.*

Key words : Streamline Diffusion Method, Hyperbolic Problems, Wave Equation, Error Estimate, Finite Elements.

AMS Subject Classifications : 65M12, 65M15, 65M60, 82D10, 35L80

1. Introduction

The finite element method is a well known numerical procedure for solving partial differential BVPs in a variety of applications that may encompass, e.g., structural mechanics, dynamics, acoustics, heat transfer, hydrodynamics, electric and magnetic fields, and electromagnetics. Streamline diffusion (SD) methods, addressed in this work, perform slightly better than standard finite element methods for smooth and non-smooth solutions of first order hyperbolic problems ([32] and [33]) for mass-spring systems of various dimensions. As with the finite element method, the main theoretical bases for the streamline diffusion method are variational and weighted residuals. They exhibit distinctively a comparatively higher order of accuracy and improved stability properties (see [2], [3], [6], [12]-[16], [19], [22] and [28]). It is believed that these relative merits result from the fact that the artificial diffusion intrinsic in the SD method is added only in the characteristic direction so that internal layers are not smeared out, while this added diffusion removes oscillations near boundary layers([4], [5], [7], [8], [9] and [11]).

Generally speaking, a spring is to be conceived as a system that tends to return to its equilibrium position when displaced from that position. Accordingly the spring concept has a much wider application than simply the metal spirals commonly called springs. A concept that is modelled by a wave equation with special boundary conditions ([23]-[25]). The SD method, we intend to use in this work, was introduced for the first time by Hughes and Brooks in [22] to the case of stationary linear problems. The mathematical analysis of this method together with its extensions to time-dependent problems, using space-time elements, was started by Johnson and Navert in [28] and was refined in [29], [34], [26] and [27]. The purpose of this paper is to develop an extension of the SD method to the mass-spring system which has been studied by several authors in various settings (see [10], [31], [21], [36] and [35]). One of these settings, which relates to arch bridges and viaducts, deals with static and dynamic loads due to a moving train. On another note, a mass-spring system can also simulate a facial soft tissue which is of much interest in biomedical and visualization applications([17], [18]). In order to gain a physical insight into the procedures of the SD method we start this paper by establishing the equation of motion for a spring with a single spatial dimension, then for a coupled mass spring system.

1.1. **Hyperbolic equation for the mass-spring system**

The main physical problem (see [18] and [17]) entertained in this work is introduced by means of the following two examples.

Example 1.1. Single spatial dimension.

Let us consider a spring in a non-deformed condition and denote by L its natural length. We define an auxiliary parameter x to describe properties of the spring such as its tension or its density at a given point. Let there be a one-to-one correspondence between the spring viewed as a one-dimensional smooth matter distribution and the closed interval $[a,b]$ in such a way that $x = a$ corresponds to the left end of the spring and $x = b$ to its right end. To an arbitrary point P on the spring there corresponds a point $x \in [a,b]$. The parameter x shall not be viewed as a regular spatial coordinate but can be thought of as an internal degree of freedom of the spring that is not subject to the transformations associated with the one-dimensional Galileo group. It could for instance mean non-relativistic boosts or translations. When the string is made up of N discrete masses, labeled by a discrete index j, running from 1 to N, this index would play a role analogous to x. We shall assume that the correspondence established here holds for any state of motion of the spring, exactly as in the case of the discrete model. Now consider an inertial reference frame Ω with a suitable coordinate system and suppose that the spring moves along the u-axis such that the position of a point of the spring with respect to Ω is given by the function $u(x,t)$. The tension T at a point of the spring is given by [36],

$$T(x,t) = e(x)\left(\frac{\partial u(x,t)}{\partial x} - 1\right),$$

where $e(x,t) = e_1(x,t) - x\widetilde{F}(x,t) > 0$ is the elastic function of the spring on Ω. In this way, at any given point, the force that the right portion of the spring exerts on the left portion will be $T(x,t)$ and conversely the force that the left portion of the spring exerts on the right portion will be $-T(x,t)$. The resultant force acting on this element is

$$\frac{\partial T(x,t)}{\partial x} = \frac{\partial}{\partial x}\left(e(x)\left(\frac{\partial u(x,t)}{\partial x} - 1\right)\right),$$

which when substituted in Newton's second law yields the following hyperbolic equation

$$\frac{\partial}{\partial x}\left(e_1(x,t)\left(\frac{\partial u(x,t)}{\partial x}-1\right)\right)-\psi(x)\frac{\partial^2 u(x,t)}{\partial t^2} = F(x,t), \tag{1}$$

where $F(x,t) = \frac{\partial \widetilde{F}(x,t)}{\partial x}$ and ψ is the linear mass density of the spring.

Example 1.2. Coupled mass spring system.

A coupled mass spring system in the spatial variable x with damping and external force is modeled by a second order partial differential equation:

$$\begin{cases} M.\nabla_x^2 u + C\nabla_t u.I + K.u = F(x,t), & (x,t) \in \Omega, \\ u = u_0, & (x,t) \in \partial\Omega, \end{cases} \tag{2}$$

in which displacement from the equilibrium position is $u(x,t)$. Here, M is the system mass matrix, u is the displacement vector (the vector of unknown displacements), K is the system stiffness matrix, $\nabla_t u$ is the vector of velocity, I is unity matrix, $\partial\Omega$ is boundary of Ω and $F(x,t)$ is the force vector.

In the next section we show that the above second order partial differential equation can be equivalently written as a coupled system of two ordinary differential equations.

1.2. Main problem

According to (1), we observe that the well-known two points boundary value problem for a mass-spring system has an analogue to the continuum case which was first formulated in [30]and [33]as follows:

$$\begin{cases} u_{tt} - u_{xx} = F(x,t), & (x,t) \in \Omega \\ u(x,0) = 0, & 0 \le x \le 2 \\ u(t,t) = u(1+t,1-t), & 0 \le t \le 1. \end{cases} \tag{3}$$

Here, Ω is a region bounded by the characteristics of the line segment $t = 0, \quad 0 \le x \le 2$, as described below

$$\Omega = \{(x,t) : 0 \le t \le 1, t \le x \le 2-t\}.$$

In [30], it is shown that there is a unique solution for (3) in a Sobolev space ([1]) : $W_2^1(\Omega) \cap W_2^1(\partial\Omega) \cap C(\overline{\Omega})$. It was demonstrated additionally that one can replace the boundary condition in (3) with

$$\begin{cases} u_t(x,0) = 0, & 0 \le x \le 2 \\ u(1,1) = 0. \end{cases} \tag{4}$$

It is possible then to introduce the variables $v = \partial u/\partial t$ and $\dot{v} = \partial v/\partial t$ to rewrite (3) as

$$\begin{cases} \dot{u} - v = 0, & \text{in} \quad \Omega \\ \dot{v} - \frac{\partial^2 u}{\partial x^2} = F(x,t), & \text{in} \quad \Omega. \end{cases} \tag{5}$$

Further combination of (5) with (4) leads to the following coupled system of equations:

$$\begin{cases} Aw(x,t) + \dot{w}(x,t) = f(x,t) & \text{in } \Omega \\ w(x,0) = 0, & \text{in } \Omega \\ w(t,t) = \begin{pmatrix} u(1+t,1-t) \\ -v(1+t,1-t) \end{pmatrix}, & \text{in } \Omega, \end{cases} \tag{6}$$

in which the notation

$$w(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}, \ \dot{w}(x,t) = \begin{pmatrix} \dot{u}(x,t) \\ \dot{v}(x,t) \end{pmatrix}, \ A = \begin{pmatrix} 0 & -1 \\ -\frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$$

and $f(x,t) = \begin{pmatrix} 0 \\ F(x,t) \end{pmatrix}$ is consistent with (2).

The rest of this article is organized as follows. In Section 2, we present and analyze the SD method for above mass-spring system as modelled by (6). Finally, in section 3, we apply the SD method to investigate stability and obtain an error estimate for the solution of the system (6).

2. The Streamline Diffusion Method

In this method, instead of using the standard Galerkin approach (which is usual in the one-dimensional Finite Element Method) we shall employ the Galerkin method both in space and time simultaneously. That is, the employed finite elements and interpolation functions shall both depend on time and space.

When carefully designed, this space-time SD method can aim at improving the stability of the solution. It should be noted however that, when employed without care, this method could lead to a very large linear systems that require solution. One of the reasons for this is the need to use continuous (in time) test and trial functions for all time scales. A possible way to avoid this difficultly, and to decrease the size of the corresponding linear system, is to work in slabs of space-time. These invoke interpolation functions that will be continuous in the spatial variables but will be discontinuous in the time variables at the common boundary of every two slabs.

In summary the SD method for solving (6) will be based on using finite elements over the space-time domain Ω. The method is constructed by assuming $0 = t_0 < t_1 <\ldots< t_N = 1$ to be a subdivision of the time interval $[0,1]$ into intervals $I_n = (t_n, t_{n+1})$, with time steps $k_n = t_{n+1} - t_n$, $n = 0,1,\ldots,N-1$ and introducing the corresponding space-time slabs, viz

$$S_n = \left\{ (x,t) : \begin{array}{l} t_{n+1} \leq x \leq 2 - t_{n+1}, \\ t \leq x \leq t_{n+1}, \\ 2 - t_{n+1} \leq x \leq t, \\ t_n < t < t_{n+1} \end{array} \right\}, \tag{7}$$

for $n = 0,1,\ldots,N-2$ and

$$S_{N-1} = \{(x,t) : t \leq x \leq 2-t, \quad t_n < t < t_{n+1}\}, \tag{8}$$

illustrated by Fig 1. Furthermore, for each n we let W^n to be a finite element subspace of $H^1(S_n) * H^1(S_n)$, based on triangularization of the slab S_n with elements of size h, where

$$\dot{W}^n = \{w \in W^n | \ w(t,t) = \begin{pmatrix} u(1+t,1-t) \\ -v(1+t,1-t) \end{pmatrix}, 0 \leq t \leq 1, \quad 0 \leq x \leq 2\}. \tag{9}$$

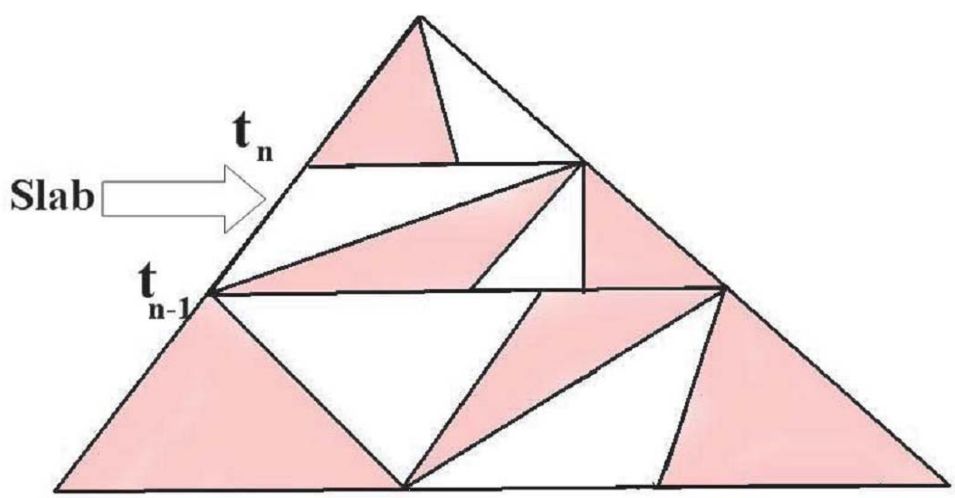

Figure 1: The slabs on triangle

Only for the sake of simplicity, we consider the boundary condition for $\dot{W}^n$ equal to zero. Consequently the SD method can be formulated on the slab S_n for (6), as follows. For $n = 0,\ldots,N-1$, find $w^n \in \dot{W}^n$ such that

$$(\dot{w}^n + Aw^n, g + \delta(\dot{g} + Ag))_n + < w^n_+, g_+ >_n - < w^n_+, g_+ >_{\Gamma_n} =$$

$$= (f, g + \delta(\dot{g} + Ag))_n + < w^n_-, g_+ >_n ,$$

where $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $w = \begin{pmatrix} u \\ v \end{pmatrix}$ and

$$(\dot{u}^n - v^n, g_1 + \delta(\dot{g}_1 - g_2))_n + < u_+^n, g_{1+} >_n \; - < u_+^n, g_{1+} >_{\Gamma_n} \; = < u_-^n, g_{1+} >_n \; ,$$

while

$$(\dot{v}^n - u^{n''}, g_2 + \delta(\dot{g}_2 - g_1''))_n + < v_+^n, g_{2+} >_n \; - < v_+^n, g_{2+} >_{\Gamma_n} \; =$$

$$= (F, g_2 + \delta(\dot{g}_2 - g_1''))_n + < v_-^n, g_{2+} >_n \; . \tag{10}$$

Our test functions are of the form $g + \delta(\dot{g} + Ag)$, where $\delta = \bar{C}h$ with $\bar{C}$ is a suitably chosen (sufficiently small, see [29]) positive constant, and (10) contains the notations that follow.

$$(u, v)_n = \int_{S_n} uvdxdt,$$

$$< u, v >_n \; = \int u(x, t_n)v(x, t_n)dx,$$

$$v_+(x, t) = lim_{s \to 0^+} v(x, t + s),$$

$$v_-(x, t) = lim_{s \to 0^-} v(x, t + s),$$

also

$$< u_+, v_+ >_\Gamma \; = \int_\Gamma u_+ v_+ d\sigma,$$

$$< u_+, v_+ >_{\Gamma_n} \; = \int_{\Gamma_n} \; < u_+, v_+ > ds,$$

where $\Gamma = \partial\Omega$ and $\Gamma = \bigcup_{n=0}^{N-1} \Gamma_n$. The terms containing $<,>$ in the relations above are jump conditions which impose a weakly enforced continuity condition across the slab interfaces, at t_n . This happens to be the mechanism by which information is propagated from one slab to another. In a more concise fashion, after summing up over n, we may rewrite (10) as follow: Find $w \in \prod_{n=0}^{N-1} W^n$, such that:

$$B(w, g) = L(g), \tag{11}$$

where $B(.,.)$ is a bilinear function satisfying, when $w \in \prod_{n=0}^{N-1} \dot{W}^n$,

$$B(w, g) = \sum_{n=0}^{N-1} \{(\dot{w}^n + Aw^n, g + \delta(\dot{g} + Ag))_n - < w_+^n, g_+ >_{\Gamma_n} \; \}$$

$$+ \sum_{n=1}^{N-1} \; < [w^n], g_+ >_n \; + < w_+^n, g_+ >_0 \; ,$$

in which

$$[w] = w_+ - w_-,$$

and $L(.)$ is a linear function satisfying

$$L(g) = \sum_{n=0}^{N-1}(f, g + \delta(\dot{g} + Ag))_n.$$

It is assumed throughout that T_h^n is a triangularization of the slab S_n into triangles K, for $h > 0$, and

$$W_h^n = \{w \in W^n : w|_K \in P_k(K) * P_k(K), K \in T_h^n\}.$$

Also, $P_k(K)$ denotes the set of polynomials in K of degree less than or equal k and

$$W_h = \prod_{n=0}^{N-1} W_h^n.$$

Thus (11) can be formulated as follows.

Find $w_h = \begin{pmatrix} u_h \\ v_h \end{pmatrix} \in W_h$ such that

$$B(w_h, g) = L(g), \tag{12}$$

for $g \in W_h$.

Moreover, we know that the exact solution of (11) satisfies

$$B(w, g) = L(g),$$

for $g \in \dot{W}^n$, and by subtraction we have the following error equation

$$B(e, g) = 0, \tag{13}$$

where $e = w - w_h$ and $w \in W_h$.

3. Stability of the SD Method

Based on the properties of of the bilinear funtion $B(.,.)$, we derive a stability estimate for the SD method (11). This estimate will be of crucial importance for the present finite element analysis.

Proposition 3.1. *For any* $w = \begin{pmatrix} u \\ v \end{pmatrix} \in \prod_{n=0}^{N-1} W^n$ *satisfying* $uv \leq 0$ *and* $\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \geq 0$ *, we have:*

$$B(w, w) \geq |||w|||^2 = \frac{1}{2}\{|\ w_-\ |_N^2 - |\ w_+\ |^2 + \delta \ \|\ \dot{w} + Aw\ \|_\Omega^2\} + |\ w_+\ |_\Gamma^2\ . \tag{14}$$

Proof. Using the definition of the bilinear form B and setting $g = w$ it follows that

$$B(w, w) = (\dot{w}, w)_\Omega + (Aw, w)_\Omega + \delta \ \|\ \dot{w} + Aw\ \|_\Omega^2 + |\ w_+\ |_\Gamma^2$$

$$+ \sum_{n=1}^{N-1} < [w], w_+ > + < w_+, w_+ >_0\ .$$

Integration by parts yields

$$(\dot{w},w)_\Omega+\sum_{n=1}^{N-1} <[w],w_+>+<w_+,w_+>_0 \;=\; \frac{1}{2}\{|\; w_- \;|_N^2\; +|\; w_+\; |_0^2\; +\sum_{n=1}^{N-1}\; |\; [w]\; |_n^2\;\}$$

Moreover, we have:

$$(Aw,w)_\Omega = \begin{pmatrix} (v,u) \\ (\frac{-\partial^2 u}{\partial x^2}, v\,) \end{pmatrix} = -\begin{pmatrix} \int_0^1\int_t^{2-t} uvdxdt \\ \int_0^1\int_t^{2-t} \frac{\partial^2 u}{\partial x^2} vdxdt, \end{pmatrix}$$

and,

$$\int_0^1\int_t^{2-t} \frac{\partial^2 u}{\partial x^2} vdxdt = -\int_0^1\int_t^{2-t} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dxdt.$$

The fact that $(Aw,w)_\Omega \geq 0$ completes the proof. ■

We then use a standard argument for finite element and introduce the linear nodal interpolate $I_h w \in W_h$ of the exact solution w to set $\zeta = w - I_h w$ and $\xi = w_h - I_h w$. Thus, we have:

$$e := w - w_h = (w - I_h w) - (I_h w - w_h) = \zeta - \xi.$$

After recalling the Galerkin orthogonality relation (13),

$$B(e,w) = 0,$$

we can prove a result on the basic global error estimate of this method.

Proposition 3.2. *If $w_h \in W_h$ satisfies (12) and w is the exact solution of the mass-spring (6), and if*

$$\|\; A\; \|_{\infty,\Omega}\; \leq C,$$

then, there is a constant C such that

$$|||w - w_h||| \leq Ch^{k+1/2}\|w\|_{k+1}. \tag{15}$$

Proof. Using the basic stability estimate with $w = e$ and invoking (14), with $w = \xi$, leads to

$$|||e|||^2 \leq B(e,e) = B(e,\eta) - B(e,\xi) = B(e,\eta)$$

$$= (\dot{e} + Ae, \zeta + \delta(\dot{\zeta} + A\zeta))_\Omega + \sum_{n=0}^{N-1} <[e],\zeta_+>_n \; +< e_+,\zeta_+>_\Gamma\; .$$

Taking into consideration that $2ab \leq \epsilon a^2 + \epsilon^{-1}b^2$ for a, b real numbers and $\epsilon > 0$, allows for

$$B(e,\zeta) \le \frac{\delta}{8} \| \dot{e} + Ae \|_{\Omega}^{2} + \frac{2}{\delta} \| \zeta \|_{\Omega}^{2} + \frac{\delta}{8} \| \dot{e} + Ae \|_{\Omega}^{2} + 2\delta \| \dot{\zeta} + A\zeta \|_{\Omega}^{2} +$$

$$+ \frac{1}{4} \sum_{n=1}^{N-1} | [e] |_{n}^{2} + \sum_{n=1}^{N-1} | \zeta_{+} |_{n}^{2} + \frac{1}{4} | e_{+} |_{0}^{2} + | \zeta_{+} |_{0}^{2} + \frac{1}{4} \| e_{+} \|_{\Gamma}^{2} + \| \zeta_{+} \|_{\Gamma}^{2} \le$$

$$\le \frac{1}{4} |||e|||^{2} + \left\{ \frac{2}{\delta} \| \zeta \|_{\Omega}^{2} + 2\delta \| \dot{\zeta} + A\zeta \|_{\Omega}^{2} + \sum_{n=1}^{N-1} | \zeta_{+} |_{n}^{2} + | \zeta_{+} |_{0}^{2} + \| \zeta_{+} \|_{\Gamma}^{2} \right\}.$$

Moreover

$$\| \dot{\zeta} + A\zeta \|_{\Omega} \le \| \dot{\zeta} \|_{\Omega} + \| A \|_{\infty,\Omega} \| \zeta \|_{\Omega} ,$$

and

$$\| \zeta' \|_{\Omega} \le Ch^{-1} \| \zeta \|_{\Omega} .$$

Therefore, upon assumption that $\delta = \bar{C}h$, we obtain:

$$|||e|||^{2} \le C \left\{ \| \zeta_{+} \|_{\Gamma}^{2} + h^{-1} \| \zeta \|_{\Omega}^{2} + \sum_{n=0}^{N-1} | \zeta_{+} |_{n}^{2} + h \| \zeta \|_{1,\Omega}^{2} \right\}.$$

Finally, by standard interpolation theory it follows that (see e.g. Ciarlet [11])

$$\left[h \| \zeta_{+} \|_{\Gamma}^{2} + \| \zeta \|_{\Omega}^{2} + h \sum_{n=0}^{N-1} | \zeta_{+} |_{n}^{2} + h^{2} \| \zeta \|_{1,\Omega}^{2} \right]^{1/2} \le Ch^{k+1} \| w \|_{k+1,\Omega} ,$$

which proves the claimed estimates. ■

4. Numerical Experiments

In this section we provide a numerical example for testing the SD method. The algorithm (4) is carried out using an AMD Opteron computer having a 15 Gegabytes RAM memory with 2.2 GHz CPU. For each slab S_n, let x_i^n be a mesh, portitioned into intervals $J_i^n = (x_{i-1}^n, x_i^n)$, with $h_i^n = x_i^n - x_{i-1}^n$. We define the global mesh function $h = h(x,t)$ viz $h(x,t) = h^n(x)$ and the time mesh function $k = k(t)$ viz $k(t) = k_n$ for $t \in (t_n, t_{n+1})$. For $h > 0$ let T_h^n be a triangularization of the slab S_n into triangle K (cf. Fig. 1.), satisfying as usual the minimum angle condition, and indexed by the parameter h of the triangle $K \in T_h^n$. The triangularization of S_n may be chosen independently of that of S_{n-1}, but for the sake of simplicity it must satisfy quasi-uniformity conditions for finite element meshes [12]. To arrive at numerical results using the SD method, we use the finite element approximation on a space time slab with a trial function which is piecewise polynomial in space and linear in time; that is, for $(x,t) \in S_n$, we assume

$$u_h^n = \sum_{i=1}^{M} \phi_i(x)(\theta_1(t)\widetilde{u_i^n} + \theta_2(t)u_i^{n+1}) \tag{16}$$

and

$$v_h^n = \frac{\partial u_h^n}{\partial t} = \sum_{i=1}^{M} \phi_i(x)(\theta_1'(t)\widetilde{v_i^n} + \theta_2'(t)v_i^{n+1}) \tag{17}$$

such that $\{\phi_i(x_j) = \delta_{ij}\}$, $i,j = 0,\ldots,M$ are the spatial shape functions at node i and $\{\theta_1 = \frac{t_{n+1}-t}{k}, \theta_2 = \frac{t-t_n}{k}\}$ are the time linear interpolation functions. We assume moreover the nodal values of u at node i with$(t_n)_+$ and $(t_{n+1})_+$, denoted by $\widetilde{u_i^n}$ $(= \widetilde{v_i^n})$ and u_i^{n+1} $(= v_i^{n+1})$ respectively. This algorithm is subsequently applied to the test problem that follows.

Problem 4.1. The streamline diffusion method is computed for a given δ, $h = 0.01$, $k = 0.005$ (see Fig. 2). In this figure we verify numerically the rate of convergence of $E_n = \|u(x,t) - u_h^n(x,t)\|_\infty$. The results are generated after 10, 20, 30, 40, … 200 time steps. The obtain $F(x,t)$ corresponds to an exact solution which is $u(x,t) = \sin(\pi x)\cos(\pi x)$. The error of the SD approximate solution is displayed in Table 1, where the order of error is calculated using the following formula:

$$\text{Order of error} \approx \ln \frac{E_{n-1}}{E_n}.$$

Table 1. Error of the SD method at t=0.5 for h=0.1

x	$\delta = 0.00$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.10$	$\delta = 0.15$
0	0.581e-9	0.352e-8	0.233e-9	0.731e-10	0.521e-9
0.5	0.431e-10	0.423e-9	0.323e-12	0.563e-9	0.342e-9
1.0	0.206e-9	0.911e-8	0.654e-9	0.983e-9	0.532e-9
2.0	0.134e-7	0.143e-8	0.113e-11	0.701e-12	0.411e-9
	order	4.1543			

5. Conclusions

In this work we have studied the numerical solution of a linear second order hyperbolic BVP subject to the the special initial-boundary conditions (4). The streamline diffusion method is employed to obtain this solution with a priori error estimates. We have shown how these a priori error estimates emerge naturally in the SD method. Optimal error bounds for this method are currently being investigated and shall be reported in a new paper in the future.

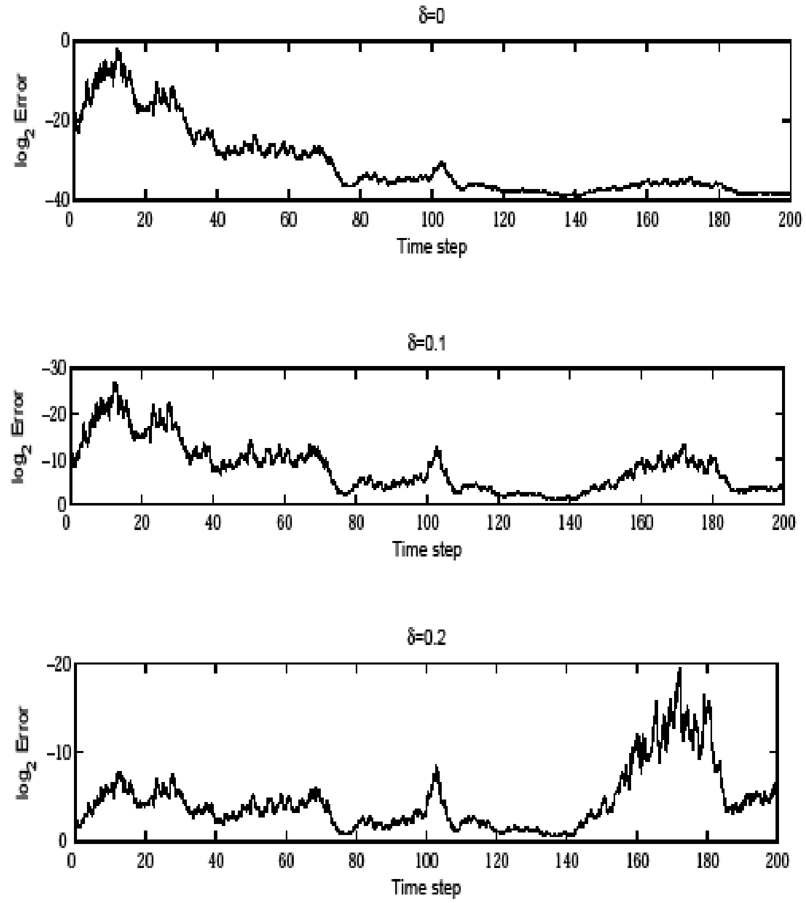

Figure 2: Convergence order of error for time steps with h=0.5 and x=0.5

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