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Numerical Solution of a Stochastic Lorenz Attractor

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Abstract. *The aim of this paper is to extend the -deterministic- Lorenz attractor to a stochastic system and to numerically solve it. We propose and implement the Milstein scheme for solving multidimensional nonlinear Itô stochastic differential systems, with particular emphasis on the Lorenz attractor. In order to assure the first convergence order of the Milstein scheme, we use Fourier series to approximate the double Itô integrals. Furthermore, numerical behaviors of the stochastic Lorenz attractor solutions are presented and analyzed.*

Key words : System of Stochastic Differential Equations, Milstein Scheme, Random Lorenz Attractor, Random Dynamical System.

AMS Subject Classifications: 35R60, 60H15

1. Introduction

The Lorenz attractor invokes a solution of a three dimensional deterministic system of differential equations given in [4]. It was first studied by Edward N. Lorenz, a meteorologist, around 1963. The system was derived from a simplified model of convection in the earth's atmosphere. It also arises naturally in lasers and dynamos models. Its beautiful three dimensional plots are most commonly expressed as a solution of the three coupled non-linear differential equations, which are also popular in the field of Chaos. The equations describe the flow of fluid in a box which is heated along the bottom. This model was intended to simulate medium-scale atmospheric convection. Lorenz simplified in his model some of the Navier-Stokes equations in the area of fluid dynamics, and obtained the following non-linear three dimensional ordinary differential system:

$$\begin{aligned} \partial_t x &= p(y(t) - x(t)), \qquad (1)\\ \partial_t y &= rx(t) - y(t) - x(t)z(t),\\ \partial_t z &= x(t)y(t) - cz(t), \end{aligned}$$

where the parameter p is the Prandtl number in a dimensionless parameter of a convecting system that characterizes the regime of convection. $r = \frac{Ra}{Ra_c}$ is the quotient of the Rayleigh number to the critical Rayleigh number and b is a geometric factor. In his study, Lorenz used the values $p = 10,\ r = 28$ and $c = \frac{8}{3}$.

2. The Stochastic Lorenz Attracror (SLA)

In this section, we extend the Lorenz system to a stochastic one. For $t_0, T \in R^+$, we examine the numerical solution of the stochastic differential system (SDE) given as:

$$d\mathbf{X}_t = a(t,\mathbf{X}_t)dt + b(t,\mathbf{X}_t)\, d\mathbf{W}_t\ , \qquad \mathbf{X}_0 = x, \tag{2}$$

where $a : [t_0,T] \times R^d \to IR^d$ represents the drift vector, $b : [t_0,T] \times R^d \to IR^{m\times d}$ the diffusion matrix, $(\mathbf{W}_t)_{t\in[t_0,T]}$ is an m–dimensional Brownian motion (every coordinate is a Brownian motion, and the coordinates are pairwise independent) and $(\mathbf{X}_t)_{t\in[t_0,T]}$ is a d–dimensional real valued stochastic process solution of the SDE (2). This is known as an Itô process. For more details we refer to [1, 6, 5, 11].

Using the general form of Itô differential equations, like those studied in [7, 1, 6, 5, 11], we can extend the deterministic system (1) to a stochastic one. Hence by setting

$$\mathbf{X}_t = (x(t),\ y(t),\ z\,(t))^T,$$

let us define the drift vector (a^1, a^2, a^3), given in the general form of the SDE (2) as:

$$\begin{aligned} a^1(t,x(t),y(t),z(t)) &:= p(y(t) - x(t)), \\ a^2(t,x(t),y(t),z(t)) &:= rx(t) - y(t) - x(t)z(\,t), \\ a^3(t,x(t),y(t),z(t)) &:= x(t)y(t) - cz(t). \end{aligned} \tag{3}$$

The stochastic Lorenz system can therefore be written in the form of a stochastic non-linear differential system of equations (2) with a drift vector defined by (3), $b(t,\mathbf{X}_t)$, a real-valued diffusion matrix in $IR^{3\times 3}$, and by $\mathbf{W}_t = (\mathbf{W}_t^1, \mathbf{W}_t^2, \mathbf{W}_t^3)$, a three dimensional Brownian motion. For more details on the analysis and simulations of stochastic processes and SDEs, we suggest the references [2, 3, 11].

3. Milstein Scheme for SLA

The simplest strong Taylor approximation for SDEs is the stochastic Euler-Maruyama method, which has a convergence order $\gamma = 0.5$. An other scheme is the Milstein, which is weakly and strongly convergent with order $\gamma = 1$. As for examples we refer to [5, 6, 1]. The Milstein scheme is similar to the Euler-Maruyama method in the one dimensional case. It only requires some additional terms, as it is detailed in [11]. But in the multi-dimensional case, the order one is assured by application of the stochastic Taylor expansion developed by [5] as:

$$f(\tau,\mathbf{X}_\tau) = \sum_{\alpha\in A} \mathbf{I}_\alpha[f_\alpha(\rho,\mathbf{X}_\rho)] + \sum_{\alpha\in\mathcal{B}(A)} \mathbf{I}_\alpha[f_\alpha(.,\mathbf{X}.)]_{\rho,\tau}\ , \tag{4}$$

where the functional $f : [t_0,T] \times R^d \to IR,$ and all its differential is of smooth functions. ρ and τ are two stopping time processes, α is a Multi-Index, A is a Hierarchical set and $\mathcal{B}(A)$ is the corresponding Rest set. For details of such constructions, we refer for example to [5].

For the construction of Taylor schemes of higher order, one has to add more terms to the Hierarchical set in equation (4). Thus the time discrete scheme, derived from the Taylor expansion, is given as:

$$f(t_{n+1},\mathbf{Y}_{t_{n+1}}) = \underbrace{\sum_{\alpha\in A}\mathbf{I}_\alpha[f_\alpha(t_n,\mathbf{Y}_{t_n})]_{t_n,t_{n+1}}}_{Main-Approximation} + \underbrace{\sum_{\alpha\in\mathcal{B}(A)}\mathbf{I}_\alpha[f_\alpha(.,\mathbf{Y}.)]_{t_n,t_{n+1}}}_{Rest-Set}, \qquad (5)$$

where $\mathbf{I}_\alpha$ are the multiple Itô integrals and $\mathbf{Y}_{t_n} = (x(t_n),\ y(t_n),\ z(\ t_n))$ is the approximation of the solution of the SLA at the time point $t_n \in [t_0,T]$, and L^j the Itô differential operators defined as follows

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{3} a_t^k \frac{\partial}{\partial x^k} + \frac{1}{2}\sum_{k,i=1}^{d}\sum_{j=1}^{3} b^{i,j}b^{k,j}\frac{\partial^2}{\partial x^i\partial x^k}, \qquad (6)$$

$$L^j = \sum_{i=1}^{3} b^{i,j}\frac{\partial}{\partial x^i} \quad \text{for} \quad i,j,k,= 1,2,3. \qquad (7)$$

However, for $j_1 \neq j_2$, the independent one-dimensional Brownian motions $(\mathbf{W}_t^{j_1})_{t\geq 0}$ and $(\mathbf{W}_t^{j_2})_{t\geq 0}$, and the double Itô integrals $\mathbf{I}_{(j_1,j_2)}$ which satisfy

$$\mathbf{I}_{(j_1,j_2)t_0,t} = \int_{t_0}^{t}\int_{t_0}^{s_1} d\mathbf{W}_s^{j_1} d\mathbf{W}_{s_1}^{j_2}, \qquad (8)$$

can not be exactly computed, see [5, 11]. Hence for its estimation we have used the Fourier series in the inherent approximation. While the Stratonovich integral have the same properties as the Riemann one, one has to approximate the Stratonovich integral $\mathbf{J}_{(j_1,j_2)}$ for $j_1 \neq j_2$ and to use the relationship between the Itô and Stratonovich integrals to arrive at a numerical approximation to the double Itô integrals. The approximation rewrites as:

$$\mathbf{I}^p_{(j_1,j_2)} = \frac{1}{2}\Delta\epsilon_{j_1}\epsilon_{j_2} + \Delta\sqrt{\rho_p}\,(\mu_{j_1,p}\epsilon_{j_2} - \mu_{j_2,p}\epsilon_{j_1}) + \frac{\Delta}{2\pi}\sum_{r=1}^{p}\frac{1}{r}(\zeta_{j_1,r}(\sqrt{2}\,\epsilon_{j_2} + \eta_{j_2,r}) - \zeta_{j_2,r}(\sqrt{2}\,\epsilon_{j_1} + \eta_{j_1,r})), \qquad (9)$$

where for $j_1 \neq j_2 = 1,\dots,m$, $r = 1,\dots,p$ and $p \in IN$, the random variables $\epsilon_{j_1},\epsilon_{j_2},\mu_{j_1,p},\mu_{j_2,p}$, $\zeta_{j_1,r},\zeta_{j_2,r},\eta_{j_2,r}$ and $\eta_{j_1,r}$ are pairwise independent and standard normally distributed. We have to note that for $j_1 \neq j_2 = 1,\dots,m$; $p \in IN$. The approximation of the double Itô integrals $\mathbf{I}_{(j_1,j_2),\Delta}$ by $\mathbf{I}_{(j_1,j_2)^p,\Delta}$ is given in second moment as

$$IE\left(\left|\mathbf{I}^p_{(j_1,j_2),\Delta} - \mathbf{I}_{(j_1,j_2),\Delta}\right|^2\right) = \rho_p\Delta^2, \qquad (10)$$

$$\rho_p = \frac{1}{2\pi^2}\sum_{r=p+1}^{\infty}\frac{1}{r^2}.$$

For a numerical examination of this approximation using Fourier series, we refer for example to the work [11], while for the computation of $\mathbf{I}_{(j,j)}$ for $j = 0,1,2,3$ we refer the reader to [5, 6, 11].

It should be noted that for a given f, as previously defined, the multiple Itô integrals are

$$\mathbf{I}_\alpha[f(.)]_{t_0,t} := \begin{cases} f(t), & \text{if } l = 0 \\ \int_{t_0}^{t}\mathbf{I}_{\alpha-}[f(.)]_{t_0,s}ds, & \text{if } l \geq 1 \text{ and } j_l = 0 \\ \int_{t_0}^{t}\mathbf{I}_{\alpha-}[f(.)]_{t_0,s}d\mathbf{W}_s^{j_l}, & \text{if } l \geq 1 \text{ and } j_l \geq 1, \end{cases}$$

where $\alpha-$ is the the multi-index without the last component, and l represents the number of entries of α. Here are some examples of the multiple Itô integrals

$$\mathbf{I}_{(1,2)}[f(.)]_{t_0,t} = \int_{t_0}^{t}\int_{t_0}^{s} f(z)\, d\mathbf{W}_z^1 d\mathbf{W}_s^2,$$
$$\mathbf{I}_{(1,2,0)}[f(.)]_{t_0,t} = \int_{t_0}^{t} \mathbf{I}_{(1,2)}[f(.)]_{0,s} ds$$
$$= \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} f(s_2) d\mathbf{W}_{s_2}^1 d\mathbf{W}_{s_1}^2 ds.$$

The first order Milstein scheme is given for summation over all Multi-indicies of the Hierarchical set A. The hierarchical set is in literature also called main set of the approximation. In fact the Milstein scheme can be derived from the stochastic Taylor expansion (5) for any system of equations to satisfy

$$\mathbf{Y}_{n+1}^k = \mathbf{Y}_n^k + a^k(t_n, \mathbf{Y}_n)\Delta + \sum_{j=1}^{3} b^{k,j}(t_n, \mathbf{Y}_n)\Delta\mathbf{W}_{t_n}^j + \sum_{j_1,j_2=1}^{3} L^{j_1} b^{k,j_2}(t_n, \mathbf{Y}_n)\mathbf{I}_{(j_1,j_2)}, \tag{11}$$

where Δ is a time step of the scheme. It should be underlined that we deal here only with a fixed time step.

To construct the numerical Milstein scheme, derived from the stochastic Taylor expansion and the corresponding three dimensional Lorenz system of differential equations, let us consider the three dimensional index set $F = 0, 1, 2, 3$. The corresponding Hierarchical $\mathcal{A}$ and Rest set $\mathcal{B}$ for the three dimensional system of equations are

$$\mathcal{A} = \{v, (0), (1), (2), (3), (1,1), (1,2), (1,3), (2,2), (2,1), (2,3), (3,3), (3,1), (3,2)\}, \tag{12}$$
$$\begin{aligned}\mathcal{B}(\mathcal{A}) = \big\{&(0,0), (0,1), (0,2), (0,3), (1,0), (2,0), (3,0), (0,1,1), (0,1,2), (1,2,2),\\ &(0,1,3), (0,2,2), (0,2,1), (0,2,3), (0,3,3), (0,3,1), (0,3,2), (1,1,1), (1,1,2), (1,1,3),\\ &(1,2,1), (1,2,3), (1,3,3), (1,3,1), (1,3,2), (2,1,1), (2,1,2), (2,1,3), (3,2,1), (2,2,2),\\ &(2,2,1), (2,2,3), (2,3,3), (2,3,1), (2,3,2), (3,1,1), (3,1,2), (3,1,3), (3,2,2), (3,2,3),\\ &(3,3,3)(3,3,1), (3,3,2)\big\},\end{aligned} \tag{13}$$

where v is the so called empty index, see [6, 5, 11]. The stochastic convergence order of the Milstein scheme can be assured by considering some necessary conditions. The method which defines the relationship between the multi-index and the convergence order are detailed in several works; for instance those reported by Kloeden and Platen [5]. The following theorem presents the conditions required to assure the convergence order one.

Theorem 3.1. *Let $Y^\Delta = \left\{\mathbf{Y}_t^\Delta,\ t\,]t_0, T\,[\right\}$ be a strong Milstein approximation of (2). If*

$$IE(|\mathbf{X}_{t_0}|^2) < \infty \quad \textit{and} \quad (IE(|\mathbf{X}_{t_0} - \mathbf{Y}_{t_0}^\Delta|^2))^{\frac{1}{2}} \leq K_1\Delta^{\frac{1}{2}}; \tag{14}$$

(Lipschitz condition)

$$|a(t,x) - a(t,y)| \leq K_2|x-y|, \tag{15}$$
$$|b^{j_1}(t,x) - b^{j_1}(t,y)| \leq K_2|x-y|, \tag{16}$$
$$|L^{j_1}b^{j_2}(t,x) - L^{j_1}b^{j_2}(t,y)| \leq K_2|x-y|, \tag{17}$$

(Bounded increment)

$$|a(t,x)|+|L^j a(t,x)| \leq K_3(1+|x|), \tag{18}$$

$$|b^{j_1}(t,x)|+|L^{j_1} b^{j_2}(t,y)| \leq K_3(1+|x|), \tag{19}$$

$$|L^j L^{j_1} b^{j_2}(t,x)| \leq K_3(1+|x|), \tag{20}$$

(Bound and Differentiability with resp.t)

$$|a(s,x) - a(t,x)| \leq K_4(1+|x|)|s-t|^{1/2}, \tag{21}$$

$$|b^{j_1}(s,x) - b^{j_1}(t,x)| \leq K_4(1+|x|)|s-t|^{1/2}, \tag{22}$$

$$|L^{j_1} b^{j_2}(s,x) - L^{j_1} b^{j_2}(t,x)| \leq K_4(1+|x|)|s-t|^{1/2}, \tag{23}$$

then for all $s,t \in [t_0,T], x,y \in IR^d, j = 0,\ldots,m$ *and* $j_1,j_2 = 1,\ldots,m,$ *where the constants* $K_i > 0, i = 1,\ldots,4$ *are independent of* Δ. *There exists a positive constant and independent of* Δ, K_5, *such that*

$$IE(|\mathbf{X}_T - \mathbf{Y}_T^{\Delta}|) \leq K_5\Delta, \tag{24}$$

i.e. The Milstein-Approximation converges (first moment convergence) with convergence order $\gamma = 1$ *to the exact solution of (2).*

For the proof of this theorem we refer to [5] or [11].

Lemma 3.1. *The components of the drift vector of the SLA satisfy the Lipschitz condition (15), the bounded increment condition (18), and the bound-differential condition (21).*

Proof. Similar constructions are given by [5]. For this reason the computational details are left to the reader. ■

Note finally that since the drift vector satisfies the convergence conditions required in the lemma above and under a suitable choice of the the diffusion matrix, we can assure the convergence order one, $\gamma = 1$, of the SLA.

4. Numerical Results

Here we numerically examine the performance of the Milstein scheme for solving the stochastic Lorenz system. The drift is the usual deterministic part of the Lorenz system of differential equations (3). For our simulations, the diffusion matrix is given as:

$$b : \mathbb{T} \times IR^3 \to IR^{3\times3}, \quad b(t,x(t),y(t),z(t)) = \lambda\big(x(t)+y(t)+z(t)\big)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{25}$$

where $\lambda << 1$ is a stabilization factor. Equations (12), (13) and the Itô-Taylor expansion lead to the one-step Milstein scheme for a three dimensional stochastic differential system. To solve the system (2), with a drift vector (3), and a diffusion matrix (25), we use the sets (12) and (13). The double Itô integrals (8) are approximated using the truncated Fourier series method given by equation (9). We refer, for example to [5], for more details.

When $\lambda = 5 \cdot 10^{-3}, \ 5 \cdot 10^{-2}, \ 10^{-1},$ we simulate 1000 realizations of the SLA. For further analysis, statistical moments, such as the mean and standard deviation, need also to be

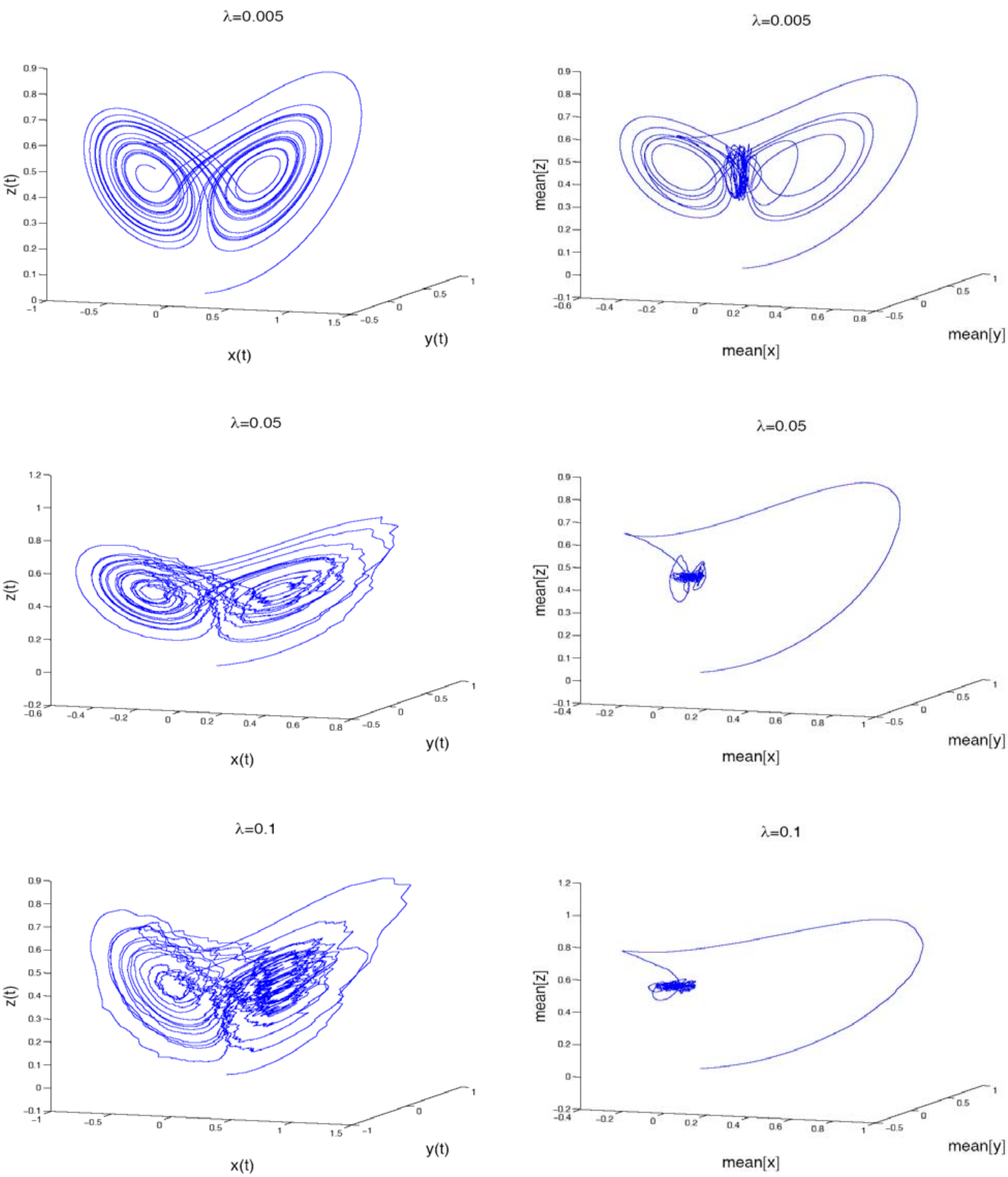

Figure 1: SLA simulation (first column) and the corresponding mean solutions (second column).

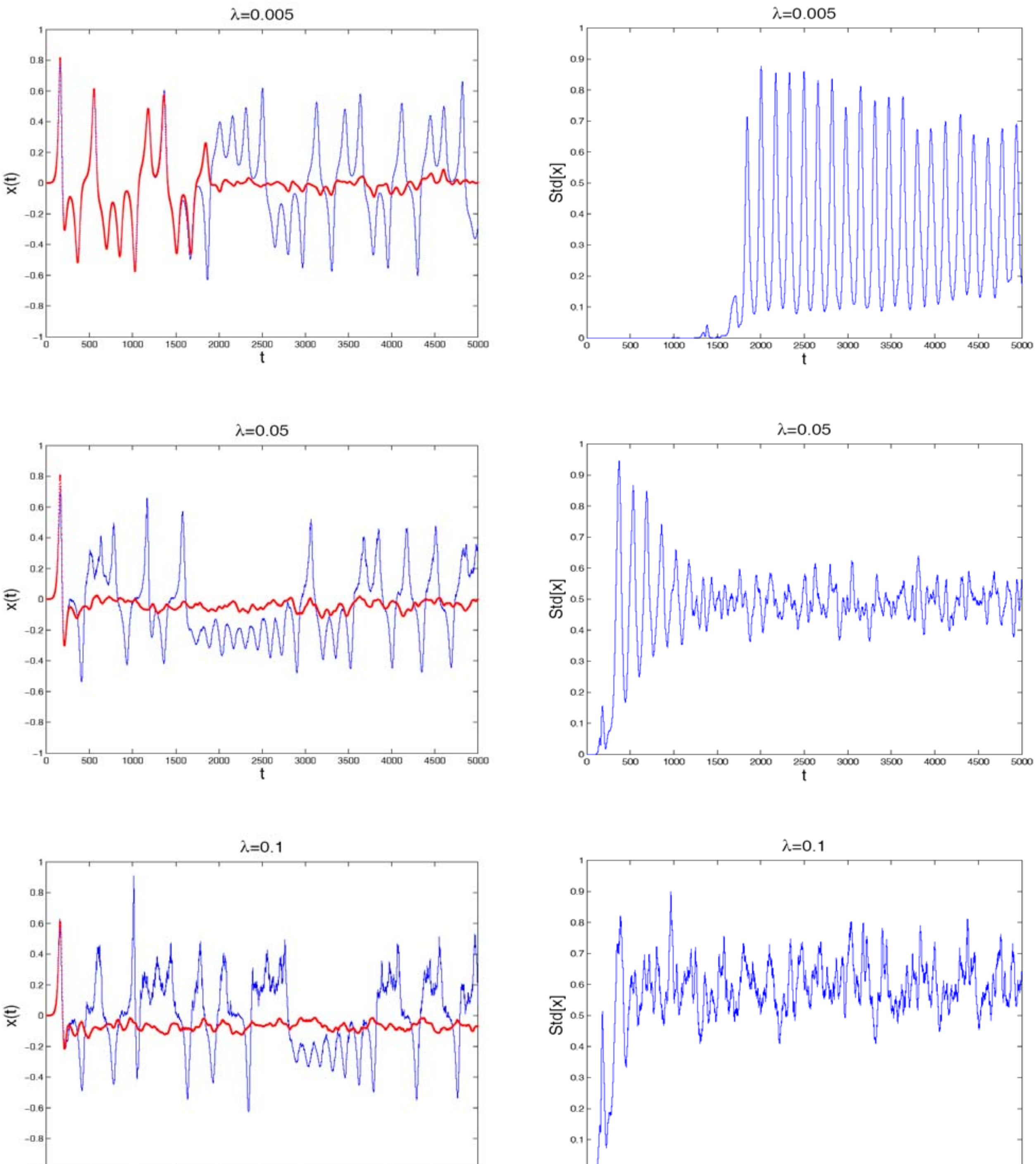

Figure 2: (first column) First component (x(t)) of SLA (solid line) and the corresponding mean (dotted line). (second row) the corresponding standard deviation of x(t).

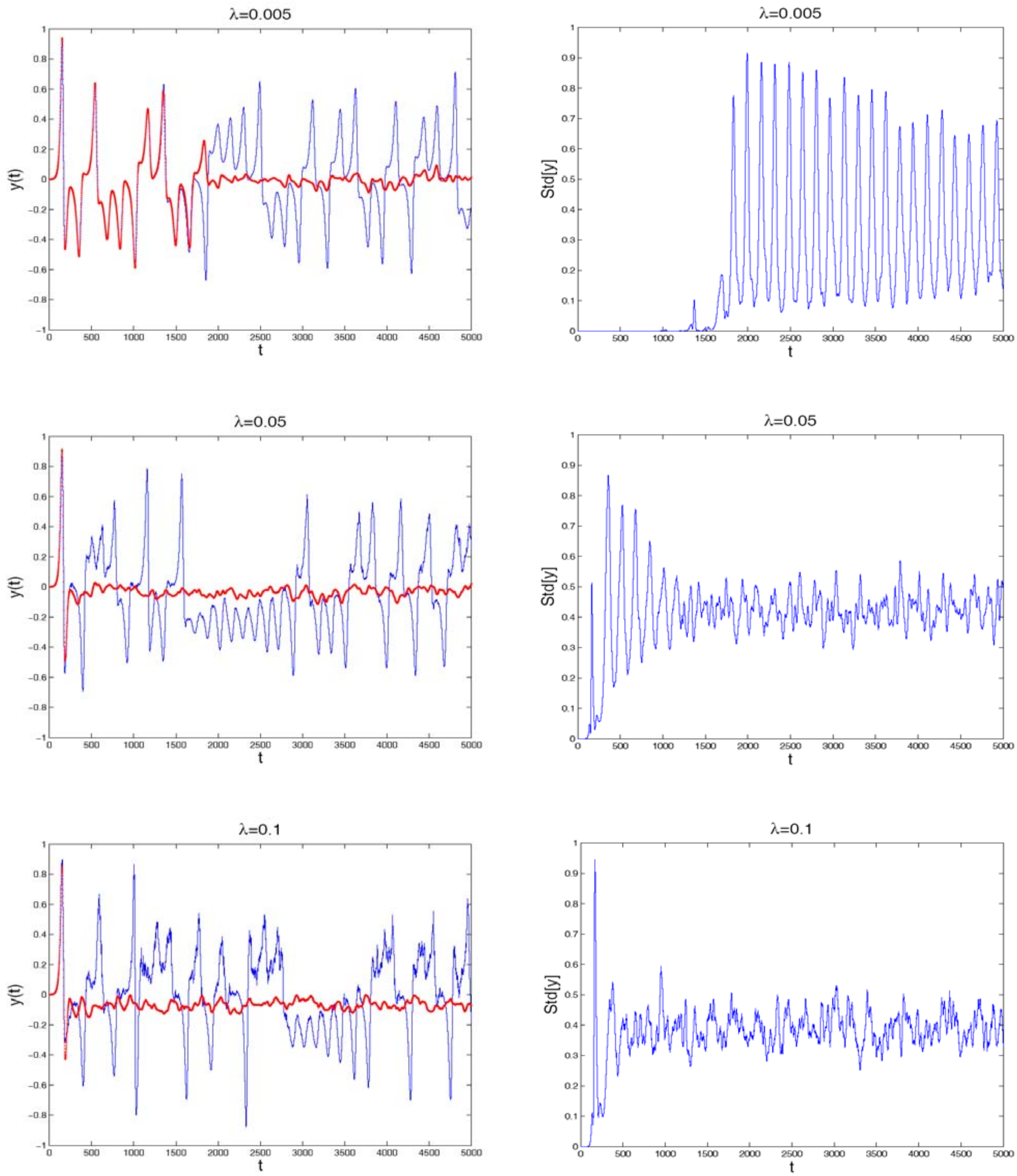

Figure 3: (first row) Second component (y(t)) of SLA (solid line) and the corresponding mean (dotted line). (second column) the corresponding standard deviation of y(t).

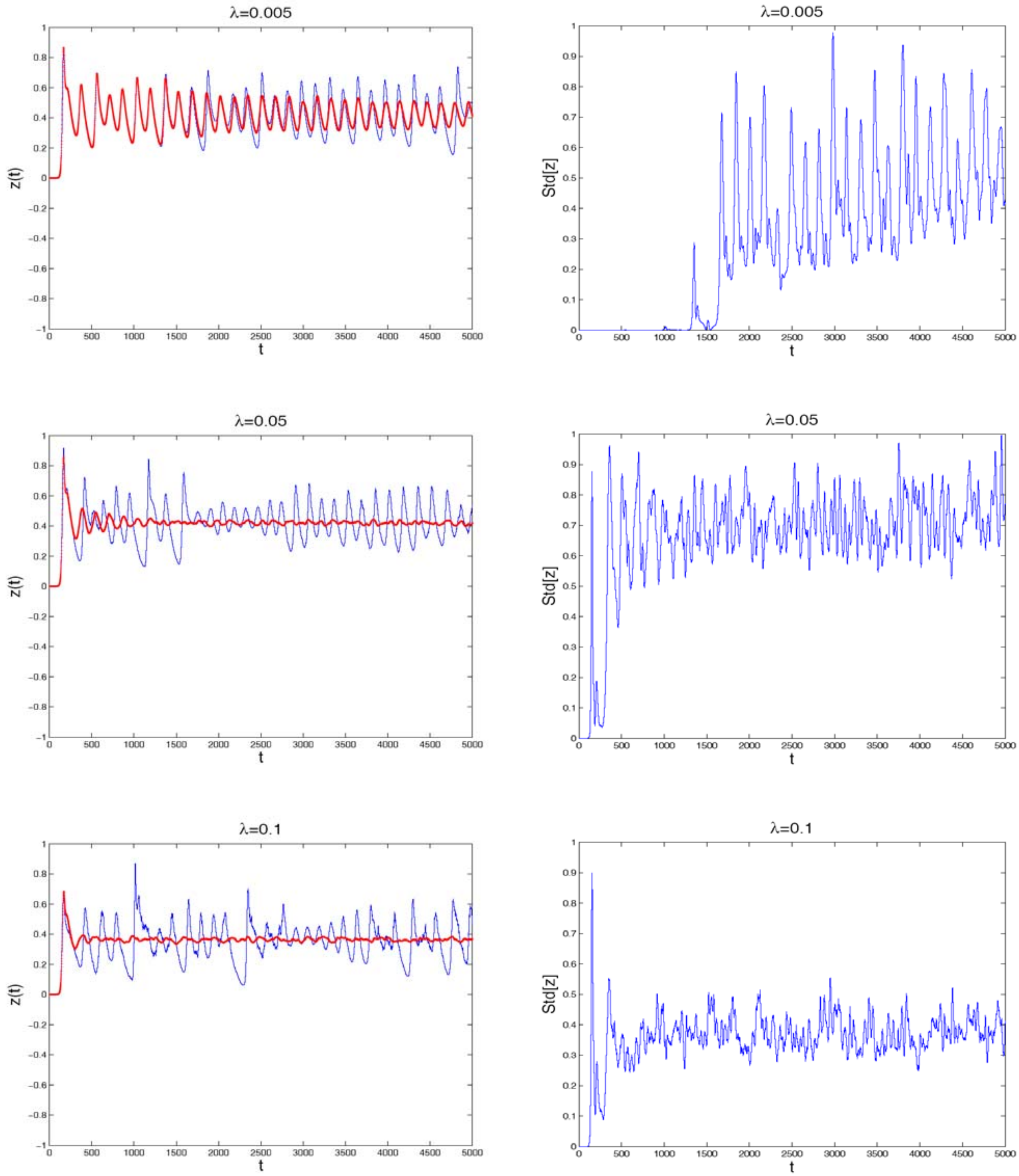

Figure 4: (first row) Third component (z(t)) of SLA (solid line) and corresponding mean (dotted line). (second row) the corresponding standard deviation of z(t).

computed. In all our simulations, we use a fixed time step $\Delta = 5 \cdot 10^{-3}$ and the same number of time iterations $T = 5 \cdot 10^{4}$. The numerical solutions are summarized as follows.

Figure 1 presents for $\lambda = 5 \cdot 10^{-3}, 5 \cdot 10^{-2}$ and 10^{-1} one simulation on the first row and the mean of 1000 realizations of the stochastic Lorenz system. We remark that even if the realizations have the behavior of the Lorenz Attractor, the mean solution is totaly different. This result is expected, since the lorenz system is a coupled one.

In Figures 2, 3 and 4 we plot the time dependent components of SLA $x(t)$, $y(t)$ and $z(t)$ respectively. The corresponding means are also shown. The dotted red line represent the means of 1000 realizations. In the second column of Figure 2, 3 and 4 we show time series of the standard deviation of the corresponding four simulations. It is clearly shown that the first one has a larger deviation than the second and third.

According to these numerical results, we remark that the mean solution of the numerical stochastic approach to the SLA does not necessarily have the same behavior as the beautiful one of the deterministic Lorenz attractor. The stochastic system in the Lorenz attractor context asymptotically converges to a single point. This can be seen at the second row of Figure 1.

It is important to note that the components of the diffusion matrix (25) are correlated. Consequently the mean behavior of the SLA tends to converge to a stationary state.

5. Concluding Remarks

We have presented the Milstein scheme to numerically solve the stochastic Lorenz differential system. Owing to the fact that the solution is coupled and because of the correlation between the diffusion matrix entries, the Lorenz system is perturbed. We believe that the mean solution of the stochastic system is an attractor of another type and therefore the numerical approach of SLA has in general different behaviors. The Milstein scheme has a strong convergence order one. For its implementation, we have used an extension of the deterministic approach and the Fourier series method to approximate the double Itô integrals. We also have to note that our approach is more academic than practical. A realistic interpretation of this extension is nevertheless helpful in explaining realistic phenomena that can be modeled by a SLA.

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