Journal of Numerical Mathematics and Stochastics, **2** (1) : 12-33, 2010
http://www.jnmas.org/jnmas2-2.pdf
Euclidean Press, LLC
Online: ISSN 2151-2302

Topological Optimization for a Controlled Dirichlet Problem in a Polygonal Domain

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Abstract. *We consider a problem of control of a distributed system in a polygonal domain and prove that its singular coefficient is expressible solely by means of its data. We subsequently study a numerical aspect of this problem using the topological optimization method. The paper provides for a representation of the topological gradient of the Dirichlet problem in this polygonal domain.*

Key words : Polygonal Domain, Partial Differential Equations, Topological Optimization, Crack, Singularity, Topological Gradient, Truncation Domain, Adjoint Method, Numerical Simulation.

AMS Subject Classifications : 65J15, 49J20, 49M27

1. Introduction

This paper focuses on the theoretical and numerical aspects of representation of the singularity coefficient for a controlled Dirichlet problem in a polygonal domain. Our goal in this work is two fold. On the one hand, we prove that the singularity coefficient of the considered Dirichlet problem, in a polygonal domain can be expressed solely with the data of the problem. To reach this goal, we utilize some results that have been established by P. Grisvard [9,10] and by the Lax-Milgram theorem. On the other hand, we employ a topological sensitivity analysis in order to represent the distribution of impact of the singularities in the considered domain. The topological sensitivity analysis aims at providing an asymptotic expansion of a shape functional on the neighborhood of a small hole created inside the domain. The reported analysis shall be based on the principle that follows.

For a criterion $j(\Omega) = J_\Omega(u_\Omega), \Omega \subset \mathbb{R}^n$, with u_Ω as the solution of a boundary value problem (BVP) defined over Ω, the pertaining expansion can generally be written in the form:

$$j(\Omega\backslash\overline{x_0+\varepsilon\varpi})-j(\Omega) = \rho(\varepsilon)g(x_0)+o(\rho(\varepsilon)),\ \ \rho(\varepsilon) > 0, \lim_{\varepsilon\to 0}\rho(\varepsilon) = 0.$$

In this expression, ε and x_0 denote respectively the radius and the center of the hole, ω is a reference domain inside Ω and containing the origin. The function $g(x_0)$ is called topological derivative and will be used as descent direction in the optimization process.

The paper is organized as follows: in the second section we introduce some notations used throughout the paper and present the problem to be addressed. In the third section, we report on the main result of this paper and it's detailed proof. In section 4 we study the topological optimization problem associated with the optimal control problem (2) together with the obtained numerical result. The paper concludes in section 5.

2. Notation and Presentation of Problem

Let Ω be an open bounded domain in $\mathbb{R}^2$, Ω has a polygonal boundary $\partial\Omega$, which is the union of the segments Γ_j for $j \in \{0,\ 1,\ \ldots, N,\ \ N \in \mathbb{N}^*\}$.
We denote by S_j the vertex between Γ_{j+1} and Γ_j for $j \in \{0,\ 1,\ \ldots, N-1\}$ and S_N the vertex between Γ_N and Γ_0. Let ω_j, for $j \in \{0,\ 1,\ 2,\ \ldots, N-1\}$ be the measure of the angle between the vertices Γ_{j+1} and Γ_j, ω_N the measure of the angle between Γ_0 and Γ_N. For $M \in \mathbb{R}^2$ we denote by θ_j the angle between $\overrightarrow{S_jM}$ and Γ_{j+1}, $1 \le j \le N-1$ and θ_N the angle between $\overrightarrow{S_NM}$ and Γ_0, as illustrated in Figure 1.
Additionally $\forall\, j \in J,\ J = \{j \in \{0,1,\ldots,N\}\ /\ \omega_j > \pi\}$, a nonempty set, we are able to introduce a truncation function $\eta_j \in \mathcal{D}(\overline{\Omega})$ which depends only on the distance r_j to S_j such that $\eta_j \equiv 1$ near S_j, and η_j vanishes near all $\overline{\Gamma_k}$ for $k = j$ and $k = j+1$.

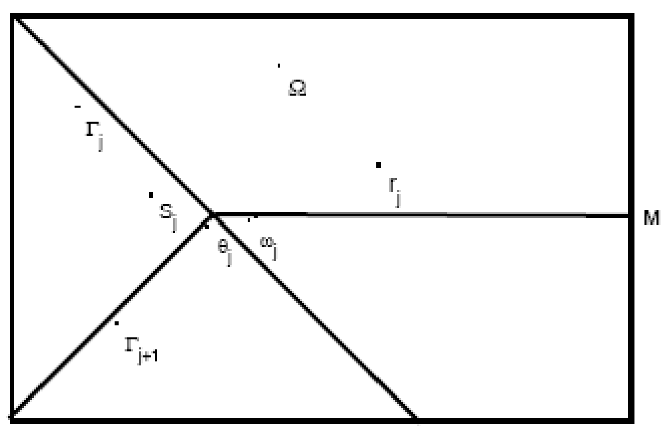


Figure 1: Sketch of a polygonal domain with vertices and angles

We suppose also that the support of η_j does not intersect the Γ_k's if $j \neq k$. Therefore

$$\theta_j = \widehat{\left(\overrightarrow{S_jM},\Gamma_{j+1}\right)}; \qquad \omega_j = \widehat{(\Gamma_j,\Gamma_{j+1})}$$

If we consider that U is a nonempty, closed and convex part of $L^2(\Omega)$, then for $f \in L^2(\Omega)$ and

$u \in U$, we can denote by $y(u)$ the unique solution in $H_0^1(\Omega)$ of the system

$$\begin{cases} -\Delta y(u) &= f+u \quad in \quad \Omega \\ \gamma y(u) &= 0 \quad on \quad \partial\Omega\,. \end{cases} \tag{1}$$

By applying the Lax-Milgram theorem, there is a unique solution $y(u)$ of the variational problem:

$$\forall\ v \in H_0^1(\Omega),\ \int_\Omega \nabla y(u).\nabla v dx = \int_\Omega (f+u)v dx\,.$$

Moreover, let y_d be an element of $L^2(\Omega)$ and $\alpha > 0$ to define the cost functional

$$J(u) = \frac{1}{2}\int_\Omega |y(u) - y_d(u)|^2 dx + \frac{\alpha}{2}\int_\Omega u^2 dx\,,$$

and invoke the following optimal control problem

$$\begin{cases} \min_{u\in U} \quad J(u) \quad , \quad subject\ to\ : \\ \quad -\Delta y(u) \quad = \quad f+u \qquad in \quad \Omega \\ \quad \gamma(y(u)) \quad = \quad 0 \qquad on \quad \partial\Omega\,. \end{cases} \tag{2}$$

When Ω is regular (for example $\mathcal{C}^2$) $y(u) \in H^2(\Omega) \cap H_0^1(\Omega)$ and if $U = L^2(\Omega)$, the control $\bar{u}$ solution of (2) belongs in $H^2(\Omega) \cap H_0^1(\Omega)$; see for example [12]. But if the boundary of Ω is polygonal, $y(\bar{u})$ and $\bar{u}$ have a singular part, i.e. they don't belong in $H^2(\Omega)$.

Our first aim here would be to find the singularity coefficient of the solution to the optimal control problem (2).

Remark 2.1. Let $g \in L^2(\Omega)$ to consider the following problem

$$\begin{cases} \quad -\Delta y(u) \quad = \quad g \\ \quad \gamma(y(u)) \quad = \quad 0 \\ y(u) \in H_0^1(\Omega)\,. \end{cases} \tag{3}$$

One result due to P. Grisvard [9] shows in fact that there exists $y_r \in H^2(\Omega) \cap H_0^1(\Omega)$ and a real C_j , $j \in J$ such that the solution could be written as

$$y = y_r + \sum_{j\in J} C_j \eta_j r_j^{\pi/\omega_j} sin\left(\frac{\pi}{\omega_j}\theta_j\right)$$

where $J = \{j \in \{0,1,\ldots,N\}\ /\ \omega_j > \pi\}$.

In addition, $\exists\ K$, a positive constant (which does not depend on g), such that

$$||y_r||_{H^2(\Omega)} + \sum_{j\in J}|C_j| \leq K||g||_{L^2(\Omega)}.$$

Also $\exists$ functions $w_j \in L^2(\Omega)$, $\forall\ j \in J$, such that

$$C_j = \int_\Omega g\ w_j\ dx.$$

which are of the form, see e.g. [9],

$$w_j = \gamma(e^{-r\sqrt{\lambda}} r^{-\alpha} \sin \alpha\theta_j + \phi),$$

$\pi < w_j < 2\pi,\ 1/2 < \alpha < 1,\ r = \sqrt{x^2 + y^2}$. where $0 < \theta_j < \pi$ and the construction of ϕ follows from several lemmas of [9].

3. Main Result

Let us consider the J functional defined by

$$J(u) = \frac{1}{2}\int_\Omega |y(u) - y_d|^2\, dx + \frac{\alpha}{2}\int_\Omega u^2 dx\,,$$

where $y(u)$ is the solution of the optimal control problem

$$\begin{cases} \min_{u\in U} \quad J(u) \quad , \quad \textit{subject to} : \\ -\Delta y(u) = f + u \quad \textit{in} \quad \Omega \\ \gamma(y(u)) = 0 \quad \textit{on} \quad \partial\Omega\,, \end{cases} \tag{4}$$

and

$$\begin{cases} -\Delta z_d = y_d \quad \textit{in} \quad \Omega \\ \gamma z_d = 0 \quad \textit{on} \quad \partial\Omega\,, \end{cases} \tag{5}$$

with u as the control function. We are then directly led to the following main result of this paper.

Theorem 3.1. *The optimal control problem (4) admits a unique solution $\bar{u} \in U$, and the optimal state corresponding to $y(\bar{u})$ may be written as:*

$$y(\bar{u}) = y_r(\bar{u}) + \sum_{j\in J} C_j \eta_j r_j^{\pi/\omega_j} \sin\left(\frac{\pi}{\omega_j}\theta_j\right)$$

where:
i) $y_r \in H^2(\Omega) \cap H_0^1(\Omega)$
ii) $C_j = \left(\frac{4\pi}{\omega_j} + 4\right)\int_\Omega w_j(f + \frac{1}{\alpha}z_d)dx, \forall j \in J,$
and w_j is defined in Remark 2.1.

Proof. Let us consider the optimal control problem (4). This problem admits a unique solution $\bar{u}$ in U satisfying the following system

$$\begin{cases} -\Delta y(\bar{u}) = f + \bar{u} \quad \textit{in} \quad \Omega \\ -\Delta \bar{p} = y(\bar{u}) - y_d \quad \textit{in} \quad \Omega \\ \langle \bar{p} + \alpha\bar{u}, v - \bar{u}\rangle_{L^2(\Omega)} \geq 0 \quad \forall v \in U \\ y(\bar{u}),\ \ \bar{p} \in H_0^1(\Omega)\,. \end{cases} \tag{6}$$

In the case where $U = L^2(\Omega)$, we obtain

$$\begin{cases} -\Delta y(\bar{u}) &= f+\bar{u} & in \quad \Omega \\ -\Delta \bar{p} &= y(\bar{u})-y_d & in \quad \Omega \\ \bar{p}+\alpha\bar{u} &= 0 & \\ y(\bar{u}), \quad \bar{p} &\in H_0^1(\Omega)\ . \end{cases} \tag{7}$$

Setting $y(\bar{u}) = \bar{y}$ leads to

$$\begin{cases} -\Delta \bar{y} &= f-\frac{1}{\alpha}\bar{p} & in \quad \Omega \\ -\Delta \bar{p} &= \bar{y}-y_d & in \quad \Omega \\ \bar{y}, \quad \bar{p} &\in H_0^1(\Omega)\ . \end{cases} \tag{8}$$

Now if $\bar{z}$ is a solution in $H_0^1(\Omega)$ of the equation

$$\begin{cases} -\Delta \bar{z} &= \bar{y} & in \quad \Omega \\ \gamma\bar{z} &= 0 & on \quad \partial\Omega\ , \end{cases} \tag{9}$$

then

$$\begin{cases} -\Delta \bar{y} &= f-\frac{1}{\alpha}\bar{p} & in \quad \Omega \\ -\Delta \bar{p} &= -\Delta(\bar{z}-z_d) & in \quad \Omega \\ \gamma\bar{z} &= \gamma(\Delta\bar{z}) = 0 & on\ \partial\Omega \\ \bar{y},\ \bar{p},\ \bar{z} \quad z_d, \in H_0^1(\Omega)\ . \end{cases} \tag{10}$$

The Laplacian is an injective operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, this implies that $\bar{p} = \bar{z}-z_d$.
From the equations (9) and (10) it follows that :

$$\begin{cases} -\Delta^2\bar{z}+\frac{1}{\alpha}\bar{z} &= f+\frac{1}{\alpha}z_d & in\ \Omega \\ \gamma\bar{z} &= \gamma(\Delta\bar{z}) = 0 & on\ \partial\Omega \\ \bar{z},\ \Delta\bar{z}, \ \in H_0^1(\Omega)\ . \end{cases} \tag{11}$$

Letting $\mathcal{V}$ to be the space defined by:

$$\mathcal{V} = \left\{ v \in H^3(\Omega) \cap H_0^1(\Omega) \ / \ \gamma(\Delta v) = 0 \ \ on \ \partial\Omega \right\} ,$$

allows using the Lax Milgram theorem to prove that equation (11) has a unique solution z in $\mathcal{V}$.
Now, consider the equation

$$\begin{cases} \Delta^2\bar{z}+\frac{1}{\alpha}\bar{z} &= f+\frac{1}{\alpha}z_d \quad in\ \Omega \\ \bar{z} \in \mathcal{V}\ , \end{cases}$$

together with $F = f+\frac{1}{\alpha}z_d-\frac{1}{\alpha}\bar{z}$,

$$\begin{cases} -\Delta\bar{z} &= \bar{y} & in\ \Omega \\ \gamma\bar{z} &= 0 = \gamma(\Delta\bar{z}) & on\ \ \partial\Omega, \end{cases}$$

allows us to deduce that

$$\begin{cases} -\Delta\bar{y} & = & F \quad in \ \Omega \\ \gamma\bar{y} & = & 0 \quad on \ \partial\Omega \\ F \ \in L^2(\Omega) \end{cases} \tag{12}$$

One result of P. Grisvard [9] shows that there exist real $C_j, \quad j \in J$ and y_r belonging to $H^2(\Omega) \cap H_0^1(\Omega)$ such that the solution could be written as:

$$\bar{y} = y_r + \sum_{j \in J} C_j \eta_j r_j^{\pi/\omega_j} \sin\left(\frac{\pi}{\omega_j}\theta_j\right)$$

with

$$\pi < \omega_j < 2\pi$$

We are going to use this decomposition and the linearity of the equation to obtain two equations to be solved

$$\begin{cases} -\Delta\overline{z_1} & = & y_r \quad in \ \Omega \\ \gamma\overline{z_1} & = & 0 \quad on \ \partial\Omega \end{cases} \quad y_r \ \in \ H^2(\Omega). \tag{13}$$

This implies that

$$\overline{z_1} = \overline{z_{1r}} + \sum_{i=1}^{N} \sum_{k \in \mathbb{Z}} \gamma_{ki} \eta_i r_i^{\frac{k\pi}{\omega_i}} \sin\left(\frac{k\pi}{\omega_i}\theta_i\right)$$

with

$$z_{1r} \in H^4(\Omega) \quad and \quad r_i^{\frac{k\pi}{\omega_i}} \sin\left(\frac{k\pi}{\omega_i}\theta_i\right) \in H^3(\Omega)_{|H^4(\Omega)}$$

and

$$\begin{cases} -\Delta\overline{z_2} & = & \eta_j r_j^{k\pi/\omega_j} \sin\left(\frac{k\pi}{\omega_j}\theta_j\right) \quad in \ \Omega \\ \gamma\overline{z_2} & = & \gamma_j\left(-r_j^{k\pi/\omega_j} \sin\left(\frac{k\pi}{\Omega_i}\theta_i\right)\right) \quad on \ \partial\Omega \end{cases} \tag{14}$$

with $\bar{z} = \bar{z}_1 + \bar{z}_2$.
The solution of the problem (13) is

$$\overline{z_1} = \overline{z_{1r}} + \sum_{i=0}^{N} \sum_{k=3}^{5} \gamma_{ki} \eta_i r_i^{\frac{k\pi}{\omega_i}} \sin\left(\frac{k\pi}{\omega_i}\theta_i\right)$$

Where $\overline{z_{1r}}$ is the regular part of $\overline{z_1}$, $z_{1r} \in H^4(\Omega)$, and the sum from 3 to 5 follows from the fact that :

$$r^\beta \sin(\alpha\theta) \in H^s(\Omega)_{|H^{s'}(\Omega)} \Leftrightarrow s-1 < \beta < s'-1$$

This implies

$$r_i^{\frac{k\pi}{\omega_i}} \sin\left(\frac{k\pi}{\omega_i}\theta_i\right) \in H^3(\Omega)_{|H^4(\Omega)} \Leftrightarrow 3-1 < \frac{k\pi}{\omega_i} < 4-1$$

$$\Rightarrow \begin{cases} \frac{2\omega_i}{\pi} & < & k & < & \frac{3\omega_i}{\pi} \\ \pi & < & \omega_i & < & 2\pi \end{cases} \Leftrightarrow \begin{cases} 2 < \frac{2\omega_i}{\pi} & < & k & < & \frac{3\omega_i}{\pi} < 6 \\ 1 & < & \frac{\omega_i}{\pi} & < & 2. \end{cases}$$

Thus $2 < k < 6 \Rightarrow k = 3,4,5$.
For the resolution of the equation (14), we look for at first a solution around crack S_j

$$\begin{cases} -\Delta w_{1,j}^k &= r_j^{k\pi/\omega_j} \sin\left(\frac{k\pi}{\omega_j}\theta_j\right) \quad in\ \Omega \\ \gamma w_{1,j}^k &= 0 \quad on\ \partial\Omega \end{cases}$$

we obtain

$$w_{1,j}^k = \frac{1}{4\frac{k\pi}{\omega_j}+4} r_j^{\frac{k\pi}{\omega_j}+2} \sin\left(\frac{k\pi}{\omega_j}\theta_j\right), \quad w_{1,j}^k \in H^3(\Omega)_{|H^4(\Omega)}, \tag{15}$$

$$r_j^{\frac{k\pi}{\omega_j}+2} \sin\left(\frac{k\pi}{\omega_j}\theta_j\right) \in H^3(\Omega)_{|H^4(\Omega)} \Leftrightarrow 3-1 < \frac{k\pi}{\omega_j}+2 < 4-1.$$

It follows that

$$\begin{cases} 0 < \frac{k\pi}{\omega_j} \leq 1 \\ 1 < \frac{\omega_j}{\pi} < 2 \end{cases} \Leftrightarrow \begin{cases} 0 < k < \frac{\omega_j}{\pi} \\ 1 < \frac{\omega_j}{\pi} < 2 \end{cases}$$

$\Rightarrow 0 < k < 2 \Leftrightarrow k = 1$.
We attempt then solving the boundary value problem

$$\begin{cases} -\Delta v_j &= \eta_{(j-1)} r_j^{k\pi/\omega_j} \sin\left(\frac{k\pi}{\omega_j}\theta_j\right) \quad in\ \Omega \\ \gamma v_j &= \gamma\left(r_j^{k\pi/\omega_j} \sin\left(\frac{k\pi}{\omega_j}\theta_j\right)\right) \quad on\ \partial\Omega\,. \end{cases}$$

The second member is equal to zero around the crack S_j and is regular on $\partial\Omega$. So the solution $\overline{z_{2r}}$ belongs in $H^4(\Omega)$. Letting $\overline{z_r} = \overline{z_{1r}} + \overline{z_{2r}} \quad \Rightarrow$

$$\begin{aligned} \bar{z} &= \overline{z_r} + \sum_{i\in J'} \sum_{k=3}^{5} \gamma_{ki}\eta_i r_i^{\frac{k\pi}{\omega_i}} \sin\left(\frac{k\pi}{\omega_i}\theta_i\right) \\ &+ \sum_{j\in J'}^{N} \left(\frac{C_j}{\frac{4k\pi}{\omega_j}+4}\right) \eta_j r_j^{\frac{\pi}{\omega_j}+2} \sin\left(\frac{\pi}{\omega_j}\theta_j\right), \quad \overline{z_r} \in H^4(\Omega) \end{aligned}$$

where

$$J' = \left\{ j = 1,2,\ldots,N \middle/ \begin{array}{c} r_j^{k\pi/\omega_j+2} \sin\left(\frac{k\pi}{\omega_j}\theta_j\right) \in H^3(\Omega)_{|H^4(\Omega)} \\ r_j^{k\pi/\omega_j} \sin\left(\frac{k\pi}{\omega_j}\theta_j\right) \in H^3(\Omega)_{|H^4(\Omega)} \end{array} \right\}$$

and

$$\|\overline{z_r}\|_{H^4(\omega)} + \sum_{i=0}^{N} \sum_{k=3}^{5} |\gamma_{ki}| + \sum_{j=0}^{N} \frac{|C_j|}{4\frac{\pi}{\omega_j}+4} \leq \|f + \frac{1}{\alpha} z_d\|_{L^2(\Omega).}$$

Using the Riesz theorem and one result of P. Grisvard [9], we prove the existence of functions $w_j \in L^2(\Omega)$ satisfying

$$\frac{C_j}{4\frac{\pi}{\omega_j}+4} = \int_\Omega w_j\Big(f+\frac{1}{\alpha}\,z_d\Big)dx, \quad \forall\, j \in J.$$

This implies that

$$C_j = \Big(4\frac{\pi}{\omega_j}+4\Big)\int_\Omega w_j\Big(f+\frac{1}{\alpha}\,z_d\Big)dx, \quad \forall\, j \in J. \qquad \blacksquare$$

If we are going to consider only one vertex, we can set $\omega_j = \omega, \theta_j = \theta, \eta_j = \eta, C_j = C$. Consequently $\bar{z}$ is written as f

$$\begin{aligned}\bar{z} &= \overline{z_r} + \sum_{k=3}^{5}\gamma_k \eta r^{\frac{k\pi}{\omega}}\sin\Big(\frac{k\pi}{\omega}\theta\Big)\\ &+ \Bigg(\frac{C}{\frac{4\pi}{\omega}+4}\Bigg)\eta r^{\frac{\pi}{\omega}+2}\sin\Big(\frac{\pi}{\omega}\theta\Big), \quad \overline{z_r} \in H^4(\Omega).\end{aligned}$$

Remark 3.1. Let $u \in H_0^1(\Omega)$ be the solution of $-\Delta u = f$, which is Fourier transformable to $-\Delta\hat{u}+\xi^2\hat{u} = \hat{f}$ *in* Ω. This allows considering the problem with a complex parameter λ; $-\Delta v+\lambda v = g$ where $v \in H_0^1(\Omega)$ with $\xi = \sqrt{\lambda}$. For given λ the mapping $g \to C_j$ is a continuous linear functional on $L^2(\Omega)$. Therefore $\exists\, w_j \in L^2(\Omega)$ such that

$$C_j = \int_\Omega g\, w_j\, dxdy,$$

where w_j is defined as in Remark 2.1.

4. Topological Optimization

Topological optimization seems to be more general and efficient than other classical shape optimization techniques such as global optimization methods, genetic algorithms or the level set method [11]; which are all confined to a quite restricted field of application.
We present here a general framework for topological sensitivity which is based on a method introduced by Schumacher [17] in shape optimization and applied by J. Cea, M. Masmoudi and al. The beginning of our analysis is a presentation of the fundamental steps in this method and we refer the interested reader to [4], [14], [12] for more details.

Topological sensitivity analysis aims at providing an asymptotic expansion of a shape functional acting on the neighborhood of a small hole created inside the domain. The underlying principe is the following : For a criterion $j(\Omega) = J_\Omega(u_\Omega), \Omega \subset \mathbb{R}^n$ and u_Ω is the solution of a BVP defined over Ω. The pertaining expansion of the cost function $j(\Omega)$ can be generally written in the form:

$$j(\Omega\backslash\overline{x_0+\varepsilon\omega}) - j(\Omega) = \rho(\varepsilon)g(x_0)+o(\rho(\varepsilon)),$$

$$\lim_{\varepsilon\to 0}\rho(\varepsilon) = 0, \quad \rho(\varepsilon) > 0.$$

Remark 4.1. The function $\rho(\varepsilon)$ depends on the boundary conditions and the dimension of the

$\mathbb{R}^n$ space, see [7].
For example, the results obtained for the homogenous Laplace equation with a cost function defined on the boundary of the domain are displayed in the following table, see [14].

Boundary condition in the hole	$\rho(\varepsilon)$	$g(x)$
Neumann 2D	$\pi\varepsilon^2$	$-2\nabla u\nabla p$
Neumann 3D	$\frac{4}{3}\pi\varepsilon^3$	$-\frac{3}{2}\nabla u\nabla p$
Dirichlet 2D	$\frac{-1}{2\pi log(\varepsilon)}$	$u.p$
Dirichlet 3D	$4\pi\varepsilon$	$u.p$

The now posing problem consists in using topological optimization tools in order to minimize the functional

$$J(y) = \int_\Omega \left[|y(u) - y_d(u)|^2 + \frac{\alpha}{2}u^2\right]dx, \tag{16}$$

where $y(u)$ is the solution of the BVP

$$\begin{cases} -\Delta y(u) &= f+u \quad in \quad \Omega \\ y(u) &= 0 \quad on \quad \partial\Omega = \Gamma\,. \end{cases} \tag{17}$$

The variational formulation associated with (17) is: find $y(u) \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla y(u)\nabla z dx = \int_\Omega (f+u)z dx, \quad \forall \quad z \in H_0^1(\Omega). \tag{18}$$

Our further analysis shall require the additional notation.

$$a_0(y,z) = \int_\Omega \nabla y(u)\nabla z dx, \quad l_0(z) = \int_\Omega (f+u)z dx.$$

4.1. A generalized adjoint method

The mathematical framework for domain parameterization introduced by the Murat-Simon work [15] cannot be used here. Alternatively, it is possible however to invoke the adjoint method, as described in [14], in application to topological optimization. A basic feature of the adjoint method is its yield of an asymptotic expansion of a functional $J(\Omega,u_\Omega)$ which depends of a parameter u_Ω, using an adjoint state v_Ω which does not depend on the parameter. This implies the need to solve a certain system of equations in order to obtain an approximation of the topological gradient $g(x)$, $\forall x \in \Omega$. Accordingly we let $\mathcal{V}$ be a fixed Hilbert space and $\mathcal{L}(\mathcal{V})$ (respectively $\mathcal{L}_2(\mathcal{V})$) denotes the spaces of linear (respectively bilinear) forms on $\mathcal{V}$. We are able then to state the following hypotheses:
•**H-1**: There exists a real function ρ, a bilinear form $\delta_a \in \mathcal{L}_2(\mathcal{V})$ and a linear form $\delta_l \in \mathcal{L}(\mathcal{V})$ such that:

$$\rho(\varepsilon) \to 0, \quad \varepsilon \to 0^+, \tag{19}$$

$$\|a_\varepsilon - a_0 - \rho(\varepsilon)\delta_a\|_{\mathcal{L}_2(\mathcal{V})} = o(\rho(\varepsilon)), \tag{20}$$

$$\|l_\varepsilon - l_0 - \rho(\varepsilon)\delta_l\|_{\mathcal{L}(\mathcal{V})} = o(\rho(\varepsilon)). \tag{21}$$

•**H-2**: The bilinear form a_0 is coercive: There exists a constant $\alpha > 0$ such that

$$a_0(y,y) \geq \alpha\|y\|^2, \quad \forall y \in \mathcal{V}.$$

According to (4), the bilinear form a_ε depends continuously on ε, hence $\exists\ \varepsilon_0$ and $\beta > 0$ such that for $\varepsilon \in [0,\varepsilon_0]$ the following uniform coercivity condition holds.

$$a_\varepsilon(y,y) \geq \beta\|y\|^2 \quad \forall y \in \mathcal{V}.$$

Moreover, according to Lax-Milgram's theorem, for $\varepsilon \in [0,\varepsilon_0]$, the problem find $y_\varepsilon \in \mathcal{V}$, such that

$$a_\varepsilon(y_\varepsilon,z) = l_\varepsilon(z) \quad \forall z \in \mathcal{V}. \tag{22}$$

has a unique solution.

Lemma 4.1. [8] *If the hypotheses H-1 and H-2 hold, then*

$$\|y_\varepsilon - y_0\| = O(\rho(\varepsilon)).$$

Proof. It follows from the coercivity of a_ε that

$$\alpha\|y_\varepsilon - y_0\|^2 \leq a_\varepsilon(y_\varepsilon - y_0, y_\varepsilon - y_0)$$

which implies that

$$\alpha\|y_\varepsilon - y_0\|^2 \leq a_\varepsilon(y_\varepsilon, y_\varepsilon - y_0) - a_\varepsilon(y_0, y_\varepsilon - y_0)$$

$$= l_\varepsilon(y_\varepsilon - y_0) - a_\varepsilon(y_0, y_\varepsilon - y_0)$$

$$= l_0(y_\varepsilon - y_0) + (l_\varepsilon - l_0)(y_\varepsilon - y_0) - a_\varepsilon(y_0, y_\varepsilon - y_0)$$

$$= a_0(y_0, y_\varepsilon - y_0) - a_\varepsilon(y_0, y_\varepsilon - y_0) + (l_\varepsilon - l_0)(y_\varepsilon - y_0)$$

$$= f(\varepsilon)(\delta_a(y_0, y_\varepsilon - y_0) + \delta_l(y_\varepsilon - y_0)) + (\|y_0\|{+}1)(\|y_\varepsilon - y_0\|)o(f(\varepsilon)$$

•**H-3**: Consider a cost function $j(\varepsilon) = J(y_\varepsilon)$, where the functional J is differentiable. For $y \in \mathcal{V}$ there exists a linear and continuous form $DJ(y) \in \mathcal{L}(\mathcal{V})$ and δ_J such that:

$$J(z) - J(y) = DJ(y)(z-y) + \rho(\varepsilon)\delta_J(y) + o(\|z-y\|_{\mathcal{V}}). \tag{23}$$

Here we may define the Lagrangian $\mathcal{L}_\varepsilon$, when $\varepsilon \geq 0$, see for example [12,18], as

$$\mathcal{L}_\varepsilon(y,z) = a_\varepsilon(y,z) - l_\varepsilon(z) + J(y) \quad \forall y\, z \in \mathcal{V}.$$

to be led to the next theorem which gives an asymptotic expansion for $j(\varepsilon)$.

Theorem 4.1. [8, 6, 18] *If the hypotheses H-1, H-2, and H-3 are satisfied, then*

$$j(\varepsilon) - j(0) = \rho(\varepsilon)\delta\mathcal{L}(y_0,z_0) + o(\rho(\varepsilon)), \tag{24}$$

where y_0 is the solution of (22) with $\varepsilon = 0$, z_0 is the solution to the adjoint problem : Find z_0 such that

$$a_0(w,z_0) = -DJ(y_0)w \quad \forall w \in \mathcal{V},$$

and

$$\delta\mathcal{L}(y,z) = \delta_a(y,z) - \delta_l(z) + \delta_J(y). \tag{25}$$

Proof. For all $z \in \mathcal{V}$, one has

$$j(\varepsilon) = \mathcal{L}_\varepsilon(y_\varepsilon, y_0)$$

Hence

$$j(\varepsilon) - j(0) = \mathcal{L}_\varepsilon(y_\varepsilon,z) - \mathcal{L}_0(y_0,z)$$

$$= a_\varepsilon(y_\varepsilon,z) - a_0(y_0,z) + J_\varepsilon(y_\varepsilon) - J_0(y_0) - l_\varepsilon(y_\varepsilon) + l_0(y_0).$$

It follows from (7) and the Lemma 4.1 that

$$J(y_\varepsilon) - J(y_0) = DJ(y_0)(y_\varepsilon - y_0) + f(\varepsilon)\delta_J(y_0) + o(f(\varepsilon).$$

Next, choosing z_0 as the solution to (9), we obtain with (5)

$$j(\varepsilon) - j(0) = a_\varepsilon(y_\varepsilon,z_0) - a_0(y_0,z_0) + DJ(y_0)(y_\varepsilon - y_0)$$

$$+ f(\varepsilon)(\delta_J(y_0) - \delta_l(z_0)) + o(f(\varepsilon))$$

$$= a_\varepsilon(y_\varepsilon,z_0) - a_0(y_\varepsilon,z_0) - a_0(y_\varepsilon - y_0,z_0) + DJ(y_0)(y_\varepsilon - y_0)$$

$$+ f(\varepsilon)(\delta_J(y_0) - \delta_l(z_0)) + o(f(\varepsilon))$$

Then, it follows from (3), (4) and the Lemma 4.1 (with $\|y_\varepsilon\|$ bounded) that

$$j(\varepsilon) - j(0) = f(\varepsilon)\delta_a(y_\varepsilon,z_0) + f(\varepsilon)(\delta_J(y_0) - \delta_l(z_0)) + o(f(\varepsilon))$$

$$= f(\varepsilon)(\delta_a(y_0,z_0) + \delta_a(y_\varepsilon - y_0,z_0)) + f(\varepsilon)(\delta_J(y_0) - \delta_l(z_0)) + o(f(\varepsilon))$$

$j(\varepsilon) - j(0) = f(\varepsilon)\delta_{\mathcal{L}}(y_0, z_0) + o(f(\varepsilon)).$ ■

4.2. Perturbation of the domain

For all $\varepsilon \geq 0$, we set $\Omega_\varepsilon = \Omega \backslash \omega_\varepsilon$ where $\omega_\varepsilon = x_0 + \varepsilon\omega$, $\omega \in \mathbb{R}^n$ in a reference domain (see figure 2). Let $y_\varepsilon(u)$ be the solution of the set problem in the perturbed domain

$$\begin{cases} -\Delta y_\varepsilon(u) = f + u & in \quad \Omega_\varepsilon \\ y_\varepsilon(u) = 0 & on \quad \Gamma \\ y_\varepsilon(u) = 0 & on \quad \partial\omega_\varepsilon. \end{cases} \tag{26}$$

Our objective would then be to find the asymptotic expansion of $y_\varepsilon(u) - y(u)$ when ε goes to zero.

Remark 4.2. A Lagrangian method setup for $\varepsilon \geq 0$ cannot utilize the variational formulation of (26). This is due to the fact that there is no bilinear and continuous form such that $\|a_\varepsilon - a_0 - \rho(\varepsilon)\delta_a\|_{\mathcal{L}_2(H_0^1(\Omega))} = o(\rho(\varepsilon))$ for some adequate function ρ. Moreover, if a_ε is defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ with functions of $H_0^1(\Omega)$ extended by zeros on ω_ε, we have for example for $n = 3$ and smooth functions y, z,

$$a_\varepsilon(y,z) - a_0(y,z) = -\varepsilon^3 \nabla y(u) \nabla z(x_0) \int_\omega dx + o(\varepsilon^3).$$

But $\delta a(y,z) = \nabla y(x_0)\nabla z(x_0)$ cannot be continuously extended on $H_0^1(\Omega) \times H_0^1(\Omega)$. Besides, if y_ε is extended by zeros on ω_ε, the behavior of $\|y_\varepsilon - y_0\|_{H^1(\Omega)}$ is not of order ε^3 but only of order $\varepsilon^{1/2}$, [8].

The method to be used here (truncation technique) can, however, be applied to the the case of Neumann boundary conditions, or even to more general boundary conditions, and has two advantages:

1. It allows for construction of a fixed Hilbert space, required in order to apply the Lagrangian method.
2. It facilitates obtaining a bilinear and continuous form δa, such that for some adequate $\rho(\varepsilon)$ function, $\|a_\varepsilon - a_0 - \rho(\varepsilon)\delta_a\|_{\mathcal{L}_2(H_0^1(\Omega))} = o(\rho(\varepsilon))$ is satisfied and the associated y_ε will yield same order $\|y_\varepsilon - y_0\|_{H^1(\Omega)} = o(\rho(\varepsilon))$.

4.3. The truncated problem

As it is impossible to find an bilipschitzian mapping between Ω and Ω_ε, then the domain parameterization presented in [15] can't be used (as hinted before) when changing the topology of the geometry. This objective can be reached, however, by the domain truncation new method of [14].

This method can be motivated at least by the following two reasons. First, it allows the analysis to be made in a fixed Hilbert space. Second, it validates the application of the Lagrangian method. The variation of the Lagrangian can then be written as a continuous global bilinear expression.

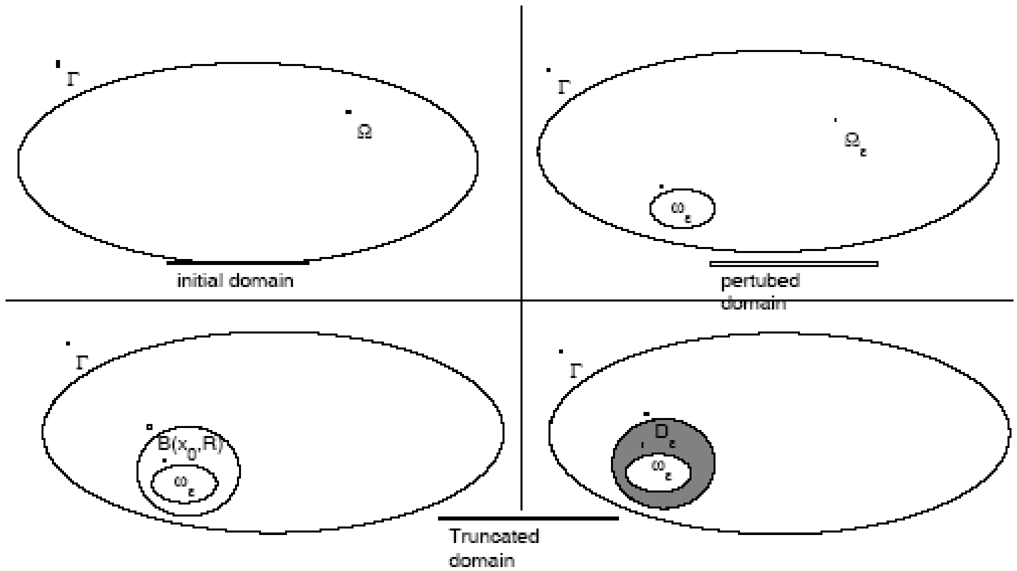


Figure 2: Initial domain, the perturbed and the truncated domains

Let $R > 0$ such that $\omega_\varepsilon \subset B(u_0,R) \subset \Omega_\varepsilon$, $D_\varepsilon = B(u_0,R)\backslash\omega_\varepsilon$ and φ the restriction of $y(u)$ through $\Gamma_R = \partial B(u_0,R)$ and $y_\varepsilon^R(R)$, the solution of the perturbed problem.

$$\begin{cases} -\Delta y_\varepsilon^{f,\varphi}(u) &= f+u \quad in \quad D_\varepsilon \\ y_\varepsilon^{f,\varphi}(u) &= \varphi \quad on \quad \Gamma_R \\ y_\varepsilon^{f,\varphi}(u) &= 0 \quad on \quad \partial\omega_\varepsilon. \end{cases} \tag{27}$$

For $\varepsilon = 0, y_0^{\varphi,0}$ is the solution of

$$\begin{cases} -\Delta y_0^{f,\varphi}(u) &= f+u \quad in \quad B(u_0,R) \\ y_0^{f,\varphi}(u) &= \varphi \quad on \quad \Gamma_R, \end{cases} \tag{28}$$

where $y_\varepsilon^{f,\varphi} = y_\varepsilon^{f,0} + y_\varepsilon^{0,\varphi}$.
For $\varepsilon \geq 0$, the Dirichlet-to-Neumann operator is defined by

$$\begin{aligned} T_\varepsilon : H^{1/2}(\Gamma_R)^n &\longrightarrow \quad H^{-1/2}(\Gamma_R)^n \\ \varphi &\longmapsto \quad T_\varepsilon\varphi = \nabla(y_\varepsilon^\varphi).\nu, \end{aligned}$$

where ν is chosen outward to D_ε on $\partial\omega_\varepsilon$, and the function $f_\varepsilon \in H^{-1/2}(\Gamma_R)$ is defined by $f_\varepsilon = -\nabla y_\varepsilon^{f,0}.\nu$.
Thus we have,

$$\nabla y_\varepsilon^{f,\varphi}.\nu = T_\varepsilon\varphi - f_\varepsilon.$$

Finally for $\varepsilon \geq 0$ the function y_ε is defined as the solution of the following problem.
Find y_ε such that:

$$\left\{\begin{array}{rcl} -\Delta y_\varepsilon(u) & = & f+u \quad in \quad \Omega_R \\ \nabla y_\varepsilon(u).\nu - T_\varepsilon y_\varepsilon & = & f_\varepsilon \quad on \quad \Gamma_R \\ y_\varepsilon(u) & = & 0 \quad on \quad \Gamma. \end{array}\right. \tag{29}$$

The variational formulation associated with (29) is the following.
Find y_ε such that:

$$a_\varepsilon(y,z) = l_\varepsilon(z), \quad \forall\ z \ \in \ \mathcal{V}_R, \tag{30}$$

where the functional space, $\mathcal{V}_R$, the bilinear form a_ε and the linear form l_ε are defined by

$$\mathcal{V}_R = \{y \ \in \ H^1(\Omega_R) \quad y = 0 \ \in \ \Gamma\}, \tag{31}$$

$$a_\varepsilon(y,z) = \int_{\Omega_R} \nabla y.\nabla z dx + \int_{\Gamma_R} T_\varepsilon yz \, d\gamma(x), \tag{32}$$

$$l_\varepsilon(z) = \int_{\Omega_R} (f+u)z dx + \int_{\Gamma_R} f_\varepsilon z d\gamma(x). \tag{33}$$

Here $u.v$ denote the usual dot product of $\mathbb{R}^n$ and $d\gamma(u)$ is the Lebesgue measure on the boundary.

Remark 4.3. Symmetry, continuity and coercivity of a_ε, and continuity of l_ε follow directly from:

$$\int_{\Gamma_R} T_\varepsilon \varphi\psi d\gamma(u) = \int_{D_\varepsilon} \nabla y_\varepsilon^{0,\varphi}.\nabla y_\varepsilon^{0,\psi} dx, \tag{34}$$

$$\int_{\Gamma_R} f_\varepsilon \psi d\gamma(u) = \int_{D_\varepsilon} f y_\varepsilon^{0,\psi} du. \tag{35}$$

Proposition 4.1. *Let $J_\Omega(y)$ be the objective functional defined by (16). Then there exist δJ and a function $\rho(\varepsilon) > 0$ such that*

$$J_{\Omega_\varepsilon}(y_\varepsilon) - J_\Omega(y) = DJ(y_\varepsilon)(y_\varepsilon - y) + \rho(\varepsilon)\delta J(y) + o(\rho(\varepsilon)) \tag{36}$$

Proof. Let

$$J_\Omega(y) = \int_\Omega \Big[y(u) - y_d(x)|^2 + \frac{\alpha}{2} |u|^2 \Big] dx,$$

$$J_{\Omega_\varepsilon}(y_\varepsilon) - J_\Omega(y) = \int_{\Omega_\varepsilon} \Big[y_\varepsilon - y_d|^2 + \frac{\alpha}{2} |u|^2 \Big] dx$$

$$- \int_\Omega \Big[y - y_d|^2 + \frac{\alpha}{2} |u|^2 \Big] dx$$

$$= \int_{\Omega_\varepsilon} \left\{ |y_\varepsilon - y_d|^2 dx - \int_{\Omega_\varepsilon} |y - y_d|^2 dx \right\}$$

$$- \int_{\omega_\varepsilon} \left\{ |y - y_d|^2 + \frac{\alpha}{2} |u|^2 \right\} dx$$

$$= \int_{\Omega_\varepsilon} (y_\varepsilon - y)(y_\varepsilon + y - 2y_d) dx - \int_{\omega_\varepsilon} \left\{ |y - y_d|^2 + \frac{\alpha}{2} |u|^2 \right\} dx.$$

Let $DJ(y_\varepsilon) = \int_{\Omega_\varepsilon} (y_\varepsilon - y) z du \quad \forall z \in \mathcal{V}_R$ and using the mean value theorem, it follows that

$$\int_{\omega_\varepsilon} \left\{ |y - y_d|^2 + \frac{\alpha}{2} |u|^2 \right\} dx = \left[|y(u_0) - y_d|^2 + \frac{\alpha}{2} |u_0|^2 \right] \rho(\varepsilon) + o(\rho(\varepsilon)).$$

To end the proof, we set

$$\delta J(y) = \left[|y(u) - y_d|^2 + \frac{\alpha}{2} |u|^2 \right]. \qquad \blacksquare$$

Theorem 4.2. *Let $j(\varepsilon) = J_{\Omega_\varepsilon}(y)$, the functional defined by (16), a_ε and l_ε are respectively the bilinear form and the linear form associated with (29), then there exist a bilinear form δa and a linear form δl such that*
1) $\| a_\varepsilon - a_0 - \rho(\varepsilon) \delta_a \|_{\mathcal{L}_2(\mathcal{V})} = o(\rho(\varepsilon))$,
2) $\| l_\varepsilon - l_0 - \rho(\varepsilon) \delta_l \|_{\mathcal{L}(\mathcal{V})} = o(\rho(\varepsilon))$,
and j has the following asymptotic expansion:

$$j(\varepsilon) - j(0) = \rho(\varepsilon)[\delta a(y_0, z_0) - \delta l(z) + \delta J(y_0)] + o(\rho(\varepsilon)),$$

where z_0 is the solution of the adjoint problem,
Find w such that:

$$a_\varepsilon(w, z_0) = -DJ(y_0) w, \quad \forall w \in \mathcal{V}_R.$$

Proof. We will utilize the fact that the variation of the Lagrangian is equal to the variation of the cost function.
According to the variational formulation of (29) and relations (34) and (35), the variations of the bilinear form and the linear form are

$$a_\varepsilon(y,z) - a_0(y,z) = \int_{\Gamma_R} (T_\varepsilon - T_0) yz \, d\gamma(x),$$

$$l_\varepsilon(z) - l_0(z) = \int_{\Gamma_R} (f_\varepsilon - f) z \, d\gamma(x).$$

The problem reduces to the analysis of $(T_\varepsilon - T_0)\varphi$ for $\varphi \in H^{1/2}(\Gamma_R)$ and of $f_\varepsilon - f_0 \in H^{-1/2}(\Gamma_R)$. It is shown in [8] that there exists a function $\delta T \in \mathcal{L}(H^{1/2}(\Gamma_R); H^{1/2}(\Gamma_R))$ and $\delta f \in H^{-1/2}(\Gamma_R)$ such that

$$\| T_\varepsilon - T_0 - \rho(\varepsilon) \delta T \|_{\mathcal{L}(H^{1/2}(\Gamma_R); H^{1/2}(\Gamma_R))} = O(\rho(\varepsilon)),$$

$$\| f_\varepsilon - f_0 - \rho(\varepsilon) \delta f \|_{H^{-1/2}(\Gamma_R)} = O(\rho(\varepsilon)).$$

Setting

$$\delta a = \int_{\Gamma_R} \delta T\, yz\, d\gamma(u), \quad \forall \;\; y,\; z \;\in\; \mathcal{V}_R, \tag{37}$$

$$\delta l = \int_{\Gamma_R} \delta fz\, d\gamma(u), \quad \forall\; z \;\in\; \mathcal{V}_R, \tag{38}$$

will yield the hypothesis (1) and (2) of the theorem.
By realizing that (37), (38) ,(36) and the fundamental hypothesis of the adjoint method are satisfied, we can apply Theorem 6 to end the proof. ■

4.4 Numerical results

In order to illustrate the theoretical result, we present here some numerical applications. In this respect, we consider a polygonal domain Ω in which we solve a Dirichlet problem and we represent the adjoint state and the topological derivative. The corresponding equations are:

$$\begin{cases} -\Delta y(\bar{u}) &= f+\bar{u} \quad in \quad \Omega \\ y(u) &= 0 \quad on \quad \partial\Omega\,, \end{cases}$$

$$\begin{cases} -\Delta v &= -2(y(\bar{u})-y_d)-2\bar{u} \quad in \quad \Omega \\ v &= 0 \quad on \quad \partial\Omega\,. \end{cases} \tag{39}$$

When $\omega_\varepsilon = B(u_0,\varepsilon)$, using potential theory and the mean value theorem allow for an explicit computation of all integrals (see [8]) . We therefore have

$$j(\varepsilon)-j(0) = -4\pi\rho(\varepsilon)\Big(y_\Omega(\bar{u}(x_0))v(x_0) + \frac{\alpha}{2}|\bar{u}(x_0)|\Big) + o(\rho(\varepsilon)),$$

v being the solution of the adjoint problem. The numerical code we use is written in Getfem ++ [16]. Getfem ++ is a free software which uses C++ language and can be used as toolbox in Matlab, Python and Scilab.
The main theorem (2) gives for C_j the expression

$$C_j = \Big(4\frac{\pi}{\omega_j}+4\Big)\int_\Omega w_j\Big(f+\frac{1}{\alpha}z_d\Big)dx, \quad \forall\; j \in\; J. \tag{40}$$

$w_j = \gamma(e^{-r\sqrt{\lambda}}r^{-\alpha}\sin\alpha\theta_j+\phi)$ is given in Remark 3. In the applications, we set in the first case $J = 1$, and in the second $J = 2$. Throughout
$\omega = \frac{4\pi}{3}$, $y_d = f = 1$, $\alpha = \frac{\pi}{\omega} = \frac{3}{4}$, $r = \sqrt{x^2+y^2}$, $\gamma = 1$, $\phi = 0$ and z_d of (51) is determined by a numerical solution of the following BVP.

$$\begin{cases} -\Delta z_d &= y_d \quad in \quad \Omega \\ z_d &= 0 \quad on \quad \partial\Omega\,. \end{cases}$$

The numerical approximation of C_j proceeds as follows. The optimal control $\bar{u}$ is given by $\bar{u} = -\frac{1}{\alpha}\bar{p}$ where $\bar{p} = \bar{z}-z_d$ and $\bar{z}$ with z_d satisfying the BVP

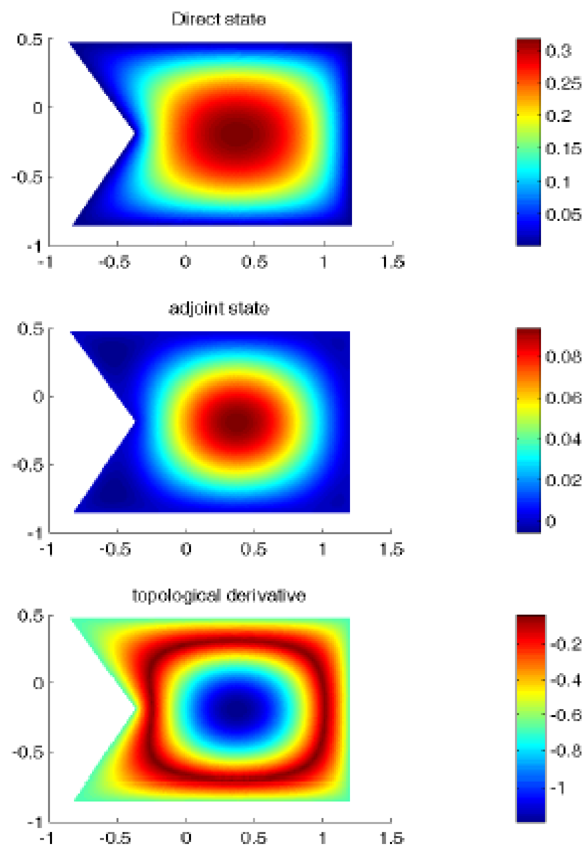
$$\begin{cases} \Delta^2\bar{z}+\frac{1}{\alpha}\bar{z} &= f+\frac{1}{\alpha}z_d \quad in \quad \Omega \\ \gamma(\bar{z}) &= \gamma(\Delta\bar{z}) \quad on \quad \partial\Omega. \end{cases}$$

Next we insert a hole in the initial polygonal domain, where the topological derivative is most negative. Correspondingly

$$C_j = (4\frac{3\pi}{4\pi} + 4)\int_{\Omega}(1 + \frac{3}{4}z_d)e^{-r}r^{-\frac{3}{4}+1}\sin(\frac{3}{4}\theta)drd\theta$$

$$= 7\int_{\frac{1}{2}}^{1}(1 + \frac{3}{4}z_d)e^{-r}r^{\frac{1}{4}}dr\int_{\pi}^{2\pi}\sin(\frac{4}{3}\theta)d\theta$$

$$= 7\left[\int_{\frac{1}{2}}^{1} e^{-r}r^{\frac{1}{4}}dr + \int_{\frac{1}{2}}^{1} \frac{3}{4}z_d e^{-r}r^{\frac{1}{4}}dr\right]\int_{\pi}^{2\pi}\sin(\frac{4}{3}\theta)d\theta.$$



$f = 1$; $y_d = 0.1415$

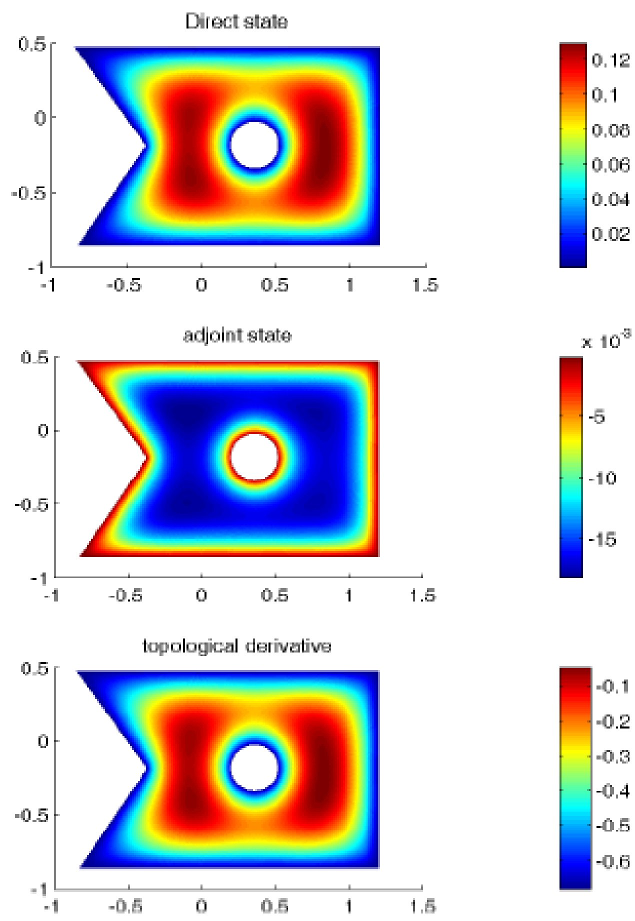
Figure 3: Domain without hole: on the top, we have direct state, in the middle, the adjoint state, on the bottom, the topological derivative.

Some symbolic calculus in Matlab gives the following approximation

$$\int_{\frac{1}{2}}^{1} e^{-r} r^{\frac{1}{4}} dr = 0,02197,$$

$$\frac{3}{4}\int_{\frac{1}{2}}^{1} z_d e^{-r} r^{\frac{1}{4}} dr = \begin{cases} = 12\times 10^{-3}, & \text{in the first case} \\ = 15\times 10^{-4}, & \text{in the second case} \end{cases}$$

$$\int_{\pi}^{2\pi} \sin(\frac{4}{3}\theta)d\theta = \frac{3}{4}\Big[-cos\frac{4}{3}\theta\Big]_{\pi}^{2\pi} = \frac{3}{4}\Big[cos\frac{4\pi}{3} - cos\frac{8\pi}{3}\Big] \simeq 5.9\times 10^{-3}.$$



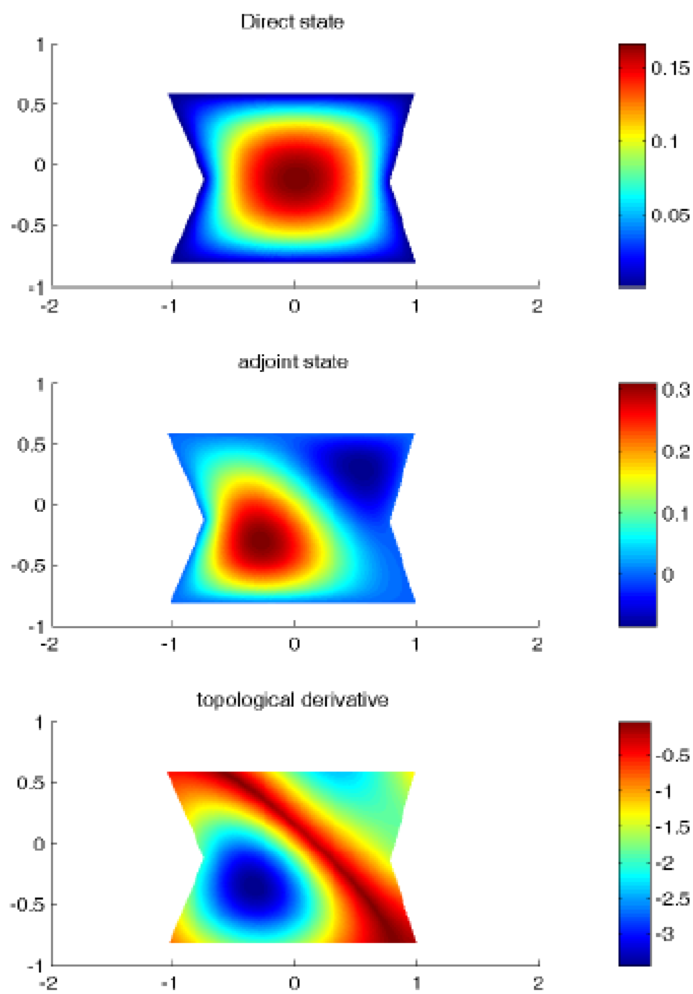
$$f = 1\ ;\ y_d = 0.1415$$

Figure 4: Domain with a hole where the topological derivative is the most negative: on the top, we have direct state, in the middle, the adjoint state, on the bottom, the topological derivative.

It follows that

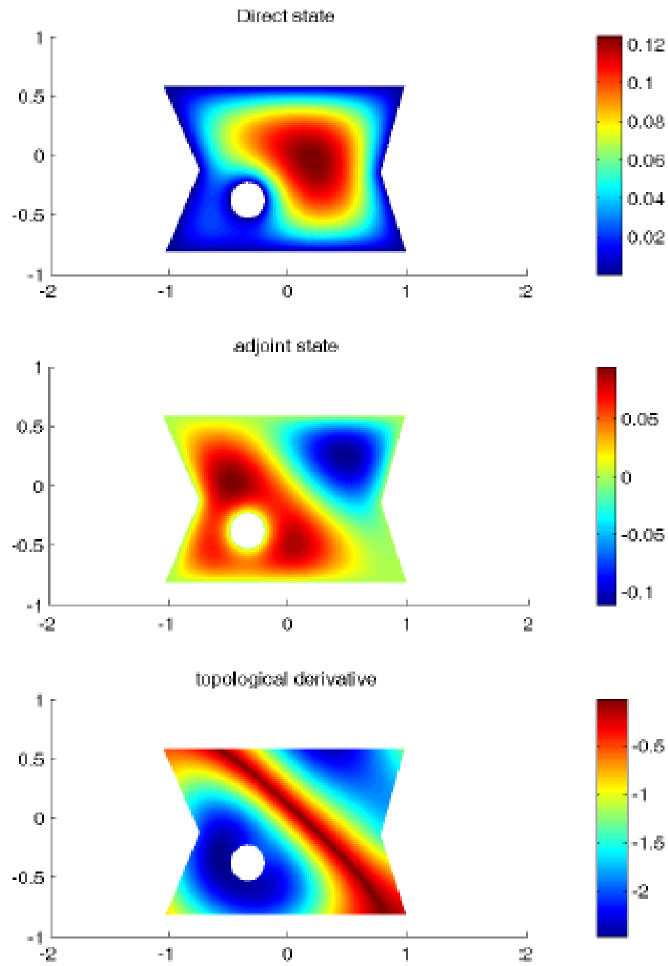
$$C_j = \begin{cases} 1,4 \times 10^{-3} & \text{in the first case} \\ 23 \times 10^{-4} & \text{in the second case} \end{cases}$$

The different cases correspond to the different values of y_d in the pertaining numerical work.



$$f = 1 \; ; \; y_d = e^{-(x^2+y^2)} sin(x+y)$$

Figure 5: Domain without hole: on the top, we have direct state, in the middle, the adjoint state, on the bottom, the topological derivative.



$$f = 1\ ;\ y_d = e^{-(x^2+y^2)} sin(x+y)$$

Figure 6: Domain with a hole where the topological derivative is the most negative: on the top, we have direct state, in the middle, the adjoint state, on the bottom, the topological derivative.

5. Conclusion and Extensions

In this paper, we have proven that the coefficients of singularity for a controlled Dirichlet problem in a polygonal domain can be expressed solely with the data of the problem. Moreover with the help of topological optimization tools, we have built an algorithm which allows calculating this coefficients and representation of its effects in the polygonal domain.

In a forthcoming paper, we intend to generalize this calculus to the case of a nonlinear operator as for example in the works of Atkinson and Champion [1],[2], or in the stationary model of image processing [19].

Acknowledgments.
The authors are grateful to an anonymous referee for a number of helpful remarks on the original typescript.

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Article history : Submitted November, 11, 2009; Received in revised form March, 4, 2010; Accepted May, 21, 2010.