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An Implicit Method for Some NSDDEs of Itô's Form

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Abstract. *We consider the problem of neutral stochastic delay differential equations (NSDDEs) of Itô's form with constant lag in the argument. The paper reports on an implicit method for solving these equations and on a detailed proof of its convergence. Some numerical model examples are provided to illustrate the distinctive features of this method in comparison with other available alternatives.*

Key words : Neutral Stochastic Delay Differential Equations, Itô's Form, Implicit Method, Mean-square Convergence, Absolute Mean Convergence, Stiff Equations, Numerical Experiment.

AMS Subject Classifications : 65C20

1. Introduction

We consider the evolution problem of the numerical solution of a system of neutral stochastic delay differential of equations (NSDDEs) of Itô form

$$d[X(t) - D(X(t-\tau))] = f(t, X(t), X(t-\tau))dt + g(t, X(t), X(t-\tau))dW(t), \quad t \in [t_0, T],$$

$$X(t) = \phi(t), t \in [t_0 - \tau, t_0]. \tag{1}$$

NSDDEs appear naturally in chemical engineering systems and aeroelasticity [1.2]. Convergence of implicit methods for solving these equations can be said to be currently quite well understood. In actual fact until now the convergence of only the stochastic θ-method for solving (1) has been satisfactorily studied. The reader is referred on that to [3]. Moreover the convergence of the Euler method for (1), with Markovian switching, has been discussed in [4]. Both of these methods are explicit, but the situation with implicit methods is however much more complex. As in the deterministic case, implicit methods are necessary to integrate stiff systems, where these methods are deterministically well adapted. But in those situations when the stochastic part plays an essential role in the dynamics, application of fully implicit methods, also involving implicit stochastic terms, turns out to be unavoidable. This paper deals restrictively with the construction of a new (fully) implicit method for solving (1) with rather strong convergence characteristics.

The paper is organized as follows. In section 2 we give a brief review of the basis for the proposed implicit method, together with a time discretization scheme for the system (1). Here also we study the convergence of this method. In section 3 we present some illustrative numerical results for this method, which confirm the claimed theoretical convergence characteristics. The paper concludes in section 4.

2. Analysis of the Implicit Method

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, with a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions; that is, the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, contains all P-null sets in $\mathcal{F}$.W(t) is supposed to be a standard l-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$ with mutually independent coordinates.

Let $0 \leq t_0 < T < \infty$, $\wp^d$ be the Borel σ-algebra and $f : [t_0, T] \times R^d \times R^d \to R^d, g : [t_0, T] \times R^d \times R^d \to R^{d\times l}, D : R^d \to R^d, t \in [t_0, T]$, be all Borel measurable functions.

Consider the d-dimensional system (1) of NSDDEs in Itô form ,where the initial function $\phi(t)$ is assumed to be continuous, $(\mathcal{F}_t, \wp^d)$-measurable with the finite expectation

$$E\left(\sup_{t_0-\tau\leq t\leq t_0} |\ \phi(t)\ |^2\right) < \infty. \tag{2}$$

According to Itô, satisfaction of (1) means that for $t_0 \leq t \leq T$ the relation

$$X(t) - D(X(t-\tau)) = \phi(0) - D(X(t_0-\tau)) + \int_{t_0}^{t} f(s, X(s), X(s-\tau))ds$$
$$+\int_{t_0}^{t} g(s, X(s), X(s-\tau))dW(s), \tag{3}$$

should hold.

2.1. **Assumptions**

The following assumptions concerning the function, f , g and D are subsequently made.

A1. (Lipschitz condition):

$$|\ f(t, x_1, y_1) - f(t, x_2, y_2)\ | \leq L_1(|\ x_1 - x_2\ | + |\ y_1 - y_2\ |), \tag{4}$$

$$|\ g(t, x_1, y_1) - g(t, x_2, y_2)\ | \leq L_2(|\ x_1 - x_2\ | + |\ y_1 - y_2\ |), \tag{5}$$

where L_1, L_2 are constant.

A 2. (Linear growth condition):

$$|\ f(t, x_1, x_2)\ |^2 \leq C_2(1 + |\ x_1\ |^2 + |\ x_2\ |^2), \tag{6}$$

$$|\ g(t, x_3, x_4)\ |^2 \leq C_3(1 + |\ x_3\ |^2 + |\ x_4\ |^2), \tag{7}$$

where C_2, C_3 are constant.

A 3. There exists a constant $k \neq 0$ such that,

$$|\ D(x_1) - D(x_2)\ | \leq k\ |\ x_1 - x_2\ |. \tag{8}$$

Lemma 2.1[3]. *Let the conditions (6) and (8) both hold. If $X(t)$ is a solution to the system (1) with the initial function $\phi(t)$ satisfying (2) then*

$$E\left(\sup_{t_0-\tau\le t\le T} |X(t)|^2\right) \le C_1\left(1 + E\left(\sup_{t_0-\tau\le t\le t_0} |X(t)|^2\right)\right). \tag{9}$$

2.2. **Time discretization**

Here we are presenting the Time-Discretization of the method for k^{th} component of the scheme with a uniform step on the interval $[0,T], h = \frac{T}{N}, t_n = n \cdot h$, where $n = 0,\ldots\ldots,N$, and we also assume that for the given h there is a corresponding integer m such that the lag can be expressed in the terms of the step size as $\tau = m \cdot h$,

$$x^k_{n+1} = D^k_{n+1-m} - D^k_{n-m} + x^k_n + hf^k_{n,n-m} + (g_{n,n-m}\Delta W_n)^k$$

$$+ \alpha C^k_n[(ha_1 | f^k_{n,n-m} | +a_2 | (g_{n,n-m}\Delta W_n)^k |](x^k_n - x^k_{n+1} + D^k_{n+1-m} - D^k_{n-m}), \tag{10}$$

where a_1, a_2, α are non-negative parameters and

$$C^k_n = \begin{cases} 1, \quad if \quad \| -ha_1 | f_{n,n-m} | -a_2 | g_{n,n-m} | \|_n \le \alpha \\ \frac{1}{\| -ha_1 | f^k_{n,n-m} | -a_2 | g^k_{n,n-m} | \|_n}, \qquad otherwise, \end{cases} \tag{11}$$

Here $| f_{n,n-m} |, | g_{n,n-m} |$ are absolute values of each component, i.e. for the vector $y, | y | = (|y_1|, |y_2|, \ldots, |y_n|)$. The $\| \cdot \|_n$ norm is computed using MATLAB FUNCTION "norm()".

Let K_n is a diagonal matrix whose k_{ll}-th entry is the l-th component of the vector h $a_1 | f_{n,n-m} | +a_2 | g_{n,n-m}W_n |$. This allows for $(I + \alpha C_n K_n)^{-1}$ to be uniformly bounded,viz.

$$\| (I + \alpha C_n K_n)^{-1} \| \le M,$$

where $C_n = (C^1_n, \ldots, C^d_n)$.

In the applications to follow we shall adopt the values $\alpha \in [2,3]$, $a_1 = 1$, $a_2 = 1$, the notation

$$M_n = I + \alpha C_n K_n,$$

$$B_n = \alpha C_n K_n, \tag{12}$$

and the increment function

$$\psi(h, t_n, x_n, x_{n-m}, I_\psi) = M^{-1}_n(hf_{n,n-m} + (g_{n,n-m}\Delta W_n)). \tag{13}$$

Lemma 2.2. Under the assumptions **A**1 – **A**3, there exists a constant C_4 such that for $x_1, y_1 \in R^d$ there holds

$$E(| \psi(h, t_n, x_1, y_1, I_\psi) |^2) \le C_4 h (1 + | (x_1) |^2 + | (y_1) |^2). \tag{14}$$

Lemma 2.3. *If Lemma 2.2. holds,then*

$$E(\mid X_n \mid^2) < \infty,$$

for all $n \leq N$.

The proof of these two lemmata is straightforward and we refer the interested reader to [1]. Furthermore if $x_1, x_2, y_1, y_2 \in R^d$ are analytic or numerical solutions to the system (1), the lemma that follows should hold for them.

Lemma 2. 4. *Under the assumptions* $A1 - A3$, *there exist constants* C_5 *and* C_6 *for* $x_1, x_2, y_1, y_2 \in R^d$ *such that*

$$\mid E(\psi(h, t_n, x_1, y_1, I_\psi) - \psi(h, t_n, x_2, y_2, I_\psi)) \mid$$

$$\leq C_5 h(\mid (x_1 - x_2) \mid + \mid (y_1 - y_2) \mid) + O(h), \tag{15}$$

$$E(\mid \psi(h, t_n, x_1, y_1, I_\psi) - \psi(h, t_n, x_2, y_2, I_\psi) \mid^2)$$

$$\leq C_6 h(\mid (x_1 - x_2) \mid^2 + \mid (y_1 - y_2) \mid^2) + O(h). \tag{16}$$

Consider next the inequalities,

$$p_2 \geq \frac{1}{2}, \tag{17}$$

$$p_1 \geq p_2 + \frac{1}{2}, \tag{18}$$

to state what follows.

Definition 2. 1. Let

$$\varepsilon_n = X(t_n) - x_n \qquad n = 0, 1, \ldots\ldots, N-1.$$

We say x_n converges to $X(t_n)$ in the mean-square sense with order p if,

$$\max_{1 \leqslant n \leqslant N} \Big(E(\mid \varepsilon_n \mid^2)\Big)^{\frac{1}{2}} \leq \alpha_1 h^p,$$

where α_1 is a certain constant.

Assume further that $X(t_n)$ is the solution of the approximation (10) to the system (1)-(2) at $t = t_n$. Clearly then convergence of $X(t_n)$ in $\mathcal{L}^2$ (as $h \to 0$ when $\frac{\tau}{h} \in N$) with order $p = p_2 - \frac{1}{2}$, is a convergence is in the mean-square sense.

Theorem 2. 1. *If the approximation (10) satisfies the assumptions* $A1 - A3$ *and* ψ *satisfies the estimates* (15) *and* (16), *then* $X(t_n)$ *is convergent in the mean-square sense and*

$$\max_{1 \leqslant n \leqslant N} \Big(E(\mid \varepsilon_n \mid^2)\Big)^{\frac{1}{2}} \leqslant C h^p + O(\sqrt{h}), \tag{19}$$

as $h \to 0$.

Theorem 2.2. *Under the assumption* $A1 - A3$, *there exist a positive constant* C_7 *such that the scheme* (10) *has strong order of convergence 0.5; that is*

$$\max_{1 \leqslant n \leqslant N} \Big(E(\mid \varepsilon_n \mid^2)\Big)^{\frac{1}{2}} \leqslant C_7 h^{\frac{1}{2}}.$$

Proof. Invoke the Euler discretization

$$X_{n+1}^E = X_n + D_{n+1-m} - D_{n-m} + hf_{n,n-m} + g_{n,n-m}\Delta W_n ,$$

considered in [5]. By the triangle inequality,

$$\begin{aligned} H_1 &:= \mid E\,(X(t_{n+1}) - x_{n+1})|\mathcal{F}_{t_n} \mid \\ &= \mid E\,(X(t_{n+1}) - X_{n+1}^E + X_{n+1}^E - x_{n+1})|\mathcal{F}_{t_n} \mid \\ &\le \mid E\,(X(t_{n+1}) - X_{n+1}^E)|\mathcal{F}_{t_n} \mid + \mid E\,(X_{n+1}^E - x_{n+1})|\mathcal{F}_{t_n} \mid \\ &\le O(h^{\frac{3}{2}}) + H_2, \end{aligned}$$

where,

$$H_2 = \mid E\,(X_{n+1}^E - x_{n+1})|\mathcal{F}_{t_n} \mid .$$

Using the symmetry property of W_n and the definition of B_n,

$$\left(E\,(\mid B_n \mid^2 |\mathcal{F}_{t_n})\right)^{\frac{1}{2}} \le O(h^{\frac{1}{2}}), \tag{20}$$

allows rewriting the the preceding relation as

$$\begin{aligned} H_2 &= \mid E\,(X_{n+1}^E - x_{n+1})|\mathcal{F}_{t_n} \mid \\ &= \mid E\left((I - M_n)^{-1}(hf_{n,n-m} + g_{n,n-m}\Delta W_n|\mathcal{F}_{t_n})\right) \mid \\ &= \mid E\,((M_n^{-1}B_n)(hf_{n,n-m} + g_{n,n-m}\Delta W_n)|\mathcal{F}_{t_n}) \mid \\ &= \mid E\,((M_n^{-1}B_n)(hf_{n,n-m})|\mathcal{F}_{t_n}) \mid . \end{aligned}$$

Consider next equation (12) in (20) together with Hölder's inequality to arrive at

$$\begin{aligned} H_2 &= \mid E\,((M_n^{-1}B_n)(hf_{n,n-m})|\mathcal{F}_{t_n}) \mid \\ &\le M \mid E\,(B_n(hf_{n,n-m})|\mathcal{F}_{t_n}) \mid \le O(h^{\frac{3}{2}}). \end{aligned}$$

In a similar fashion it is possible to establish that

$$\begin{aligned} H_3 &= \left(E\,(\mid X(t_{n+1}) - x_{n+1} \mid^2 |\mathcal{F}_{t_n})\right)^{\frac{1}{2}} \\ &= \left(E\,(\mid X(t_{n+1}) - X_{n+1}^E + X_{n+1}^E - x_{n+1} \mid^2 |\mathcal{F}_{t_n})\right)^{\frac{1}{2}} \\ &\le \left(E\,(\mid X(t_{n+1}) - X_{n+1}^E) \mid)^2|\mathcal{F}_{t_n}\right)^{\frac{1}{2}} + \left(E\,(\mid X_{n+1}^E - x_{n+1} \mid^2 |\mathcal{F}_{t_n})\right)^{\frac{1}{2}} \\ &\le O(h), \end{aligned}$$

and here the proof completes. ■

3. Numerical Experiments

Example 3.1. Solve the following NSDDE, subject to the accompanying initial function.

$$d(x(t) - 0.5x(t-1)) = \left(-32\frac{x(t)^2}{2 + x(t)^2} + 6.2\frac{x(t-1)^2}{2 + x(t-1)^2}\right)dt$$
$$+\left(5\frac{x(t)^2}{2 + x(t)^2} + \frac{x(t-1)^2}{2 + x(t-1)^2}\right)dW,\ t \in [0,2];\ x(t) = t + 1, \quad t \in [-1,0].$$

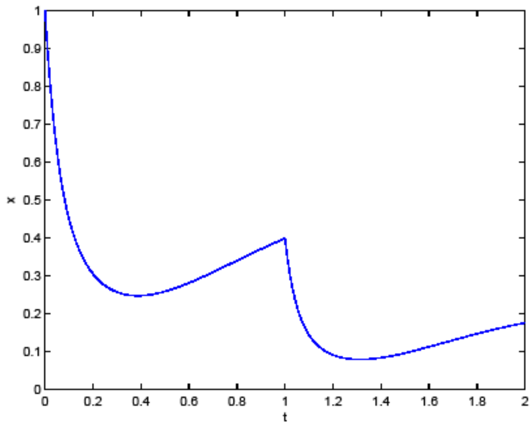

Figure 1. Sketch of the deterministic solution.

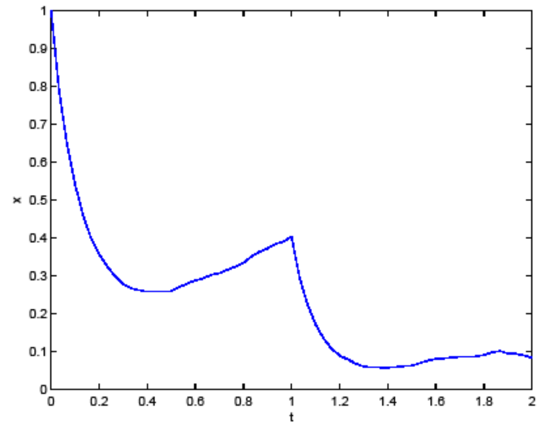

Figure 2. Average solution via (10).

Figures 1-3 are plots of various solutions to example 3.1 when $\alpha = 2$ is a common assumption. The stochastic solutions of figures 2 and 3 are both made for 600 independent sample paths, with the same step size $h = \frac{1}{100}$.

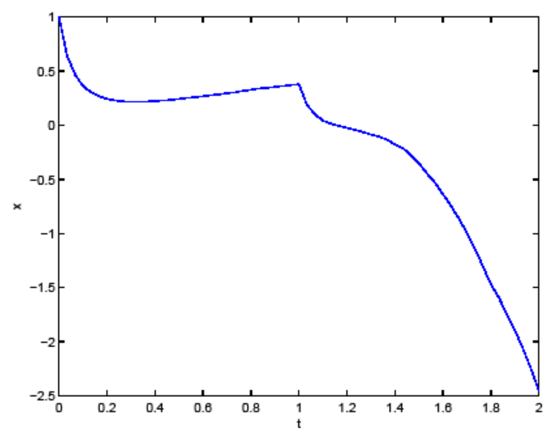

Figure 3. Average solution by Euler's method.

Example 3.2. Solve the following NSDDE, subject to the accompanying initial function.

$$d(x(t) - 0.5x(t-1)) = \left(-31\frac{x(t)^2}{2+x(t)^2} + 6.2\frac{x(t-1)^2}{2+x(t-1)^2}\right)dt$$
$$+ \left(7\frac{x(t)^2}{2+x(t)^2} + \frac{x(t-1)^2}{2+x(t-1)^2}\right)dW, \quad t \in [0,2];$$
$$x(t) = t+1, \quad t \in [-1,0].$$

Plots of various solutions to this example are given in Figures 4-6. These correspond respectively to the plots of Figures 1-3 for example 3.1 with the only difference that here $\alpha = 3$ instead of the previous $\alpha = 2$.

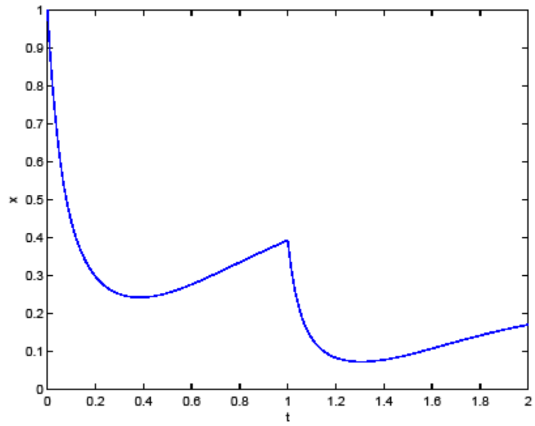

Figure 4. Sketch of the deterministic solution.

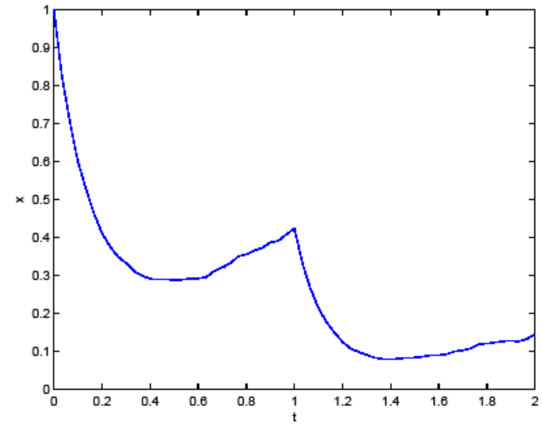

Figure 5. Average solution via (10).

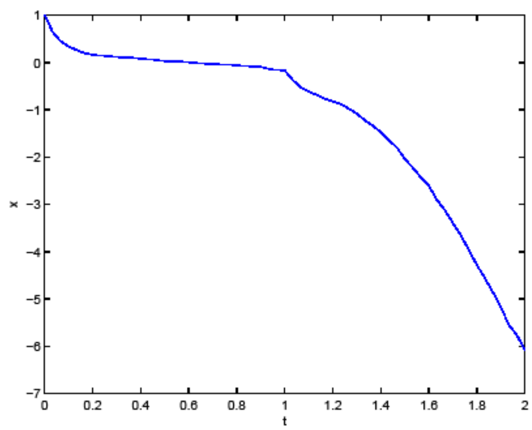

Figure 6. Average solution by Euler's method.

Comparison of the plots in Figures 3.3 and 3.6 with the respective plots of Figures 3.2 and 3.5 reveals some relative limitations of the Euler explicit method versus the present implicit scheme. We move on now to the last example of this paper.

Example 3.3. Study the accuracy of solving the following NSDDE, subject to the accompanying initial function.

$$d(x(t) - 0.6x(t-\tau)) = [-8x(t) - 8x(t-\tau)]dt + [3x(t) + 1.2x(t-\tau)]dW(t), t \in [0,2];$$

$$x(t) = t + 1, \ \ t \in [-1, 0].$$

Let us consider the implicit scheme (10) with $\alpha = 2$. Alternatively, an explicit solution by

Euler's method [4,5] is considered over the interval[1,2]. In both schemes we may employ the same idea, as in [4], of just taking a constant in the balanced part.

The mean square error when $T = 2$ can be estimated in the following way. A set of 60 blocks, each containing 100 outcomes $(\omega_{i,j}; 1 \leq i \leq 60, 1 \leq j \leq 60)$, is identified. Clearly the block estimator is

$$\varepsilon_i = (1/100)\Sigma_{j=1}^{100} \mid X(T,\omega_{i,j}) - x_N(\omega_{i,j}) \mid^2 ,$$

and the mean of this estimator is

$$\varepsilon = (1/60)\Sigma_{i=1}^{60}\varepsilon_i.$$

Results of the computations of this ε, when the constant in the balanced part is 4, are summarized in the next table.

Table 3.1. Estimated errors ε for various step sizes

Method \ h	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{30}$
Implicit (10)	0.01437	0.00620	0.000857
Explicit	0.24096	0.07371	0.02091

These results illustrate clearly the superior comparative accuracy of the employed implicit scheme, which turns out to increase with decreasing the step size.

4. Conclusions

We have advanced a new fully implicit method for an approximate solution to NSDDEs. The performed numerical tests indicate that this method appears to be effective and accurate in comparison with possible alternatives.

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