Journal of Numerical Mathematics and Stochastics, **14** (1) : 49-66, 2023
http://www.jnmas.org/jnmas14-4.pdf
Euclidean Press, LLC
Online: ISSN 2151-2302

Controllability Result for Neutral Stochastic Integro-differential Equations Driven by a Rosenblatt Process

E. KPIZIM[1], O. NDIAYE[1], H. HMOYED[2] and M. A. DIOP[1,3]
[1]Laboratoire d'Analyse Numerique et Informatique, Département de Mathématiques, Université Gaston Berger de Saint-Louis, UFR de Sciences appliqués et de Technologie, Université Gaston Berger, BP 234, Saint-Louis, Sénégal, [2]Département de Mathématiques, Faculté de Sciences et Techniques de Nouakchott, Al-Aasariya, Mauritanie,[3]UMMISCO UMI 209 IRD/UPMC, Bondy, France, E-mail: mamadou-abdoul.diop@ugb.edu.sn

Abstract. *This paper studies the controllability of neutral stochastic integrodifferential evolution equations in Hilbert spaces. By employing the resolvent operator theory in the sense of Grimmer, stochastic analysis, and a fixed point approach, the sufficient conditions of exact controllability, for such a system, are established. We provide an example to illustrate the effectiveness of the proposed result.*

Key words: C_0-Semigroup, Grimmer Resolvent Operator, Stochastic Integrodifferential Equations, Fixed Point Theorem, Controllability, Rosenblatt Process.

AMS Subject Classifications: 47A10, 93B05, 47H10, 60H05

1. Introduction

Using stochastic differential equations to model dynamic phenomena is helpful when predicting precisely how the modeled system will behave. Modeling the noises that occur in a variety of fields, such as financial mathematics, hydrology, medicine, and telecommunications networks can be performed with stochastic differential equations that are driven by fractional Brownian motion (see, for example, [4, 6, 23] and the references therein). Moreover, the Rosenblatt process is a helpful tool when the Gaussianity property of the model does not make sense when dealing with data that does not fit a normal distribution. As is well-known, the Rosenblatt process is non-Gaussian with many exciting properties, such as stationarity of the increments, long-range dependence, and self-similarity. There exists consistent literature that focuses on different theoretical aspects of the Rosenblatt processes. In the past ten years, it has undergone substantial development; for example, see the papers [9, 38 – 40] in which the numerous properties of the Rosenblatt process are examined and detailed. Because of this, it would be interesting to research a new class of fractional stochastic differential equations driven by the Rosenblatt process. We want to direct the reader to the extensive work in [1, 3, 9 – 11, 13, 16, 18, 25, 28, 29, 33, 34, 42] and [5] and references listed in those sources for additional information. Recent research carried out by Shen et al. [35] established the existence and uniqueness of a mild solution for a neutral stochastic partial differential equation with a finite delay driven by the Rosenblatt process in a real separable Hilbert space. Shen et al. [36] investigated the controllability and exponential stability in the pth moment for stochastic differential systems driven by the Rosenblatt process. Sakthivel et al. [32] used fixed point theory to investigate the existence results for retarded SDEs with infinite delay driven by the Rosenblatt process. Caraballo et al. [6] investigated the existence and uniqueness of a mild solution for an impulsive stochastic system driven by a Rosenblatt process. They did this using the Banach fixed point theorem and the theory of resolvent operators developed by R. Grimmer in [19]. Also, they got exponential stability in the mean square for the mild solutions by using an integral inequality.

Controllability is one of the most essential and fundamental concepts in mathematical control theory. Both deterministic and stochastic systems stand to gain a great deal from their presence and application (for additional details, see [2, 41] and the references there in). The study of controllability of stochastic partial differential equations is one of the topics that researchers are currently focusing on. We refer the reader to [8, 20, 23].

Our main goal in this study is to look into the controllability of nonlinear neutral evolution equations, driven by the Rosenblatt process, of following form.

$$\begin{cases} d[\vartheta(t) - h(t,\vartheta(t))] = \Big[\mathrm{A}[\vartheta(t) - h(t,\vartheta(t))] + \int_0^t \Upsilon(t-s)[\vartheta(s) - h(s,\vartheta(s))]\,ds \\ \qquad\qquad + \mathrm{Cu}(t) + F\Big(t, \int_0^t \rho(s,\vartheta(s))ds\Big)\Big]dt + g(t)\,d\mathrm{Z}_Q^{\mathrm{H}}(t), t \in I = [0,b], \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{1}$$

where the state $\vartheta(\cdot)$ takes values in a separable real Hilbert space X with inner product $(\cdot, \cdot)$, $\mathrm{A} : \mathrm{D}(\mathrm{A}) \subset \mathrm{X} \to \mathrm{X}$ is the infinitesimal generator of a C_0-semigroup $(\mathrm{T}(t))_{t\geq 0}$ with domain D(A), $\Upsilon(t)$ is a closed linear operator on X with domain $\mathrm{D}(\Upsilon(t)) \supset \mathrm{D}(\mathrm{A})$ which is independent of t. The control function $\mathrm{u}(\cdot)$ takes values in $\mathfrak{L}^2(I, \mathrm{U})$, the Hilbert space of admissible control functions for a separable Hilbert space U. C is a bounded linear operator from U into X. $F : I \times \mathrm{X} \to \mathrm{X}$; $h : I \times \mathrm{X} \to \mathrm{X}$; $\rho : I \times \mathrm{X} \to \mathrm{X}$ and $g : I \to \mathfrak{L}_2^0(\mathbb{K}, \mathrm{X})$ are appropriate functions to be specified later, Z^{H} is a Rosenblatt process in real separable Hilbert spaces with Hurst parameter $\mathrm{H} \in (\frac{1}{2}, 1)$.

In [19], R. Grimmer considered the following integrodifferential equation:

$$\begin{cases} \vartheta'(t) = \mathrm{A}\vartheta(t) + \int_0^t \Upsilon(t-s)\vartheta(s)ds + \Xi(t) \text{ for } t \geq 0 \\ \vartheta(0) = \vartheta_0 \in \mathbb{Y}, \end{cases} \tag{2}$$

where, $\mathbb{Y}$ is a Banach space and $\Xi : \mathbb{R}_+ \to \mathbb{Y}$ is a continuous function. The author was able to get some results about the existence, regularity, and asymptotic behavior of solutions to the equation (2) by using resolvent operator theory. These findings are presented in the form of a variation constant of formula. Note that in [12], the authors discussed via α-norm the controllability results of (1) with $\Upsilon = 0$, and the presence of fractional Brownian motion. In this study, we ignore these two assumptions and instead make the assumption that the linear component possesses a resolvent operator in the sense that it is described by Grimmer and that the system is driven by a Rosenblatt process.

The contributions made by this manuscript fall into the following categories:

- The formulation of nonlinear stochastic functional integrodifferential control system with the Rosenblatt process incorporated.

• The resolvent operator theory, in the sense of Grimmer, is utilized, together with the Grammian, for the purpose of achieving sufficient conditions in a stochastic setting and to ensure that the system (1) is controllable.
• An example is given to show the findings of the theoretical analysis that was carried out.
The following is the paper's structure: In the second section, we will review some fundamental ideas about integrodifferential equations in Banach spaces and the Rosenblatt process. In Section 3, we investigate the controllability of the stochastic system (1). Finally, we provide a working example that illustrates the proposed theory in this paper.

2. Preliminaries

In this section, we present some basic concepts, definitions, and lemmas that are required to obtain the results. Throughout this paper, it is assumed that $\frac{1}{2} < \mathrm{H} < 1$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and for $t \geq 0$, $\mathcal{F}_t$ denote the σ-field generated by $\{\mathrm{Z}^{\mathrm{H}}(t)(s), s \in [0,t]\}$ and the $\mathbb{P}$-null sets. Let X be a separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\| \cdot \|$. The collection of all strongly measurable, square integrable X- valued random variables denoted by $\mathfrak{L}^2(\Omega, \mathcal{F}, \mathbb{P}, \mathrm{X}) \equiv \mathfrak{L}^2(\Omega, \mathrm{X})$ stands for the space of all X-valued random variables $\mathcal{G}$ such that $\mathbb{E} \| \mathcal{G} \|^2 = \int_\Omega \| \mathcal{G} \|^2 \, d\mathbb{P} < \infty$. Let $\mathfrak{L}(\mathbb{K}, \mathrm{X})$ denotes the space of all bounded linear operators from $\mathbb{K}$ to X and $Q \in \mathfrak{L}(\mathbb{K}, \mathbb{K})$ represents a non-negative self-adjoint operator. Let $\mathfrak{L}_Q^0(\mathbb{K}, \mathrm{X})$ be the space of all functions $\Gamma \in \mathfrak{L}^2(\mathbb{K}, \mathrm{X})$ such that $\Gamma Q^{1/2}$ is a Hilbert-Schmidt operator. The norm is given by $\| \Gamma \|^2_{\mathfrak{L}_Q^0} = \| \Gamma Q^{1/2} \|^2 = Tr(\Gamma Q \Gamma^*)$ and Γ is called a Q-Hilbert-Schmidt operator from $\mathbb{K}$ to X.

2.1. **Rosenblatt process**

Let $[0,b]$ denote a time interval with arbitrary fixed horizon b and let $\{\mathrm{Z}^{\mathrm{H}}(t), t \in [0,b]\}$ be a one-dimensional Rosenblatt process with parameter $\mathrm{H} \in (\frac{1}{2}, 1)$. Now, the Rosenblatt process with parameter $\mathrm{H} > \frac{1}{2}$ can be written as [40]

$$\mathrm{Z}_{\mathrm{H}}(t) = d(\mathrm{H}) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial \mathrm{K}^{\mathrm{H}'}}{\partial u}(u, y_1) \frac{\partial \mathrm{K}^{\mathrm{H}'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2), \tag{3}$$

where $\mathrm{K}^{\mathrm{H}}(t,s)$ is given by

$$\mathrm{K}^{\mathrm{H}}(t,s) = m(\mathrm{H}) s^{\frac{1}{2}-\mathrm{H}} \int_s^t (u-s)^{\mathrm{H}-3/2} u^{\mathrm{H}-1/2} du \text{ for } t > s,$$

with

$$m(\mathrm{H}) = \sqrt{\frac{\mathrm{H}(2\mathrm{H}-1)}{\beta(2-2\mathrm{H}, \mathrm{H}-\frac{1}{2})}},$$

$\beta(\cdot,\cdot)$ denotes the Beta function, $\mathrm{K}^{\mathrm{H}}(t,s) = 0$ when $t \leq s$, $(B(t), t \in [0,b])$ is a Brownian motion, $\mathrm{H}' = \frac{\mathrm{H}+1}{2}$ and $d(\mathrm{H}) = \frac{1}{\mathrm{H}+1} \sqrt{\frac{\mathrm{H}}{2(2\mathrm{H}-1)}}$ is a normalizing constant. The covariance of the Rosenblatt process $\{\mathrm{Z}_{\mathrm{H}}(t), t \in [0,b]\}$ is

$$\mathbb{E}(Z_H(t)Z_H(s)) = \frac{1}{2}(s^{2H} + t^{2H} - | s-t |^{2H}).$$

The covariance structure of the Rosenblatt process allows to construct Wiener integral with respect to it. We refer to Maejima and Tudor [27] for the definition of Wiener integral with respect to general Hermite processes and to Kruk, Russo, and Tudor [31] for a more general context (see also Tudor [40]).

Notice that

$$Z_H(t) = \int_0^b \int_0^b \mathcal{I}(1_{[0,t]})(y_1, y_2) dB(y_1) dB(y_2),$$

where the operator $\mathcal{I}$ is defined on the set of functions $\mathcal{G} : [0,b] \to \mathbb{R}$, which takes its values in the set of functions $\mathcal{G} : [0,b]^2 \to \mathbb{R}^2$ and is given by

$$\mathcal{I}(\mathcal{G})(y_1, y_2) = d(H) \int_{y_1 \vee y_2}^b \mathcal{G}(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

Let $\mathcal{G}$ be an element of the set $\mathcal{E}$ of step functions on $[0,b]$ of the form

$$\mathcal{G} = \sum_{i=0}^{n-1} a_i 1_{(t_i, t_{i+1}]}, \; t_i \in [0,b].$$

Then, it is natural to define its Wiener integral with respect to Z_H as

$$\int_0^b \mathcal{G}(u) dZ_H(u) := \sum_{i=0}^{n-1} a_i (Z_H(t_{i+1}) - Z_H(t_i)) = \int_0^b \int_0^b \mathcal{I}(\mathcal{G})(y_1, y_2) dB(y_1) dB(y_2).$$

Let $\mathcal{H}$ be the set of functions $\mathcal{G}$ such that

$$\|\mathcal{G}\|_{\mathcal{H}}^2 := 2 \int_0^b \int_0^b (\mathcal{I}(\mathcal{G})(y_1, y_2))^2 dy_1 dy_2 < \infty.$$

It follows from [40] that

$$\|\mathcal{G}\|_{\mathcal{H}}^2 = H(2H-1) \int_0^b \int_0^b \mathcal{G}(u)\mathcal{G}(v) \; | u - v |^{2H-2} \, du dv,$$

and it has been proved in [27] that the mapping

$$\mathcal{G} \to \int_0^b \mathcal{G}(u) dZ_H(u)$$

defines an isometry from $\mathcal{E}$ to $\mathfrak{L}^2(\Omega)$. Because $\mathcal{E}$ is dense in $\mathcal{H}$, it can be extended continuously to an isometry from $\mathcal{H}$ to $\mathfrak{L}^2(\Omega)$. We call this extension as the Wiener integral of $\mathcal{G} \in \mathcal{H}$ with respect to Z_H. It is noted that the space $\mathcal{H}$ contains not only functions but its elements could be also distributions. Therefore it is suitable to identify subspaces $| \mathcal{H} |$ of $\mathcal{H}$: $| \mathcal{H} | = \left\{ \mathcal{G} : [0,b] \to \mathbb{R} \; | \; \int_0^b \int_0^b | \mathcal{G}(u) | | \mathcal{G}(v) | \; | u - v |^{2H-2} \, du dv < \infty \right\}$. The space $| \mathcal{H} |$ is not complete with respect to the norm $\| \cdot \|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|\mathcal{G}\|_{|\mathcal{H}|}^2 = H(2H-1) \int_0^b \int_0^b | \mathcal{G}(u) | | \mathcal{G}(v) | \; | u - v |^{2H-2} \, du dv.$$

As a consequence, we have

$$\mathfrak{L}^2([0,b]) \subset \mathfrak{L}^{1/\mathrm{H}}([0,b]) \subset |\ \mathcal{H}\ | \subset \mathcal{H}.$$

For any $\mathcal{G} \in \mathfrak{L}^2([0,b])$, we have

$$\|\mathcal{G}\|^2_{|\mathcal{H}|} \leq 2\mathrm{H}b^{2\mathrm{H}-1} \int_0^b |\ \mathcal{G}(s)\ |^2\ ds$$

and

$$\|\mathcal{G}\|^2_{|\mathcal{H}|} \leq m(\mathrm{H})\|\mathcal{G}\|^2_{\mathfrak{L}^{1/\mathrm{H}}([0,b])}, \tag{4}$$

for some constant $m(\mathrm{H}) > 0$. Let $m(\mathrm{H}) > 0$ stands for a positive constant depending only on $\mathcal{H}$ and its value may be different in different settings.
Define the linear operator K^*_H from $\mathcal{E}$ to $\mathfrak{L}^2([0,b])$ by

$$(\mathrm{K}^*_\mathrm{H}\mathcal{G})(y_1,y_2) = \int_{y_1 \vee y_2}^b \mathcal{G}(t)\frac{\partial \mathcal{K}}{\partial t}(t,y_1,y_2)dt,$$

where $\mathcal{K}$ is the kernel of Rosenblatt process in representation (3)

$$\mathcal{K}(t,y_1,y_2) = 1_{[0,t]}(y_1)1_{[0,t]}(y_2)\int_{y_1\vee y_2}^t \frac{\partial \mathrm{K}^{\mathrm{H}'}}{\partial u}(u,y_1)\frac{\partial \mathrm{K}^{\mathrm{H}'}}{\partial u}(u,y_2)du.$$

Note that $(\mathrm{K}^*_\mathrm{H}1_{[0,t]})(y_1,y_2) = \mathcal{K}(t,y_1,y_2)1_{[0,t]}(y_1)1_{[0,t]}(y_2)$. The operator K^*_H is an isometry between $\mathcal{E}$ to $\mathfrak{L}^2([0,b])$, which can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,b]$ we have

$$\begin{aligned}
\langle \mathrm{K}^*_\mathrm{H}1_{[0,t]}, \mathrm{K}^*_\mathrm{H}1_{[0,s]}\rangle_{\mathfrak{L}^2([0,b])} &= \langle \mathcal{K}(t,\cdot,\cdot)1_{[0,t]}, \mathcal{K}(s,\cdot,\cdot)1_{[0,s]}\rangle_{\mathfrak{L}^2([0,b])} \\
&= \int_0^{t\wedge s}\int_0^{t\wedge s} \mathcal{K}(t,y_1,y_2)\mathcal{K}(s,y_1,y_2)dy_1dy_2 \\
&= \mathrm{H}(2\mathrm{H}-1)\int_0^t\int_0^s |\ u-v\ |^{2\mathrm{H}-2}\ dudv \\
&= \langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}}.
\end{aligned}$$

Further to this , for $\mathcal{G} \in \mathcal{H}$, we have

$$\mathrm{Z}_\mathrm{H}(\mathcal{G}) = \int_0^b\int_0^b (\mathrm{K}^*_\mathrm{H}\mathcal{G})(y_1,y_2)dB(y_1)dB(y_2).$$

Let $\{\mathrm{z}_n(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one dimensional Rosenblatt process mutually independent on $(\Omega,\mathcal{F},\mathbb{P})$. Consider then a $\mathbb{K}$-valued stochastic process $\mathrm{Z}_Q(t)$ given by the following series:

$$\mathrm{Z}_Q(t) = \sum_{n=1}^{\infty} \mathrm{z}_n(t)Q^{1/2}e_n, \quad t \geq 0.$$

Moreover, if Q is a non-negative self-adjoint trace class operator, then this series converges in the space $\mathbb{K}$, that is, it holds that $\mathrm{Z}_Q(t) \in \mathfrak{L}^2(\Omega,\mathbb{K})$. Then, we say that the above $\mathrm{Z}_Q(t)$ is a $\mathbb{K}$-valued Q-Rosenblatt process with covariance operator Q. For instance, if $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$, by assuming that Q is a nuclear operator in $\mathbb{K}$, then the stochastic process

$$\mathrm{Z}_Q(t) = \sum_{n=1}^{\infty} \mathrm{z}_n(t) Q^{1/2} e_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\, \mathrm{z}_n(t) e_n, \quad t \geq 0,$$

is well-defined as a $\mathbb{K}$-valued Q- Rosenblatt process.

Definition 2.1 [40]. Let $\varphi : [0,b] \to \mathfrak{L}_Q^0(\mathbb{K}, \mathrm{X})$ such that $\sum_{n=1}^{\infty} \|\mathrm{K}_{\mathrm{H}}^*(\varphi Q^{1/2} e_n)\|_{\mathfrak{L}^2([0,b];\mathrm{X})} < \infty$. Then, its stochastic integral with respect to the Rosenblatt process $\mathrm{Z}_Q(t)$ is defined, for $t \geq 0$, as follows :

$$\begin{aligned} \int_0^t \varphi(s) d\mathrm{Z}_Q(s) &:= \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{1/2} e_n dz_n(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \int_0^t (\mathrm{K}_{\mathrm{H}}^*(\varphi Q^{1/2} e_n))(y_1, y_2) dB(y_1) dB(y_2). \end{aligned} \tag{5}$$

Lemma 2.1 [37]. For $\psi : [0,b] \to \mathfrak{L}_Q^0(\mathbb{K}, \mathrm{X})$ such that $\sum_{n=1}^{\infty} \|\psi Q^{1/2} e_n\|_{\mathfrak{L}^{1/\mathrm{H}}([0,b];\mathrm{X})} < \infty$ holds, and for any $\alpha, \beta \in [0,b]$ with $\beta > \alpha$, we have

$$\mathbb{E}\left\| \int_\alpha^\beta \psi(s) d\mathrm{Z}_Q(s) \right\|^2 \leq m(\mathrm{H})(\beta - \alpha)^{2\mathrm{H}-1} \sum_{n=1}^{\infty} \int_\alpha^\beta \|\psi(s) Q^{1/2} e_n\|^2 ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \|\psi(t) Q^{1/2} e_n\| \text{ is uniformly convergent for } t \in [0,b],$$

then, it holds that

$$\mathbb{E}\left\| \int_\alpha^\beta \psi(s) d\mathrm{Z}_Q(s) \right\|^2 \leq m(\mathrm{H})(\beta - \alpha)^{2\mathrm{H}-1} \int_\alpha^\beta \|\psi(s)\|^2_{\mathfrak{L}_Q^0(\mathbb{K},\mathrm{X})} ds.$$

2.2. **Integrodifferential equations in Banach spaces**

Let $\mathbb{Y}$ and $\mathbb{X}$ be two Banach spaces such that $\|\vartheta\|_{\mathbb{X}} = \|\mathrm{A}\vartheta\| + \|\vartheta\|$, $\vartheta \in \mathbb{X}$. A and $\Upsilon(t)$ are closed linear operators on $\mathbb{Y}$. Let $\mathcal{C}(\mathbb{R}^+, \mathbb{X})$, $\mathscr{L}(\mathbb{X}, \mathbb{Y})$ stand for the space of all continuous functions from $\mathbb{R}^+$ into $\mathbb{X}$, the set of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$, respectively. In what follows, we suppose the following assumptions:
(**H1**) A is the infinitesimal generator of a strongly continuous semigroup $\{\mathrm{T}(t)\}_{t\geq 0}$ on $\mathbb{Y}$.
(**H2**) For all $t \geq 0$, $\Upsilon(t)$ is a closed linear operator from D(A) to $\mathbb{Y}$, and $\Upsilon(t) \in \mathscr{L}(\mathbb{X}, \mathbb{Y})$. For any $\vartheta \in \mathbb{X}$, the map $t \to \Upsilon(t)\vartheta$ is bounded, differentiable and the derivative $t \to \Upsilon(t)'\vartheta$ is bounded uniformly continuous on $\mathbb{R}^+$.

According to Grimmer [19], under the assumptions (**H1**) and (**H2**), the following Cauchy problem

$$\begin{cases} \vartheta'(t) = \mathrm{A}\vartheta(t) + \int_0^t \Upsilon(t-s)\vartheta(s) ds \text{ for } t \geq 0 \\ \vartheta(0) = \vartheta_0 \in \mathbb{Y}, \end{cases} \tag{6}$$

has an associated resolvent operator of bounded linear operator valued function $\mathrm{R}(t) \in \mathscr{L}(\mathbb{Y})$, for

$t \geq 0$.

Definition 2.2 [19]. A bounded linear operator valued function $\mathrm{R}(t) \in \mathcal{L}(\mathbb{Y})$, for $t \geq 0$, is referred to be a resolvent operator associated with (6) if :
(i) $\mathrm{R}(0) = \mathrm{I}$ and $\|\mathrm{R}(t)\|_{\mathcal{L}(\mathbb{Y})} \leq \tilde{\mathrm{M}} e^{\gamma t}$ for some constants $\tilde{\mathrm{M}}$ and γ.
(ii) For all each $m \in \mathbb{Y}$, $\mathrm{R}(t)m$ is strongly continuous for $t \geq 0$.
(iii) $\mathrm{R}(t) \in \mathcal{L}(\mathbb{X})$ for $t \geq 0$. For $m \in \mathbb{X}$, $\mathrm{R}(\cdot) \in \mathcal{C}^1([0,+\infty[,\mathbb{Y}) \cap \mathcal{C}([0,+\infty[,\mathbb{X})$ and

$$\begin{aligned} \mathrm{R}'(t)m &= \mathrm{A}\mathrm{R}(t)m + \int_0^t \Upsilon(t-s)\mathrm{R}(s)m ds, \\ &= \mathrm{R}(t)\mathrm{A}m + \int_0^t \mathrm{R}(t-s)\Upsilon(s)m ds, \;\; t \geq 0. \end{aligned}$$

Now, we present some results on the existence of solutions for the following integrodifferential equation:

$$\begin{cases} \vartheta'(t) = \mathrm{A}\vartheta(t) + \int_0^t \Upsilon(t-s)\vartheta(s)ds + \Xi(t) \;\text{ for }\; t \geq 0 \\ \vartheta(0) = \vartheta_0 \in \mathbb{Y}, \end{cases} \tag{7}$$

where $\Xi : \mathbb{R}_+ \to \mathbb{Y}$ is a continuous function.

Definition 2.3. A continuous function $\vartheta : [0,\infty[\to \mathbb{Y}$ is said to be a strict solution for equation (7) if
1. $\vartheta \in \mathcal{C}^1(\mathbb{R}_+,\mathbb{Y}) \cap \mathcal{C}(\mathbb{R}_+,\mathbb{X})$,
2. ϑ satisfies equation (7) for $t \geq 0$.

Remark 2.1. From this definition, we deduce that $\vartheta(t) \in D(A)$, and the function $s \mapsto \Upsilon(t-s)\vartheta(s)$ is integrable, for all $t > 0$ and $s \geq 0$.

Theorem 2.2 [19]. *Suppose that hypotheses (H1) and (H2) hold. If ϑ is a strict solution of (7), then the following variation of constants formula holds.*

$$\vartheta(t) = \mathrm{R}(t)\vartheta_0 + \int_0^t \mathrm{R}(t-s)\Xi(s)ds, \;\text{ for }\; t \geq 0. \tag{8}$$

Consequently, we can establish the following definition.

Definition 2.4 [19]. A function $\vartheta : \mathbb{R}_+ \to \mathbb{Y}$ is called a mild solution of (7) for $\vartheta_0 \in \mathbb{Y}$, if ϑ satisfies the variation of constants formula (8).

Theorem 2.3[19]. *Let $\Xi \in C^1([0,+\infty[;Y)$ and ϑ be defined by (8). If $\vartheta_0 \in D(A)$, then ϑ is a strict solution for equation (7).*

Theorem 2.4 [22]. *Assume that $((H1)) - (H2))$ hold. Then, the resolvent operator $(R(t))_{t\geq 0}$ for equation (6) is operator-norm continuous(or continuous in the uniform operator topology) for $t > 0$ if and only if $(T(t))_{t\geq 0}$ is operator-norm continuous for $t > 0$.*

Throughout this paper, let $\mathrm{M} := \sup\{\|\mathrm{R}(t)\| : \; 0 \leq t \leq b\}$. Denote then the space of all continuous $\mathcal{F}_t$-adapted measurable processes from $[0,b]$ to $\mathfrak{L}^2(\Omega,\mathrm{X})$ satisfying $\sup_{t\in[0,b]} \mathbb{E}\|\vartheta(t)\|^2 < \infty$ by $\mathcal{C}([0,b],\mathfrak{L}^2(\Omega,\mathrm{X}))$.

Let $\mathcal{E}_b := \mathcal{C}([0,b],\mathfrak{L}^2(\Omega,\mathrm{X}))$. The space $\mathcal{E}_b$ equipped with the norm $\|\vartheta\|_{\mathcal{E}_b} = (\sup_{t\in[0,b]}\mathbb{E}\|\vartheta(t)\|^2)^{\frac{1}{2}}$ is a Banach space.

Definition 2.5. An $\mathcal{F}_t$-adapted processes $\vartheta(t)$ is called a mild solution of system (1.1) if for all t $\in I, \vartheta(t)$ satisfies:

$$\begin{aligned}\vartheta(t) &= \mathrm{R}(t)[\vartheta_0 - h(0,\vartheta_0)] + h(t,\vartheta(t)) + \int_0^t \mathrm{R}(t-s)\Big[\mathrm{Cu}(s) + F\Big(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\Big)\Big]ds \\ &\quad + \int_0^t \mathrm{R}(t-s)g(t)dZ_Q^{\mathrm{H}}(s),\end{aligned} \tag{9}$$

where $(\mathrm{R}(t))_{t\geq 0}$ denotes the resolvent operator of the linear part of (1).

Definition 2.6. The system (1) is said to be controllable on the interval $[0,b]$, if for every initial value $\vartheta_0 \in \mathrm{X}$, there exists a control function $\mathrm{u} \in \mathfrak{L}^2([0,b],\mathrm{U})$ such that the mild solution $\vartheta(t)$ of (1) satisfies $\vartheta(b) = \vartheta_1$, where ϑ_1 is a preassigned terminal state.

3. Controllability Results

The controllability results are derived in this section. We start by introducing the following assumptions:

(**A1**) The semigroup $(\mathrm{T}(t))_{t\geq 0}$ is norm-continuous for $t > 0$.

(**A2**) The functions $F : [0,b]\times \mathrm{X} \to \mathrm{X}$ satisfy the following Lipschitz conditions: that is, there is constant $L_F > 0$ such that, for any $\vartheta_1,\vartheta_2 \in \mathrm{X}$ and $t \in [0,b]$,

$$\|F(t,\vartheta_1) - F(t,\vartheta_2)\|^2 \leq L_F\|\vartheta_1 - \vartheta_2\|^2, \|F(t,\vartheta_1)\|^2 \leq L_F(1 + \|\vartheta_1\|^2)$$

For the function $\rho : [0,b]\times \mathrm{X} \to \mathrm{X}$, there exists a constant $L_\rho > 0$ such that

$$\|\rho(t,\vartheta_1) - \rho(t,\vartheta_2)\|^2 \leq L_\rho\|\vartheta_1 - \vartheta_2\|^2, \|\rho(t,\vartheta_1)\|^2 \leq L_\rho(1 + \|\vartheta_1\|^2).$$

(**A3**) $g \in \mathfrak{L}_2^0(\mathbb{K},\mathrm{X})$ with $\sum_{n=1}^{\infty} \|gQ^{1/2}e_n\|_{\mathfrak{L}_2^0(\mathbb{K},\mathrm{X})} < \infty$.

(**A4**) The function $h : I\times \mathrm{X} \to \mathrm{X}$ is satisfy the following conditions:

(**i**) h is continuous in the following sense:

$$\lim_{t\to s}\mathbb{E}\|h(t,\vartheta(t)) - h(s,\vartheta(s))\|^2 = 0;$$

(**ii**) For any $\vartheta_1,\vartheta_2 \in \mathrm{X}$, $t \in [0,b]$,

$$\mathbb{E}\|h(t,\vartheta_1) - h(t,\vartheta_2)\|^2 \leq L_h\mathbb{E}\|\vartheta_1 - \vartheta_2\|^2, \quad \mathbb{E}\|h(t,\vartheta_1)\|^2 \leq L_h(1 + \mathbb{E}\|\vartheta\|^2),$$

and $L_h < \frac{1}{3}$.

(**A5**) The linear operator $\mathbb{L}$ from $\mathfrak{L}^2(I,\mathrm{U})$ to X defined by

$$\mathbb{L}\mathrm{u} = \int_0^b \mathrm{R}(b-s)\mathrm{Cu}(s)ds,$$

has an inverse operator $\mathbb{L}^{-1}$ that takes values in $\mathfrak{L}^2(I,\mathrm{U})\backslash\ker\mathbb{L}$ see [7] and there exists finite positive constants $K_{\mathbb{L}}, K_{\mathrm{c}}$ such that $\|\mathrm{C}\|\leq K_{\mathrm{c}}$ and $\|\mathbb{L}^{-1}\|\leq K_{\mathbb{L}}$.

Now, by using assumption (**A5**), for an arbitrary function $\vartheta(\cdot)$, we can introduce the following control:

$$\begin{aligned}\mathrm{u}^b_\vartheta(t) &= \mathbb{L}^{-1}\{\vartheta_1 - \mathrm{R}(b)[\vartheta_0 - h(0,\vartheta_0)] - h(b,\vartheta(b)) \\ &\quad - \int_0^b \mathrm{R}(b-s)F\Big(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\Big)ds \\ &\quad - \int_0^b \mathrm{R}(b-s)g(s)dZ^{\mathrm{H}}_Q(s)\Big\}(t).\end{aligned}$$

Define then the operator $\Phi : \mathcal{E}_b \to \mathcal{E}_b$ by

$$\begin{aligned}(\Phi\vartheta)(t) &= \mathrm{R}(t)[\vartheta_0 - h(0,\vartheta_0)] + h(t,\vartheta(t)) + \int_0^t \mathrm{R}(t-s)[\mathrm{Cu}^b_\vartheta(s) \\ &\quad + F\Big(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\Big)]ds \\ &\quad + \int_0^t \mathrm{R}(t-s)g(s)dZ^{\mathrm{H}}_Q(s).\end{aligned}$$

Lemma 3.1. *Assume that* $(A1)-(A4)$ *hold. For every* $\vartheta \in E_b$, $t \to (\Phi\vartheta)(t)$ *is continuous on* $[0,\ b]$ *in* $L^2(\Omega, X)$*-sense.*

Proof. For any $\vartheta \in \mathcal{E}_b$, $0 < t_1 < t_2 < b$, we have

$$\begin{aligned}&\mathbb{E}\|(\Phi\vartheta)(t_2) - (\Phi\vartheta)(t_1)\|^2 \\ &\leq 5\mathbb{E}\|(\mathrm{R}(t_2) - \mathrm{R}(t_1))(\vartheta_0 - h(0,\vartheta_0))\|^2 + 5\mathbb{E}\|h(t_2,\vartheta(t_2)) - h(t_1,\vartheta(t_1))\|^2 \\ &+ 5\mathbb{E}\|\int_0^{t_2} \mathrm{R}(t_2 - s)F(s,\rho(\mu,\vartheta(\mu))d\mu)ds \\ &- \int_0^{t_1} \mathrm{R}(t_1 - s)F\Big(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\Big)ds\|^2 \\ &+ 5\mathbb{E}\|\int_0^{t_2} \mathrm{R}(t_2-s)g(s)dZ^{\mathrm{H}}_Q(s) - \int_0^{t_1} \mathrm{R}(t_1-s)g(s)dZ^{\mathrm{H}}_Q(s)\|^2 \\ &+ 5\mathbb{E}\|\int_0^{t_2} \mathrm{R}(t_2-s)\mathrm{Cu}(s)ds - \int_0^{t_1} \mathrm{R}(t_1-s)\mathrm{Cu}(s)ds\|^2 \\ &:= J_1 + J_2 + J_3 + J_4 + J_5.\end{aligned}$$

Now, we only need to check that J_1, J_2, J_3, J_4 and J_5 tends to 0 independently of $\vartheta \in \mathcal{E}_b$ when $t_1 \to t_2$.

By the strong continuity of $\mathrm{R}(t)$ we have

$$\lim_{t_2 \to t_1} \|[\mathrm{R}(t_2) - \mathrm{R}(t_1)](\vartheta_0 - h(0,\vartheta_0))\|^2 = 0.$$

It follows then that $\|[\mathrm{R}(t_2) - \mathrm{R}(t_1)](\vartheta_0 - h(0,\vartheta_0)\| \leq 2\mathrm{M}[\|\vartheta_0\| + \|h(0,\vartheta_0)\|] \in L^2(\Omega,\mathbb{R}^+)$. According to Lebesgue dominated convergence theorem (LDCT) we obtain $\lim_{t_2\to t_1} J_1 = 0$.

By assumption (**A4**), it follows that

$$J_2 = 5\mathbb{E}\|h(t_2,\vartheta(t_2) - h(t_1,\vartheta(t_1))\|^2 \to 0 \ as\ t_1 \to t_2.$$

Moreover,

$$
\begin{aligned}
J_3 &\leq 10\mathbb{E}\|\int_0^{t_1}(\mathrm{R}(t_2-s)-\mathrm{R}(t_1-s))F\Big(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\Big)ds\|^2\\
&+ 10\mathbb{E}\|\int_{t_1}^{t_2}\mathrm{R}(t_2-s)F\Big(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\Big)ds\|^2\\
&:= J_{31}+J_{32}.
\end{aligned}
$$

By (**A2**), and Hölder's inequality, it is easy to validate that

$$
\begin{aligned}
J_{31} &\leq 10\mathbb{E}\Big\|\int_0^{t_1}(\mathrm{R}(t_2-s)-\mathrm{R}(t_1-s))F\Big(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\Big)ds\Big\|^2\\
&\leq L_F^2\Big(1+L_\rho^2 b^2\Big(1+\|\vartheta\|^2_{\mathcal{E}_b}\Big)\Big)\int_0^{t_1}\|\mathrm{R}(t_2-s)-\mathrm{R}(t_1-s)\|^2 ds \underset{t_2\to t_1}{\longrightarrow} 0,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{32} &\leq 10\mathbb{E}\int_{t_1}^{t_2}\|(\mathrm{R}(t_2-s)F\Big(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\Big)\|^2 ds\\
&\leq 10\mathrm{M}^2 L_F(t_2-t_1)\int_{t_1}^{t_2}(1+\mathbb{E}\|\int_0^s \rho(\mu,\vartheta(\mu))d\mu\|^2)ds\\
&\leq 10\mathrm{M}^2 L_F(t_2-t_1)^2(1+L_\rho^2 b^2\Big(1+\|\vartheta\|^2_{\mathcal{E}_b}\Big))ds\\
&\to 0 \quad \text{as} \quad t_1\to t_2.
\end{aligned}
$$

For J_4 , it is obvious that

$$
\begin{aligned}
J_4 &\leq 10\mathbb{E}\|\int_0^{t_1}(\mathrm{R}(t_2-s)-\mathrm{R}(t_1-s))g(s)d\mathrm{Z}_Q^{\mathrm{H}}(s)\|^2\\
&\qquad + 10\mathbb{E}\|\int_{t_1}^{t_2}\mathrm{R}(t_2-s)g(s)d\mathrm{Z}_Q^{\mathrm{H}}(s)\|^2\\
&:= J_{41}+J_{42}.
\end{aligned}
$$

Application, moreover, of Lemma **1**leads to

$$
\begin{aligned}
J_{41} &\leq 10m(\mathrm{H})b^{2\mathrm{H}-1}\int_0^{t_1}\|\mathrm{R}(t_2-s)-\mathrm{R}(t_1-s)g(s)\|^2 ds\\
&\leq 10m(\mathrm{H})b^{2\mathrm{H}-1}\int_0^{t_1}\|\mathrm{R}(t_2-s)-\mathrm{R}(t_1-s)\|^2\|g(s)\|^2_{\mathfrak{L}_Q^0(\mathbb{K},\mathrm{X})}ds\ .
\end{aligned}
$$

Combing this with the norm continuity of $\mathrm{R}(t)$ and Lebesgue dominated convergence theorem, we have $J_{41}\to 0$ as $t_1\to t_2$.
In a similar way, one can obtain

$$
\begin{aligned}
J_{42} &\leq 10\mathbb{E}\Big\|\int_{t_1}^{t_2}\mathrm{R}(t_2-s)g(s)d\mathrm{Z}_Q^{\mathrm{H}}(s)\Big\|^2\\
&\leq 10m(\mathrm{H})(t_2-t_1)^{2\mathrm{H}-1}\mathrm{M}^2\int_{t_1}^{t_2}\|g(s)\|^2_{\mathfrak{L}_Q^0(\mathbb{K},\mathrm{X})}ds \underset{t_2\to t_1}{\longrightarrow} 0.
\end{aligned}
$$

Now we may write

$$
\begin{aligned}
J_5 &\leq 10\mathbb{E}\|\int_0^{t_1}(\mathrm{R}(t_2-s)-\mathrm{R}(t_1,s))\mathrm{Cu}_\vartheta^b(s)ds\|^2 + 10\mathbb{E}\|\int_{t_1}^{t_2}\mathrm{R}(t_2,s)\mathrm{Cu}_\vartheta^b(s)ds\|^2 \\
&= J_{51}+J_{52}.
\end{aligned}
$$

Combine then Lemma **1**, with (**A1**)-(**A4**), to write

$$
\begin{aligned}
\mathbb{E}\|\mathrm{u}_\vartheta^b(s)\|^2 &\leq 5K_{\mathbb{L}}^2[\mathbb{E}\|\vartheta_1\|^2 + 5\mathbb{E}\|\mathrm{R}(b)[\vartheta_0-h(0,\vartheta_0)\| + 5\mathbb{E}\|h(b,\vartheta(b))\|^2 \\
&\quad + 5\mathbb{E}\|\int_0^b \mathrm{R}(b-s)F\Big(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\Big)ds\|^2 \\
&\quad + 5\mathbb{E}\|\int_0^b \mathrm{R}(b-s)g(s)d\mathrm{Z}_Q^{\mathrm{H}}(s)\|^2\Big] \\
&\leq 5K_{\mathbb{L}}^2[\mathbb{E}\|\vartheta_1\|^2 + \mathrm{M}^2\mathbb{E}\|\vartheta_0-h(0,\vartheta_0)\| \\
&\quad + L_h(1+\sup_{t\in[0,b]}\mathbb{E}\|\vartheta(t)\|) + L_F b\mathrm{M}^2(1+b(1+\sup_{s\in[0,b]}\mathbb{E}\|\vartheta(s))\|) \\
&\quad + m(\mathrm{H})b^{2\mathrm{H}-1}\mathrm{M}^2\int_0^b\|g(s)\|^2_{\mathfrak{L}_Q^0(\mathbb{K},\mathrm{X})}ds\Big] := K_{\mathrm{u}}.
\end{aligned}
$$

(**A1**)-(**A5**), Hölder inequality and the norm continuity of $\mathrm{R}(t)$, we obtain

$$
\begin{aligned}
J_{51} &= 10\mathbb{E}\left\|\int_0^{t_1}\mathrm{R}(t_2-s)-\mathrm{R}(t_1-s)\mathrm{Cu}(s)ds\right\|^2 \\
&\leq 10K_{\mathrm{c}}^2\int_0^{t_1}\mathbb{E}\|\mathrm{R}(t_2-s)-\mathrm{R}(t_1-s)\mathrm{u}(s)\|^2 ds \\
&\leq 10K_{\mathrm{c}}^2\int_0^{t_1}\|\mathrm{R}(t_2-s)-\mathrm{R}(t_1-s)\|^2 ds\int_0^{t_1}\mathbb{E}\|\mathrm{u}(s)\|^2 ds \\
&\underset{t_2\to t_1}{\longrightarrow} 0.
\end{aligned}
$$

In a similar way, we have

$$
\begin{aligned}
J_{52} &\leq 10\mathbb{E}\left\|\int_{t_1}^{t_2}\mathrm{R}(t_2-s)\mathrm{Cu}(s)ds\right\|^2 \\
&\leq 10K_{\mathrm{c}}^2\mathbb{E}\int_{t_1}^{t_2}\|\mathrm{R}(t_2-s)\mathrm{u}(s)\|^2 ds \\
&\leq 10K_{\mathrm{c}}\mathrm{M}^2(t_1-t_2)\int_{t_1}^{t_2}\mathbb{E}\|\mathrm{u}(s)\|^2 ds \\
&\leq 10K_{\mathrm{c}}\mathrm{M}^2K_{\mathrm{u}}(t_2-t_1)^2 \\
&\underset{t_2\to t_1}{\longrightarrow} 0.
\end{aligned}
$$

Hence, $\lim_{t_1\to t_2}\mathbb{E}\|(\Phi\vartheta)(t_2)-(\Phi\vartheta)(t_1)\|^2 = 0$, which implies that $t\to(\Phi\vartheta)(t)$ is continuous on $[0,b]$ in the $\mathfrak{L}^2(\Omega,\mathrm{X})$-sense. ■

Theorem 3.2. *Let assumptions (A1) - (A5) be satisfied. Then the system (1) is controllable on* $[0,b]$.

proof. From the definition of Φ, it is easy to show that $(\Phi\vartheta)(b)=\vartheta_1$; which means that the control u_ϑ^b steers system (1) from the initial state ϑ_0 to the preassigned state ϑ_1 at time b. In that follows, we

shall show that the operator Φ has a fixed point in $\mathcal{E}_b$, which is then a mild solution of system (1), and the system is controllable. For the sake of simplicity, we shall subdivide our proof into two steps.
Step1: $\Phi(\mathcal{E}_b) \subset \mathcal{E}_b$. Let $\vartheta \in \mathcal{E}_b$. Then for any $t \in [0,b]$, we have
$\Phi(\mathcal{E}_b) \subset \mathcal{E}_b$. Let $\vartheta \in \mathcal{E}_b$. Then for any $t \in [0,b]$, we have

$$\begin{aligned}
\mathbb{E}\|(\Phi\vartheta)(t)\|^2 &\leq 6\mathbb{E}\|\mathrm{R}(t)\vartheta_0\|^2 + 6\mathbb{E}\|\mathrm{R}(t)h(0,\vartheta_0)\|^2 \\
&\quad + 6\mathbb{E}\|h(t,\vartheta(t))\|^2 + 6\mathbb{E}\left\|\int_0^t \mathrm{R}(t-s)F\left(s,\int_0^s \rho(\mu,\vartheta(\mu))d\mu\right)ds\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_0^t \mathrm{R}(t-s)g(s)dZ_Q^{\mathrm{H}}(s)\right\|^2 + 6\mathbb{E}\left\|\int_0^t \mathrm{R}(t-s)\mathrm{Cu}(s)ds\right\|^2 \\
&\leq 6\mathrm{M}^2\mathbb{E}\|\vartheta_0\|^2 + 6\mathrm{M}^2\mathbb{E}\|h(0,\vartheta_0)\|^2 + 6L_h(1+\mathbb{E}\|\vartheta(t)\|^2) \\
&\quad + 6\mathrm{M}^2L_Fb(1+b(1+\sup_{t\in[0,b]}\mathbb{E}\|\vartheta(t)\|)) \\
&\quad + 6\mathrm{M}^2m(\mathrm{H})b^{2\mathrm{H}-1}\int_0^t\|g(s)\|^2_{\mathfrak{L}_Q^0(\mathbb{K},\mathrm{X})} + 6\mathrm{M}^2K_{\mathrm{c}}^2K_{\mathrm{u}}b \\
&\leq 6\mathrm{M}^2\mathbb{E}\|\vartheta_0\|^2 + 6\mathrm{M}^2L_h(1+\mathbb{E}\|\vartheta_0\|^2) + 6L_h(1+\mathbb{E}\|\vartheta(t)\|^2) \\
&\quad + 6\mathrm{M}^2L_Fb(1+b(1+\sup_{s\in[0,b]}\mathbb{E}\|\vartheta(s)\|)) \\
&\quad + 6\mathrm{M}^2m(\mathrm{H})\|g(t)\|^2_{\mathfrak{L}_Q^0(\mathbb{K},\mathrm{X})} + 6\mathrm{M}^2K_{\mathrm{u}}^2K_{\mathrm{u}}b,
\end{aligned}$$

which implies that $\mathbb{E}\|(\Phi\vartheta)(t)\|^2_{\mathcal{E}_b} < \infty$.
Step2: Φ is a contraction mapping in $\mathcal{E}_b$.
Let $\mathrm{y},\mathrm{z} \in \mathcal{E}_b$ we obtain for any fixed $t \in [0,b]$,

$$\begin{aligned}
\mathbb{E}\|(\Phi\mathrm{y})(t) - (\Phi\mathrm{z})(t)\|^2 &\leq 3\mathbb{E}\|h(t,\mathrm{y}(t)) - h(t,\mathrm{z}(t))\|^2 \\
&\quad + 3\mathbb{E}\left\|\int_0^t \mathrm{R}(t-s)[F\left(s,\int_0^s \rho(\mu,\mathrm{y}(\mu))d\mu\right)\right. \\
&\quad \left. - F\left(s,\int_0^s \rho(\mu,\mathrm{z}(\mu))d\mu\right)]ds\right\|^2 \\
&\quad + 3\mathbb{E}\left\|\int_0^t \mathrm{R}(t-s)\mathrm{C}[\mathrm{u_y}(s)-\mathrm{u_z}(s)]ds\right\|^2 = 3\sum_{k=1}^{3}\Delta_k.
\end{aligned}$$

By (**A2**), (**A3**) and applying Hölder's inequality, we obtain the following estimates

$$\begin{aligned}
\Delta_1 &= \mathbb{E}\|h(t,\mathrm{y}(t)) - h(t,\mathrm{z}(t))\|^2 \\
&\leq L_h\mathbb{E}\|\mathrm{y}(t)-\mathrm{z}(t)\|^2 \\
&\leq L_h\|\mathrm{y}-\mathrm{z}\|^2_{\mathcal{E}_b}, \qquad (10) \\
\Delta_2 &= 3\mathbb{E}\left\|\int_0^t \mathrm{R}(t-s)\left[F\left(t,\int_0^s \rho(\upsilon,\mathrm{y}(\upsilon))d\upsilon\right) - F\left(t,\int_0^s \rho(\upsilon,\mathrm{z}(\upsilon))d\upsilon\right)\right]ds\right\|^2 \\
&\leq 3L_F\int_0^t\|\mathrm{R}(t-s)\|^2\mathbb{E}\left\|\int_0^s[\rho(\upsilon,\mathrm{y}(\upsilon)) - \rho(\upsilon,\mathrm{z}(\upsilon))]d\upsilon\right\|^2 ds
\end{aligned}$$

$$\leq 3L_F b\mathrm{M}^2 \int_0^s \mathbb{E}\|\rho(\upsilon,\mathrm{y}(\upsilon)) - \rho(\upsilon,\mathrm{z}(\upsilon))\|^2 d\upsilon$$
$$\leq 3b^2\mathrm{M}^2 L_F L_\rho \sup_{s\in[0,b]} \mathbb{E}\|\mathrm{y}(s)-\mathrm{z}(s)\|^2,$$

and

$$\begin{aligned}\Delta_3 &= 3\mathbb{E}\left\|\int_0^t \mathrm{R}(t-s)\mathrm{C}[\mathrm{u_y}(s)-\mathrm{u_z}(s)]ds\right\|^2 \\ &\leq 3\mathrm{M}^2 bK_c^2\mathbb{E}\|\mathrm{u_y}(s)-\mathrm{u_z}(s)\|^2 \end{aligned} \tag{11}$$

According to the inequalities obtained above, we obtain the following relation:

$$\begin{aligned}\mathbb{E}\|[\mathrm{u}_\mathrm{y}^b(s)-\mathrm{u}_\mathrm{z}^b(s)\|^2] \leq{}& 2K_{\mathbb{L}}^2\mathbb{E}\|h(b,\mathrm{y}(b)) - h(b,\mathrm{z}(b)\|^2 \\ &+ 2K_{\mathbb{L}}^2\mathbb{E}\left\|\int_0^b \mathrm{R}(b-s)\Big[F\Big(s,\int_0^s \rho(\mu,\mathrm{y}(\mu))d\mu\Big) \right.\\ &\left. - F\Big(s,\int_0^s \rho(\mu,\mathrm{z}(\mu))d\mu\Big)\Big]\right\| ds \\ \leq{}& 2L_h K_{\mathbb{L}}^2 \sup_{s\in[0,b]}\mathbb{E}\|\mathrm{y}(s)-\mathrm{z}(s)\|^2 \\ &+ 2L_F K_{\mathbb{L}}^2\mathrm{M}^2 L_\rho b^4 \sup_{s\in[0,b]}\mathbb{E}\|\mathrm{y}(s)-\mathrm{z}(s)\|^2.\end{aligned}$$

By combining the estimates of $\Delta_k,\ k=1,2,3,$ one has

$$\begin{aligned}\mathbb{E}\|(\Phi\mathrm{y})(t)-(\Phi\mathrm{z})(t)\|^2 \leq{}& 3L_h\mathbb{E}\|\mathrm{y}(t)) - \mathrm{z}(t))\|^2 \\ &+ 3b^4\mathrm{M}^2 L_F L_\rho \sup_{s\in[0,b]}\mathbb{E}\|\mathrm{y}(s)-\mathrm{z}(s)\|^2 \\ &+ 3b\mathrm{M}^2 K_c^2\mathbb{E}\|\mathrm{u_y}(s)-\mathrm{u_z}(s)\|^2.\end{aligned}$$

Hence,

$$\mathbb{E}\|(\Phi\mathrm{y})(t)-(\Phi\mathrm{z})(t)\|^2 \leq \gamma(t)\sup_{s\in[0,b]}\mathbb{E}\|\mathrm{y}(s)-\mathrm{z}(s)\|^2$$

where,

$$\gamma(b) = 3(L_h + b^2\mathrm{M}^2 L_F L_\rho b^2 + b\mathrm{M}^2K_c^2K_{\mathbb{L}}^2(L_h + b^4\mathrm{M}^2 L_F L_\rho).$$

Consideration of condition (ii) in (**A4**) leads to $\gamma(0) = 3L_h < 1$. Then there exists $0 < b_1 < b$ such that $0 < \gamma(b_1) < 1$ and Φ is a contractive mapping on $\mathcal{E}_b$. Thus, by repeating the procedure, one can extend the solution to the interval $[0,b]$, that is, the stochastic system (1) is completely controllable. ■

4. Illustrative Example

As an application, we consider the following neutral stochastic integrodiffrential equation, driven by a Rosenblatt process :

$$
\begin{cases}
\frac{\partial}{\partial t}\left[z(t,x)-\frac{z(t,x)}{1+z(t,x)}\right]=\Bigg[\frac{\partial^2}{\partial x^2}\left(z(t,x)-\frac{z(t,x)}{1+z(t,x)}\right)+\alpha_1\frac{\partial}{\partial x}\Big(z(t,x) \\
\qquad -\frac{z(t,x)}{1+z(t,x)}\Big)+\alpha_2\left(z(t,x)-\frac{z(t,x)}{1+z(t,x)}\right)\Bigg] \\
\qquad +\int_0^t \beta e^{-\gamma(t-s)}\Bigg[\frac{\partial^2}{\partial x^2}\left(z(t,x)-\frac{z(t,x)}{1+z(t,x)}\right) \\
\qquad +\alpha_1\frac{\partial}{\partial x}\left(z(t,x)-\frac{z(t,x)}{1+z(t,x)}\right)+\alpha_2\left(z(t,x)-\frac{z(t,x)}{1+z(t,x)}\right)\Bigg]ds \\
\qquad \mu(t,x)+\mathrm{w}(t)\left(\int_0^t q(s,z(t,x))\right)ds+\frac{dZ_Q^{\mathrm{H}}(t)}{dt} \text{ for } t\in[0,b] \\
\qquad\qquad \text{and } x\in[0,\pi], \\
z(t,0)=z(t,\pi)=0, \text{ for } t\in[0,b];\ z(0,x)=z_0(x), x\in[0,\pi],
\end{cases} \tag{12}
$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}$, $\alpha_1,\alpha_2 \in \mathbb{R}$, $Z^{\mathrm{H}}(s)$ denotes standard Rosenblatt process defined on a stochastic basis $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,1]},\mathbb{P})$, $B>0$, $\mu : [0,b]\times[0,1]\to\mathbb{R}$.

Let $X = U = L^2([0,1])$ and define the operator A on X by:

$$
\begin{aligned}
&D(A) = H^1(0,1)\cap H_0^1(0,1)\\
&A\omega = \omega'' + \alpha_1\omega' + \alpha_2\omega, \quad \alpha_1,\alpha_2\in\mathbb{R}.
\end{aligned}
$$

From [[17], p. 173] we know that A is infinitesimal generator of an analytic C_0 semigroup $(T(t))_{t\geq 0}$ on X. Then, the semigroup $(T(t))_{t\geq 0}$ is norm continuous for $t>0$.

Let $\Upsilon(t) : D(A)\subset X\to X$ be the operator defined as follows:

$$\Upsilon(t)\xi = \gamma(t)A\xi \quad \text{for } t\geq 0 \quad \text{and} \quad \xi\in D(A).$$

Define the bounded linear operator $\mathcal{C} : \mathcal{V}\to\Sigma$ as

$$(\mathcal{C}u)(x) := u(t)(x) = \mu(t,x),\ t\in[0,b], x\in[0,1].$$

In order to rewrite (4.1) in an abstract form in X, we introduce the following notations

$$
\begin{cases}
\vartheta(t)(\xi) = z(t,\xi) \quad \text{for } t\in[0,b] \text{ and } \xi\in[0,1],\\
\vartheta(0)(\xi) = z(0,\xi) \quad \text{for } \xi\in[0,1].
\end{cases}
$$

Then we introduce the following operators $h,F,\rho : [0,b]\times X\to X$ and $g : I\to\mathfrak{L}_2^0(\mathbb{Y},X)$ defined by

$$
\begin{aligned}
h(t,z(t,x)) &= \frac{z(t,x)}{1+z(t,x)},\\
F\left(t,\int_0^t \rho(s,z(t,x))\right) &= \mathrm{w}(t)\int_0^t q(s,z(t,x))ds,\\
\rho(s,z(t,x)) &= q(s,z(t,x))ds, \qquad g(t)=1,\ \mu(t,x)=u(t)(x).
\end{aligned}
$$

Using these definitions we can represent the system (12) in the following abstract form

$$\begin{cases} d[\vartheta(t) - h(t,\vartheta(t))] = \Big[\mathrm{A}[\vartheta(t) - h(t,\vartheta(t))] + \int_0^t \Upsilon(t-s)[\vartheta(s) - h(s,\vartheta(s))]ds \\ \qquad\qquad + \mathrm{Cu}(t) + F\Big(t, \int_0^t \rho(s,\vartheta(s))ds\Big)\Big]dt + g(t)dZ_Q^{\mathrm{H}}(t)t \in I = [0,b] \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{13}$$

Moreover, we suppose that γ is a bounded and $\mathcal{C}^1$ function such that γ' is bounded and uniformly continuous, then (**H2**) is satisfied and hence by Grimmer [19] Eq. (6) has a resolvent operator $(\mathrm{R}(t))_{t\geq 0}$ on X. Finally, by Theorem **2.4** the corresponding resolvent operator is operator-norm continuous for $t > 0$.
Now, for $x \in [0,1]$, the operator $\mathbb{L}$ is given by

$$\mathbb{L}\mathrm{u} = \int_0^1 \mathrm{R}(b-s)\mathrm{u}(s,x)ds.$$

Assume then that $\mathbb{L}$ verifies (**A**4), thus, it is possible to verify that the assumptions on Theorem **3.2** are fulfilled and hence, the system (1) is controllable on $[0,b]$.

Acknowledgments
The authors would like to thank the referees for their substantial comments and suggestions, which have significantly improved the paper.

References

[1] H. M. Ahmed, M. M. El-Borai, and M. E. Ramadan, Boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fractional Brownian motion and Poisson jumps. *Advances in Difference Equations* **2019** (1), (2019), 1-23.

[2] U. Arora, and N. Sukavanam, Controllability of retarded semilinear fractional system with non-local conditions, *IMA Journal of Mathematical Control and Information* **35**(3), (2018), 689-705.

[3] A. Boudaoui , T. Caraballo, and A. Ouahab, Existence of mild solutions to stochastic delay evolution equations with a fractional Brownian motion and impulses, *Stochastic Analysis & Applications* **33**(2), (2015), 244-258.

[4] B. Boufoussi, S. Hajji, and E. H. Lakhel, Functional differential equations in Hilbert spaces driven by a fractional Brownian motion, *Afrika Matematika* **23**(2), (2012), 173-194.

[5] T. Caraballo, M. J. Garrido-Atienza, and T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. *Nonlinear Analysis: Theory, Methods & Applications* **74**(11), (2011), 3671-3684.

[6] T. Caraballo, C. Ogouyandjou, F. K. Allognissode, and M. A. Diop, Existence and exponential stability for neutral stochastic integro-differential equations with impulses driven by a Rosenblatt process, *Discrete & Continuous Dynamical Systems-B* **25**(2), (2020), 507-528.

[7] N. Carmichael, and M. D. Quinn, Fixed Point Methods in Nonlinear Control. In *Distributed Parameter Systems*, Springer, Berlin, Heidelberg,1985.

[8] D. Chalishajar, K. Ramkumar, A. Anguraj, K. Ravikumar, and M. A. Diop, Controllability of neutral impulsive stochastic functional integrodifferential equations driven by a fractional Brownian motion with infinite delay via resolvent operator. *Journal of Nonlinear Sciences & Applications (JNSA)* **15**(3), (2022), 172–185.

[9] P. Coupek,T. E. Duncan, and B. A. Pasik-Duncan, A stochastic calculus for Rosenblatt processes, (2019), arXiv 1908.00296.

[10] P. Coupek, and B. Maslowski, Stochastic evolution equations with Volterra noise, *Stochastic Processes* **127**(3), (2017), 877-900.

[11] P. Coupek, Limiting measure and stationarity of solutions to stochastic evolution equations with Volterra noise, *Stochastic Analysis* **36**(3), (2018),393-412.

[12] C. Jing, and Y. Litan, Controllability of neutral stochastic evolution equations driven by fractional Brownian motion, *Acta Mathematica Scientia* **37**(1), (2017), 108-118.

[13] G. Da Prato, and J. Zabczyk, Stochastic Equations in Infinite Dimensions. In *Encyclopedia of Mathematics and its Applications(44)*, Cambridge University Press,Cambridge, 1992.

[14] A. Diop, M. Dieye, M. A. Diop, and K. Ezzinbi, Integrodifferential equations of Volterra type with nonlocal and impulsive conditions, *Journal of Integral Equations and Applications* **34**(1), (2022), 19-37.

[15] M. Dieye, M. A. Diop, and K. Ezzinbi, On exponential stability of mild solutions for some stochastic partial integrodifferential equations, *Statistics & Probability Letters* **123**, (2017), 61-76.

[16] N. T. Dung, Stochastic Volterra integro-differential equations driven by fractional Brownian motion in a Hilbert space, *Stochastics: An International Journal of Probability and Stochastic Processes* **87**(1), (2015), 142-159.

[17] K. J. Engel, R. Nagel, and S. Brendle, *One-Parameter Semigroups for Linear Evolution Equations (Vol. 194,)*, Springer, New York:, 2000.

[18] J. Gairing, P. Imkeller, R. Shevchenko, and C. Tudor, Hurst index estimation in stochastic differential equations driven by fractional Brownian motion, *Journal of Theoretical Probability* **33**(3), (2020), 1691-1714.

[19] R. C. Grimmer, Resolvent operators for integral equations in a Banach space, *Transactions of the American Mathematical Society* **273**(1), (1982), 333-349.

[20] S. Hajji, and E. H. Lakhel, A note on controllability of impulsive neutral stochastic functional differential equations driven by a Rosenblatt process in a Hilbert space, *Gulf Journal of Mathematics* **12**(1), (2022), 62-80.

[21] B. Boufoussi, and S. Mouchtabih, Controllability of neutral stochastic functional integro-differential equations driven by fractional brownian motion with Hurst parameter lesser than 1/2, *Evolution Equations and Control Theory* **10**(4), (2021), 921-935. Doi: 10.3934/eect.2020096

[22] K. Ezzinbi, G. Degla, and P. Ndambomve, Controllability for some partial functional integrodifferential equations with nonlocal conditions in Banach spaces. *Discussiones Mathematicae, Differential Inclusions*, *Control and Optimization* **35**(1),(2015), 25-46.

[23] E. H. Lakhel, Controllability of neutral stochastic functional integro-differential equations driven by fractional Brownian motion, *Stochastic Analysis and Applications* **34**(3), (2016), 427-440.

[24] N. N. Leonenko, and V. V. Anh, Rate of convergence to the Rosenblatt distribution for additive functionals of stochastic processes with long-range dependence, Journal of Applied Mathematics & Stochastic Analysis 14(1), (2001), 27-46.

[25] Z. Li, Y. Y. Jing, and L. Xu, Controllability of neutral stochastic evolution equations driven by fBm with Hurst parameter less than 1/2, *International Journal of Systems Science* **50**(9),(2019), 1835-1846.

[26] J. Liang, J. H. Liu, and T. J. Xiao, Nonlocal problems for integrodifferential equations, *Dynamics of Continuous, Discrete & Impulsive Systems, Series A* **15**(6), (2008), 815-824.

[27] M. Maejima, and C. A. Tudor, Wiener integrals with respect to the Hermite process and a non-central limit theorem, *Stochastic Analysis and Applications* **25**(5), (2007), 1043-1056.

[28] X. Mao, Stochastic Differential Equations and Applications. In *Horwood Publishing Series in Mathematics and Applications*, Chichester Horwood Publishing Limited, London,1997.

[29] P. Coupek, B. Maslowski, and M.Ondreját, L^p-valued stochastic convolution integral driven by Volterra noise, *Stochastic Dynamics* **18**(6),(2018), No. 1850048.

[30] R. Nagel, and G. Nickel, Well-Posedness for Nonautonomous Abstract Cauchy Problems. In *Evolution Equations, Semigroups and Functional Analysis*, Birkhäuser, Basel, 2002..

[31] F. Russo, and C. A. Tudor, On bifractional Brownian motion, *Stochastic Processes and Their Applications* **116**(5), (2006), 830-856.

[32] R. Sakthivel, P. Revathi, Y. Ren, and G. Shen, Retarded stochastic differential equations with infinite delay driven by Rosenblatt process, *Stochastic Analysis & Applications* **36**(2), (2018), 304-323.

[33] S. Saravanakumar, and P. Balasubramaniam, On impulsive Hilfer fractional stochastic differential system driven by Rosenblatt process, *Stochastic Analysis* **37**(6), (2019), 955-76.

[34] T. Sathiyaraj, J. Wang, and D. O'Regan, Controllability of stochastic nonlinear oscil-lating delay systems driven by the Rosenblatt distribution, *Proceedings of the Royal Society of Edinburgh* **151**(1), (2021), 217-239 .

[35] G. Shen, and Y. Ren, Neutral stochastic partial differential equations with delay driven by Rosenblatt process in a hilbert space, *Journal of the Korean Statistical Soc*iety **44**(1), (2015),123-133.

[36] G. Shen, R. Sakthivel, Y. Ren, and M. Li, Controllability and stability of fractional stochastic functional systems driven by Rosenblatt process, *Collectanea Mathematica* **71**(1), (2020), 63-82.

[37] R. A. Yahia, A. Benchaabane, and H. Zeghdoudi, Existence results for second-order neutral stochastic equations driven by Rosenblatt process, *Methods of Functional Analysis and Topology* **27**(4), (2021), 384–400.

[38] R. Shevchenko, and C. A. Tudor, Parameter estimation for the Rosenblatt Ornstein-Uhlenbeck process with periodic mean, *Statistical Inference for Stochastic Processes* **23**(1), (2020), 227-247.

[39] M. Slaoui, and C. A. Tudor, Limit behavior of the Rosenblatt Ornstein-Uhlenbeck process with respect to the Hurst index, *Theory of Probability and Mathematical Statistics* **98**(1), (2018), DOI:10.1090/tpms/1070

[40] C. A. Tudor, Analysis of the Rosenblatt process, *ESAIM: Probability and Statistics* **12**, (2008), 230-257.

[41] B. S. Vadivoo, R. Raja, R. A. Seadawy, and G. Rajchakit, Nonlinear integro-differential equations with small unknown parameters: a controllability analysis problem, *Mathematics and Computers in Simulation* **155**, (2019), 15-26.

[42] Z. Yan, and X. Jia, Approximate controllability of impulsive fractional stochastic partial neutral integrodifferential inclusions with infinite delay, *IMA Journal of Mathematical Control & Inf*ormation **34**(3), (2017), 779-820.

Article history: Submitted February, 21, 2023; Revised September, 02, 2023; Accepted October, 20, 2023.