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Multidimensional BDSDEs With Poisson Jumps Under Non-Lipschitz Coefficients

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Abstract. *This paper is devoted to the solution of a multidimensional backward doubly stochastic differential equation with jumps. Existence and uniqueness of the solution to this equation is proved by using stochastic analysis, assuming non-Lipschitz conditions for the coefficients, and via construction of an appropriate approximation sequence.*

Key words: Backward Doubly Stochastic Differential Equations, Poisson Random Measure, Stochastic Lipschitz Condition, Gronwall's Lemma, Bihari's Inequality.

AMS Subject Classifications: 60H05, 60G57

1. Introduction

Backward stochastic differential equations (BSDEs) have originally been introduced by Pardoux and Peng in [6]. After that, in order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations, they introduced in [7] a new class of BSDEs driven by two Brownian motions called backward doubly stochastic differential equations (BDSDEs in short). They have also proved the existence and uniqueness of solution under uniformly Lipschitzian conditions and gave probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs in short) where the coefficients are smooth enough.

Taking into account the previous results of Pardoux and Peng, several works have attempted to relax the Lipschitz condition and the growth of the generator function. For instance, Bahlali et al. established in [3] the existence and uniqueness of a solution for the BDSDE with superlinear growth generators. Furthermore, Z.Wu and F. Zhang [10] obtained an existence and uniqueness result for BDSDEs under locally monotone assumptions.

Recently, papers on BDSDEs with jumps started to appear in the literature. For instance, Zhu and Shi [11] studied BDSDEs driven by Brownian motion and for a Poisson process with non-Lipschitz coefficients on a random time interval. Aman [2], Aman with Owo [1], and Ren et al. [8] studied a special reflected generalized BDSDEs (driven by Teugel's martingales associated with Lévy process) with means of the penalization method and the fixed-point theorem. Existence and uniqueness of the solution to the BDSDE with jumps, in the forward integral, have been studied by Sow in [9] for the case of non-Lipschitz coefficients.

Inspired by the work of Sow [9], we consider here multidimensional BDSDEs with Poisson jumps (BDSDEP in short) under non-Lipschitz coefficients. Thus, this work can be seen as an extension of [9].

Our paper is organized as follows: in Section 2 we give some definitions and preliminary results. Using these statements we prove existence and uniqueness of the solution of BDSDEP in Section 3.

2. Definitions and Preliminary Results

Let Ω be a non-empty set, $\mathcal{F}$ a σ −algebra of subsets of Ω and $\mathbf{P}$ a probability measure defined on $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, \ \mathbf{P})$ defines a probability space, which is assumed to be complete. We assume given three mutually independent processes :

- a d −dimensional Brownian motion $(W_t)_{t\geq 0}$,
- ℓ −dimensional Brownian motion $(B_t)_{t\geq 0}$,
- a random Poisson measure μ on $E \times \mathbf{R}_+$ with compensator $\nu(dt, de) = \lambda(de)dt$, where the space $E = \mathbf{R}^\ell \setminus \{0\}$ is equipped with its Borel field $\mathcal{E}$ such that

$$\{\widetilde{\mu}([0,t] \times A) = (\mu - \nu)[0,t] \times A\}$$

is a martingale for any $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. λ is a σ −finite measure on $\mathcal{E}$ and satisfies

$$\int_E (1 \wedge |e|^2)\lambda(de) < \infty.$$

Let $0 < T < +\infty$ be a non-random horizon time. We consider the family $(\mathcal{F}_t)_{0\leq t\leq T}$ given by

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{F}_t^\mu, \ \ 0 \leq t \leq T,$$

where for any process $\{\eta_t\}_{t\geq 0}$, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s, \ s \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. Here $\mathcal{N}$ denotes the class of $\mathbf{P}$ −null sets of $\mathcal{F}$ and we assume that $\mathcal{F}_T = \mathcal{F}$. Note that the family $(\mathcal{F}_t)_{0\leq t\leq T}$ does not constitute a classical filtration.

For an integer $Q \geq 1$, $|\ .\ |$ and $\langle . \rangle$ stand for the Euclidian norm and the inner product in $\mathbf{R}^Q$. Moreover, for every random process $(a(t))_{t\geq 0}$ with positive values, such that $a(t)$ is $\mathcal{F}_t$-measurable for any $t \geq 0$, we define an increasing process $(A(t))_{t\geq 0}$ by setting $A(t) = \int_0^t a^2(s)ds$. Then for every $\beta > 0$, we consider the following sets (where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}$):

- $\mathcal{L}^2_{[0,T]}(\mathbf{R}^Q)$ the space of $\mathcal{F}_t$ −adapted càdlàg processes

$$\Psi : [0,T] \times \Omega \to \mathbf{R}^Q, \ \|\Psi\|^2_{\mathcal{L}^2(\mathbf{R}^Q)} = \mathbf{E}\left(\sup_{0\leq t\leq T} e^{\beta A(t)}|\Psi_t|^2\right) < +\infty.$$

- $\mathcal{M}^2_{[0,T]}(\mathbf{R}^Q)$ and $\mathcal{M}^{2,a}_{[0,T]}(\mathbf{R}^Q)$ the space of $\mathcal{F}_t$ −progressively measurable processes $\Psi : [0,T] \times \Omega \to \mathbf{R}^Q$ and which satisfy respectively

$$\|\Psi\|^2_{\mathcal{M}^2(\mathbf{R}^Q)} = \mathbf{E}\left(\int_0^T e^{\beta A(t)}|\Psi_t|^2\,dt\right) < +\infty,$$
$$\|a\Psi\|^2_{\mathcal{M}^{2,a}(\mathbf{R}^Q)} = \mathbf{E}\left(\int_0^T e^{\beta A(t)}a^2(t)|\Psi_t|^2\,dt\right) < +\infty.$$

• $\mathcal{L}^2(\beta,\lambda,[0,T],\mathbf{R}^Q)$ the space of mappings $U : \Omega\times[0,T]\times E \longrightarrow \mathbf{R}^Q$ which are

$\mathcal{P}\otimes\mathcal{E}$-measurable s.t.

$$\|U\|^2_{\mathcal{L}^2(\mathbf{R}^Q)} = \mathbf{E}\left(\int_0^T\int_E e^{\beta A(t)}|U_t(e)|^2\lambda(de)dt\right) < +\infty,$$

where $\mathcal{P}\otimes\mathcal{E}$ denotes the σ –algebra of $\mathcal{F}_t$ –predictable sets of $\Omega\times[0,T]$.

Notice that the space

$$\mathcal{A}^2_{[0,T]}(\beta,a,\mathbf{R}^Q) = \mathcal{M}^{2,a}_{[0,T]}(\mathbf{R}^Q)\times\mathcal{M}^2_{[0,T]}(\mathbf{R}^{Q\times d})\times\mathcal{L}^2(\beta,\lambda,[0,T],\mathbf{R}^Q)$$

endowed with the norm

$$\|(Y,Z,U)\|^2_{\mathcal{A}^2(\mathbf{R}^Q)} = \|aY\|^2_{\mathcal{M}^{2,a}(\mathbf{R}^Q)} + \|Z\|^2_{\mathcal{M}^2(\mathbf{R}^{Q\times d})} + \|U\|_{\mathcal{L}^2(\mathbf{R}^Q)}$$

is a Banach space as is the space

$$\mathcal{B}^2_{[0,T]}(\beta,a,\mathbf{R}^Q) = (\mathcal{M}^{2,a}_{[0,T]}(\mathbf{R}^Q)\cap\mathcal{L}^2_{[0,T]}(\mathbf{R}^Q))\ \times\mathcal{M}^2_{[0,T]}(\mathbf{R}^{Q\times d})\times\mathcal{L}^2(\beta,\lambda,[0,T],\mathbf{R}^Q)$$

with the norm

$$\|(Y,Z,U)\|^2_{\mathcal{B}^2(\mathbf{R}^Q)} = \|Y\|_{\mathcal{L}^2(\mathbf{R}^Q)} + \|aY\|^2_{\mathcal{M}^{2,a}(\mathbf{R}^Q)} + \|Z\|^2_{\mathcal{M}^2(\mathbf{R}^{Q\times d})} + \|U\|^2_{\mathcal{L}^2(\mathbf{R}^Q)}.$$

Let $f : \Omega\times[0,T]\times\mathbf{R}^k\times\mathbf{R}^{k\times d}\times\mathbf{R}^k\to\mathbf{R}^k$, $g : \Omega\times[0,T]\times\mathbf{R}^k\times\mathbf{R}^{k\times d}\times\mathbf{R}^k\to\mathbf{R}^{k\times\ell}$ and ξ a $\mathbf{R}^k$ –valued random vector, we are interested in the BDSDEP with parameters (ξ,f,g,T):

$$Y_t = \xi + \int_t^T f(r,\Theta_r)dr + \int_t^T g(r,\Theta_r)dB_r - \int_t^T Z_r dW_r - \int_t^T\int_E U_r(e)\widetilde{\mu}(dr,de),\quad 0\le t\le T, \tag{1}$$

where Θ_r stands for the triple (Y_r,Z_r,U_r).

Now, let us update the notion of solution to Eq.(1).

Definition 2.1. A triple of processes $(Y_t,Z_t,U_t)_{0\le t\le T}$ is called a solution to Eq.(1), if $(Y_t,Z_t,U_t)\in\mathcal{B}^2_{[0,T]}(\beta,a,\mathbf{R}^k)$ and it satisfies Eq.(1).

Then we shall recall the following result which will be useful in the sequel.

Lemma 2.1. *Let* $X\in L^2_{[0,T]}(R^k), \vartheta\in M^2_{[0,T]}(R^k), \zeta\in M^2_{[0,T]}(R^{k\times\ell}), \pi\in M^2_{[0,T]}(R^{k\times d})$ *and* $\phi\in L^2(\beta,\lambda,[0,T],R^k)$ *be such that*

$$X_t = X_0 + \int_0^t\vartheta_r dr + \int_0^t\zeta_r dB_r + \int_0^t\pi_r dW_r + \int_0^t\int_E\phi_r(e)\widetilde{\mu}(dr,de),\quad 0\le t\le T.$$

Then we have for any $\beta > 0$ *and* $0\le t\le T$,

$$
\begin{aligned}
(i)\quad |X_t|^2 &= |X_0|^2 + 2\int_0^t \langle X_r, \vartheta_r\rangle dr + 2\int_0^t \langle X_r, \zeta_r dB_r\rangle + 2\int_0^t \langle X_r, \pi_r dW_r\rangle \\
&+ 2\int_0^t \int_E \langle X_{r-}, \phi_r(e)\widetilde{\mu}(de,dr)\rangle - \int_0^t |\zeta_r|^2 dr + \int_0^t |\pi_r|^2 dr \\
&+ \int_0^t \int_E |\phi_r(e)|^2 \lambda(de)dr + \sum_{0<r\leq t} (\triangle X_r)^2,
\end{aligned}
$$

where $\triangle X_r = X_r - X_{r-}$ the size of the left jump of X at r,

$$
\begin{aligned}
(ii)\quad & e^{\beta A(t)}|X_t|^2 + \beta\int_t^T e^{\beta A(r)} a^2(r)|X_r|^2 + \int_t^T e^{\beta A(r)}|\pi_r|^2 dr + \int_t^T \int_E e^{\beta A(r)}|\phi_r(e)|^2 \lambda(de)dr \\
&+ \sum_{t<r\leq T} e^{\beta A(r)}(\triangle X_r)^2 = e^{\beta A(T)}|X_T|^2 + 2\int_t^T e^{\beta A(r)}\langle X_r, \vartheta_r\rangle dr - 2\int_t^T e^{\beta A(r)}\langle X_r, \pi_r dW_r\rangle \\
&- 2\int_t^T \int_E e^{\beta A(r)}\langle X_r, \phi_r(e)\widetilde{\mu}(de,dr)\rangle + 2\int_t^T e^{\beta A(r)}\langle X_r, \zeta_r dB_r\rangle + \int_t^T e^{\beta A(r)}|\zeta_r|^2 dr.
\end{aligned}
$$

Proof. The point (**i**) is an adaptation of the argument developed in Pardoux and Peng [7]. To prove the second point, we consider the process $Z_t = e^{A(t)}|X_t|^2$, $0 \leq t \leq T$. Applying the integration by parts formula to $(Z_t)_{0\leq t\leq T}$, we have

$$
dZ_t = e^{A(t)} d(|X_t|^2) + a^2(t)e^{A(t)}|X_t|^2 dt, \ \ 0 \leq t \leq T.
$$

Using (**i**), (**ii**) follows by integration.

3. Existence and Uniqueness

Let us introduce the following assumptions. We say that the parameters ξ, f and g satisfy assumptions (**H**), for some $\beta > 0$ if the following hold (where we define for $0 \leq t \leq T$, $h(t,0) = h(t,0,0,0)$, for $h \in \{f,g\}$ to ease the reading).

• (**H1**): f and g are jointly measurable and four integrable functions $\{\gamma(t)\}$, $\{\vartheta(t)\}$, $\{\sigma(t)\}$, $\{\upsilon(t)\}$, $[0,T] \to \mathbf{R}^+$ such that for $t \geq 0$, $\gamma(t), \vartheta(t), \sigma(t)$, and $\upsilon(t)$ are $\mathcal{F}_t^W$–measurable and $(y,y') \in (\mathbf{R}^k)^2$, $(z,z') \in (\mathbf{R}^{k\times d})^2$ and $(u,u') \in (\mathbf{R}^k)^2$,

$$
\begin{aligned}
&|f(t,y,z,u) - f(t,y',z',u')| \leq \gamma(t)\rho(|y-y'|) + \vartheta(t)|z-z'| + \sigma(t)|u-u'|, \\
&|g(t,y,z,u) - g(t,y',z',u')|^2 \leq \gamma(t)|y-y'|\rho(|y-y'|) + \upsilon(t)(|z-z'|^2 + |u-u'|^2),
\end{aligned}
$$

where ρ is a concave and nondecreasing function with $\rho(0) = 0$ and $\int_{0+} \frac{du}{\rho(u)} = +\infty$.

• (**H2**): There exists a constant $0 < \alpha < 1$ such that $\upsilon(t) \leq \alpha$, for all $t > 0$.

• (**H3**): For all $0 \leq t \leq T$, $a^2(t) = \gamma(t) + \vartheta^2(t) + \sigma^2(t) > 0$.

• (**H4**): ξ is a $\mathcal{F}$–measurable random vector, such that $\mathbf{E}[e^{\beta A(T)}|\xi|^2] < +\infty$.

• (**H5**): For any $(t,y,z,u) \in [0,T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \times \mathbf{R}^k$, $f(t,y,z,u)$ and $g(t,y,z,u)$ are $\mathcal{F}_t$–measurable and the integrability condition holds

$$
\mathbf{E}\left[\int_0^T e^{\beta A(r)} \frac{|f(r,0)|^2}{a^2(r)} dr + \int_0^T e^{\beta A(r)}|g(r,0)|^2 dr\right] < +\infty.
$$

We recall the following results, which will be useful in the proof of uniqueness.

Lemma 3.1 (Gronwall). *Assume given* $T \geq 0, K \geq 0$ *and* $\Phi, \Psi : [0,T] \to R^+$ *such that* $\int_0^T \Psi(s)ds < \infty$. *If*

$$\forall 0 \leq t \leq T, \qquad \Phi(t) \leq K + \int_0^t \Psi(s)\Phi(s)ds < \infty,$$

then we have

$$\forall 0 \leq t \leq T, \qquad \Phi(t) \leq K \exp\left(\int_0^t \Psi(s)ds\right).$$

Lemma 3.2 (Bihari's inequality). *Let* $T > 0$, u, v *continuous non-negative functions on* $[0,T]$ *and a continuous function* $H \in S$. *If*

$$u(t) \leq \int_0^t v(r)H(u(r))dr, \quad 0 \leq t \leq T,$$

then $u(t) = 0$ *for all* $0 \leq t \leq T$.

As in [4], Theorem 1, we consider the sequence $(f_n)_{n \geq 1}$ defined by

$$f^n = (f_1^n, f_2^n, \ldots, f_k^n)$$

where

$$\forall i = 1, \ldots k, \quad f_i^n(\omega, t, y, z, u) = \inf_{v \in \mathbf{R}^k} \{f_i(\omega, t, v, z, u) + (n + A)\gamma(t)|v - y|\}.$$

We have the following result, whose proof is omitted since it is an adaptation of step 1 of Theorem 1 in [4].

Lemma 3.3. *The sequence of* $\mathcal{F}_t$ *– progressively measurable function* f^n *satisfies:*
(i) $\forall (y,z,u) \in R^k \times R^{k \times d} \times R^k, \quad |f^n(\omega,t,y,z,u) - f(\omega,t,y,z,u)| \leq k\gamma(t)\rho\left(\frac{2A}{n}\right)$
(ii) $\forall (y,y') \in (R^k)^2,\ (z,z') \in (R^{k \times d})^2$ *and* $(u,u') \in (R^k)^2$,

$$|f^n(\omega,t,y,z,u) - f^n(\omega,t,y',z',u')| \leq k(n+A)[\gamma(t)(|y-y'|) + \vartheta(t)|z-z'| + \sigma(t)|u-u'|],$$
$$|f^n(\omega,t,y,z,u) - f^n(\omega,t,y',z',u''|) + k\vartheta(t)|z-z'| + k\sigma(t)|u-u'|.$$

(iii) The integrabilty condition holds $E\left[\int_0^T e^{\beta A(r)} \frac{|f^n(r,0)|^2}{a^2(r)} dr\right] < +\infty$.

We also have the following result which is important for the proof of our main result. Its proof can be seen in [4], Lemma 1.

Lemma 3.4. *If* $\rho(u)$ *is a concave and nondecreasing function with* $\rho(0) = 0$ *and* $\int_{0^+} \frac{du}{\rho(u)} = +\infty$ *there exists a concave nondecreasing function* $\phi(u)$ *with* $\phi(0) = 0$ *and* $\int_{0^+} \frac{du}{\rho(u)} = +\infty$ *moreover* $a\sqrt{u}\,\rho(\sqrt{u}) \leq \phi(u) \leq a\sqrt{u}\,\rho(\sqrt{u})$, *where* $a > 0$ *is a constant.*

Proposition 3.1. *Assume that assumptions (**H1**)-(**H5**) are true and let $(Y_t, Z_t, U_t)_{0\leq t\leq T}$ be a solution to the MBSDEP (1). Then for a large enough β, there exists a constant $c > 0$ depending only on β and T such that, for any $0 \leq t \leq T$,*

$$\mathbf{E}\left(\sup_{t\leq r\leq T} e^{\beta A(r)}|Y_r|^2\right) + \mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)|Y_r|^2dr + \mathbf{E}\int_t^T e^{\beta A(r)}|Z_r|^2dr$$
$$+\mathbf{E}\int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr$$
$$\leq C(\beta,T)\bigg[\mathbf{E}e^{\beta A(T)}|\xi|^2 + \mathbf{E}\int_t^T e^{\beta A(r)}\tfrac{|f(r,0)|^2}{a^2(r)}dr + \mathbf{E}\int_t^T e^{\beta A(r)}|g(r,0)|^2dr$$
$$+\mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)\phi(|Y_r|^2)dr.\bigg] \tag{2}$$

Proof. Applying Lemma 2.1, we deduce from (1)

$$e^{\beta A(t)}|Y_t|^2 + \beta\int_t^T e^{\beta A(r)}a^2(r)|Y_r|^2dr + \int_t^T e^{\beta A(r)}|Z_r|^2dr + \int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr$$
$$+\textstyle\sum_{t<r\leq T} e^{\beta A(r)}(\triangle Y_r)^2 = e^{\beta A(T)}|\xi|^2 + 2\displaystyle\int_t^T e^{\beta A(r)}Y_rf(r,\Theta_r)dr$$
$$+2\int_t^T e^{\beta A(r)}Y_rg(r,\Theta_r)dB_r - 2\int_t^T e^{\beta A(r)}Y_rZ_rdW_r$$
$$-2\int_t^T\int_E e^{\beta A(r)}Y_{r-}U_r(e)\widetilde{\mu}(dr,de) + \int_t^T e^{\beta A(r)}|g(r,\Theta_r)|^2dr, \quad 0\leq t\leq T. \tag{3}$$

By the assumptions (**H1**), (**H2**) and (**H3**) and the inequality $2ab \leq \theta a^2 + b^2/\theta$ for any $\theta > 0$, we have

$$\begin{aligned}2Y_rf(r,\Theta_r) &= 2Y_r[f(r,\Theta_r) - f(r,Y_r,0,0) + f(r,Y_r,0,0) - f(r,0) + f(r,0)]\\ &\leq 2|Y_r||f(r,\Theta_r - f(r,Y_r,0,0)| + 2|Y_r||f(r,Y_r,0,0) - f(r,0)| + 2|Y_r||f(r,0)|\\ &\leq (\tfrac{4}{1-\alpha} + \tfrac{\beta}{2})a^2(r)|Y_r|^2 + \tfrac{1-\alpha}{2}[|Z_r|^2 + |U_r|^2] + a^2(r)\phi(|Y_r|^2) + \frac{2}{\beta a^2 r}|f(r,0)|^2\end{aligned} \tag{4}$$

$$\begin{aligned}|g(r,\Theta_r)|^2 &= |g(r,\Theta_r) - g(r,0) + g(r,0)|^2\\ &\leq (1+\tfrac{1}{\varepsilon})a^2(r)\phi(|Y_r|^2) + \alpha(1+\tfrac{1}{\varepsilon})[|Z_r|^2 + |U_r|^2] + (1+\varepsilon)|g(r,0)|^2.\end{aligned} \tag{5}$$

Take expectation on both sides of (3), by (4) and (5), we have

$$\begin{aligned}\mathbf{E}[e^{\beta A(t)}|Y_t|^2]e^{\beta A(t)}|Y_t|^2 &+ \left(\tfrac{4}{1-\alpha} + \tfrac{\beta}{2}\right)\mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)|Y_r|^2dr + \left(\tfrac{1-\alpha}{2} - \tfrac{\alpha}{\varepsilon}\right)\mathbf{E}\int_t^T e^{\beta A(r)}|z_r|^2dr\\ &+ \left(\tfrac{1-\alpha}{2} - \tfrac{\alpha}{\varepsilon}\right)\int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr\\ &\leq \mathbf{E}[e^{\beta A(T)}|\xi|^2] + \tfrac{2}{\beta}\mathbf{E}\int_t^T e^{\beta A(r)}\tfrac{|f(r,0)|^2}{a^2(s)}dr + (1+\varepsilon)\mathbf{E}\int_t^T e^{\beta A(r)}|g(r,0)|^2dr\\ &+ (1+\tfrac{1}{\varepsilon})\mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)\phi(|Y_r|^2)dr.\end{aligned} \tag{6}$$

Let $\varepsilon > \frac{2\alpha}{1-\alpha}$ and β be large enough, there exists a nonnegative constant $C(\beta,T)$ such that

$$\begin{aligned}\mathbf{E}[e^{\beta A(t)}|Y_t|^2]e^{\beta A(t)}|Y_t|^2 &+ \mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)|Y_r|^2dr + \mathbf{E}\int_t^T e^{\beta A(r)}|z_r|^2dr\\ &+ \int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr \ \leq C(\beta,T)\ X_t^T,\end{aligned} \tag{7}$$

where

$$X_t^T = \mathbf{E}[e^{\beta A(T)}|\xi|^2] + \mathbf{E}\int_t^T e^{\beta A(r)}\frac{|f(r,0)|^2}{a^2(r)}dr + \mathbf{E}\int_t^T e^{\beta A(r)}|g(r,0)|^2dr$$
$$+\mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)\phi(|Y_r|^2)dr.$$

Futhermore using (3) once again and (7), we have

$$\begin{aligned}\mathbf{E}\left(\sup_{t\le r\le T} e^{\beta A(r)}|Y_r|^2\right) &+ \mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)|Y_r|^2dr + \mathbf{E}\int_t^T e^{\beta A(r)}|z_r|^2dr\\ &+\int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr\\ &\le C(\beta,T)\ X_t^T + 2\mathbf{E}\sup_{t\le s\le T}\left|\int_s^T e^{\beta A(r)}\langle Y_r, Z_r dW_r\rangle\right|\\ &+2\mathbf{E}\sup_{t\le s\le T}\left|\int_s^T\int_E e^{\beta A(r)}\langle Y_{r^-}, U_r(e)\widetilde{\mu}(dr,de)\rangle\right|\\ &+2\mathbf{E}\sup_{t\le s\le T}\left|\int_s^T e^{e^{\beta A(r)}}\langle Y_r, g(r)dB_r\rangle\right|. \end{aligned}\tag{8}$$

By the Burkhölder-Davis-Gundy inequality and the standard estimates $2ab \le a^2\varepsilon + b^2/\varepsilon$, for any $\varepsilon > 0$, there exists $c > 0$ such that for any $\delta > 0$,

$$2\mathbf{E}\sup_{t\le s\le T}\left|\int_s^T e^{\beta A(r)}\langle Y_r, Z_r dW_r\rangle\right| \le \delta\mathbf{E}\left(\sup_{t\le r\le T} e^{\beta A(r)}|Y_r|^2\right) + \frac{c^2}{\delta}\mathbf{E}\int_t^T e^{\beta A(r)}|Z_r|^2dr,$$
$$2\mathbf{E}\sup_{t\le s\le T}\left|\int_s^T e^{\beta A(r)}\langle Y_r, g(r,\Theta_r)dB_r\rangle\right| \le \delta\mathbf{E}\left(\sup_{t\le r\le T} e^{\beta A(r)}|Y_r|^2\right) + \frac{c^2}{\delta}\mathbf{E}\int_t^T e^{\beta A(r)}|g(r,\Theta_r)|^2dr. \tag{9}$$

Similarly, for the discontinuous martingale, we have

$$\begin{aligned}2\mathbf{E}\sup_{t\le s\le T}\left|\int_s^T\int_E e^{\beta A(r)}\langle Y_{r^-}, U_r(e)\widetilde{\mu}(de,dr)\rangle\right| &\le \delta\mathbf{E}\left(\sup_{t\le r\le T} e^{\beta A(r)}|Y_r|^2\right)\\ &+\frac{c^2}{\delta}\mathbf{E}\left(\int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr\right).\end{aligned}\tag{10}$$

From (8), (9) and (10), we deduce that

$$\begin{aligned}\mathbf{E}\left(\sup_{t\le s\le T} e^{\beta A(r)}|Y_r|^2\right) &+ \mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)|Y_r|^2dr + \mathbf{E}\int_t^T e^{\beta A(r)}|z_r|^2dr\\ &+\int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr\\ &\le C(\beta,T)\ X_t^T + 3\delta\mathbf{E}\left(\sup_{t\le s\le T} e^{\beta A(r)}|Y_r|^2\right) + \frac{c^2}{\delta}\mathbf{E}\int_t^T e^{\beta A(r)}|Z_r|^2dr\\ &+\frac{c^2}{\delta}\mathbf{E}\left(\int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr\right) + \frac{c^2}{\delta}\mathbf{E}\int_t^T e^{\beta A(r)}|g(r,\Theta_r)|^2dr. \end{aligned}\tag{11}$$

Moreover, (11) and (5) lead to

$$\mathbf{E}\left(\sup_{t\le r\le T} e^{\beta A(r)}|Y_r|^2\right)+\mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)|Y_r|^2dr+\mathbf{E}\int_t^T e^{\beta A(r)}|z_r|^2dr$$
$$+\mathbf{E}\int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr$$
$$\le C(\beta,T)\ X_t^T+3\delta\mathbf{E}\left(\sup_{t\le r\le T} e^{\beta A(r)}|Y_r|^2\right)+\frac{c^2}{\delta}(1+\varepsilon)\mathbf{E}\int_t^T e^{\beta A(r)}|g(r,0)|^2dr$$
$$+\left(\frac{c^2}{\delta}\alpha(1+\frac{1}{\varepsilon})+\frac{c^2}{\delta}\right)\mathbf{E}\left[\int_t^T e^{\beta A(r)}|Z_r|^2dr+\int_t^T\int_E e^{\beta A(r)}|U_r(e)|^2\lambda(de)dr\right]$$
$$+\frac{c^2}{\delta}\left(1+\frac{1}{\varepsilon}\right)\mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)\phi(|Y_r|^2)dr. \tag{12}$$

Finally, from (7) and (12) for $\delta<1/3$, the required result follows .■

Theorem 3.1. *Assume that assumptions (H1)-(H4) are true. Then the BDSDEP (1) has a unique solution* $(Y,Z,U)\in B^2_{[0,T]}(\beta,a,R^k)$.

Proof. *(Existence)* We consider the sequence $(f^n)_{n\ge1}$ defined in Lemma 3.3 and the following MBSDEP with parameters (ξ,f^n,g):

$$Y_t^n=\xi+\int_t^T f^n(r,\Theta_r^n)\,dr+\int_t^T g(r,\Theta_r^n)\,dB_r-\int_t^T Z_r^n dW_r-\int_t^T\int_E U_r^n(e)\tilde{\mu}(dr,de). \tag{13}$$

Since f^n satisfies Lipschitz condition, it follows from Proposition 2.4 in [5], that the sequence $\Theta^n=(Y^n,Z^n,U^n)$ is well defined. In addition for any $n\ge1$ and $m\ge1$, define for $\delta\in\{Y,Z,U\}$, $\widehat{\delta}^{n,m}=\delta^n-\delta^m$. Then the triplet $(\widehat{Y}^{n,m},\widehat{Z}^{n,m},\widehat{U}^{n,m})$ solves the following MBSDEP

$$\widehat{Y}_t^{n,m}=\int_t^T\bar{f}^{n,m}(r)\,dr+\int_t^T\bar{g}^{n,m}(r)\,dB_r-\int_t^T\widehat{Z}_r^{n,m}dW_r-\int_t^T\int_E\widehat{U}_r^{n,m}(e)\tilde{\mu}(dr,de), \tag{14}$$

where $\forall\ t\le r\le T$,

$$\bar{f}^{n,m}(r)=f^n(r,Y_r^n,Z_r^n,U_r^n)-f^m(r,Y_r^m,Z_r^m,U_r^n),$$
$$\bar{g}^{n,m}(r)=g^n(r,Y_r^n,Z_r^n,U_r^n)-g^m(r,Y_r^m,Z_r^m,U_r^n).$$

Applying Lemma 2.1, we deduce from (14) that

$$e^{\beta A(r)}|\widehat{Y}_t^{n,m}|^2+\beta\int_t^T e^{\beta A(r)}a^2(r)|\widehat{Y}_r^{n,m}|^2dr+\int_t^T e^{\beta A(r)}|\widehat{Z}_r^{n,m}|^2dr$$
$$+\int_t^T\int_E e^{\beta A(r)}|\widehat{U}_r^{n,m}(e)|^2\lambda(de)dr\quad+\sum_{t<r\le T}e^{\beta A(r)}(\triangle\widehat{Y}_r^{n,m})^2$$
$$=2\int_t^T e^{\beta A(r)}\widehat{Y}_r^{n,m}\bar{f}^{n,m}(r)dr\quad+2\int_t^T e^{\beta A(r)}\widehat{Y}_r^{n,m}\bar{g}^{n,m}(r)dB_r-2\int_t^T e^{\beta A(r)}\widehat{Y}_r^{n,m}\widehat{Z}_r^{n,m}dW_r$$
$$-2\int_t^T\int_E e^{\beta A(r)}\widehat{Y}_r^{n,m}\widehat{U}_r^{n,m}(e)\tilde{\mu}(dr,de)+\int_t^T e^{\beta A(r)}|\bar{g}^{n,m}(r)|^2dr,\quad 0\le t\le T. \tag{15}$$

Taking suitable β, by Lemma 3.4 and Burkholder-Davis-Gundy inequality, there exists a nonnegative constant C_3 such that

$$\mathbf{E}\left[\sup_{t\le s\le T} e^{\beta A(s)}|\widehat{Y}_s^{n,m}|^2\right]+\mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)|\widehat{Y}_r^{n,m}|^2dr+\mathbf{E}\int_t^T e^{\beta A(r)}|\widehat{Z}_r^{n,m}|^2dr$$
$$+\mathbf{E}\int_t^T\int_E e^{\beta A(r)}|\widehat{U}_r^{n,m}(e)|^2\lambda(de)dr, \tag{16}$$

$$\leq C_3 \ \mathbf{E}\int_t^T e^{\beta A(r)} a^2(r)\psi(|\widehat{Y}_r^{n,m}|^2)dr$$
$$\leq C_3 \ \int_t^T a^2(r)\psi\left(\mathbf{E} \sup_{t\leq s\leq T} e^{\beta A(s)}|\widehat{Y}_s^{n,m}|^2\right)dr. \tag{17}$$

where $\psi(u)$ is a concave and nondecreasing function with $\psi(0) = 0$ and $\int_{0^+} \frac{du}{\psi(u)} = +\infty$, $ku\rho(u) \leq \psi(u) \leq 2ku\rho(u),\ k > 0$.

From Bihari inequality and (16) we deduce that the sequence $(\Theta^n) = (Y^n, Z^n, U^n)$ is a Cauchy sequence in the space $\mathcal{B}^2_{[0,T]}(\beta, a, R^k)$. Letting $n \to \infty$ in (18) in uniform convergence in probability, implies that the triple (Y, Z, U) is solution to (1). This completes the proof of existence.

(Uniqueness) Let (Y, Z, U) and $(\widehat{Y}, \widehat{Z}, \widehat{U})$ be two solutions to Eq.(1). We define $\bar{Y} = Y - \widehat{Y},\ \bar{Z} = Z - \widehat{Z},\ \bar{U} = U - \widehat{U}$ and $0 \leq r \leq T$

$$\bar{f}(r) = f(r, Y_r, Z_r, U_r) - f(r, \widehat{Y_r}, \widehat{Z_r}, \widehat{U}_r),$$
$$\bar{g}(r) = g(r, Y_r, Z_r, U_r) - g(r, \widehat{Y_r}, \widehat{Z_r}, \widehat{U}_r).$$

Thus the triple $(\bar{Y}, \bar{Z}, \bar{U})$ solves the following BDSDEP

$$\bar{Y}_t = \int_t^T \bar{f}(r)dr + \int_t^T \bar{g}(r)dB_r - \int_t^T \bar{Z}_r dW_r - \int_t^T\int_E \bar{U}_r(e)\widetilde{\mu}(dr, de), \quad 0 \leq t \leq T. \tag{18}$$

Applying (**ii**) in Lemma 2.1 to Eq.(18), we have

$$e^{\beta A(t)}|\bar{Y}_t|^2 + \int_t^T e^{\beta A(r)} a^2(r)|\bar{Y}_r|^2 dr + \int_t^T e^{\beta A(r)}|\bar{Z}_r|^2 dr + \int_t^T\int_E e^{\beta A(r)}|\bar{U}_r(e)|^2\lambda(de)dr$$
$$+ \sum_{t<r\leq T} e^{\beta A(r)}(\triangle\bar{Y}_r)^2 = 2\int_t^T e^{\beta A(r)}\langle\bar{Y}_r, \bar{f}(r)\rangle dr + \int_t^T e^{\beta A(r)}|\bar{g}(r)|^2 dr$$
$$- 2\int_t^T e^{\beta A(r)}\bar{Y}_r\bar{Z}_r dW_r + 2\int_t^T e^{\beta A(r)}\bar{Y}_r\bar{g}(r)dB_r - 2\int_t^T\int_E e^{\beta A(r)}\bar{Y}_r\bar{U}_r(e)\widetilde{\mu}(dr, de)\ . \tag{19}$$

Taking expectation on both sides of (19) from (**H1**), (**H2**) and inequality $2ab \leq \theta a^2 + b^2/\theta$ for any $\theta > 0$, we have

$$\mathbf{E}\left(e^{\beta A(t)}|\bar{Y}_t|^2 + \int_t^T e^{\beta A(r)} a^2(r)|\bar{Y}_r|^2 dr + \int_t^T e^{\beta A(r)}|\bar{Z}_r|^2 dr + \int_t^T\int_E e^{\beta A(r)}|\bar{U}_r(e)|^2\lambda(de)dr\right)$$
$$= 2\mathbf{E}\int_t^T e^{\beta A(r)}\langle\bar{Y}_r, \bar{f}(r)\rangle dr + \mathbf{E}\int_t^T e^{\beta A(r)}|\bar{g}(r)|^2 dr$$
$$\leq 3\mathbf{E}\int_t^T e^{\beta A(r)} a^2(r)\rho(|\bar{Y}_r|^2)dr + \frac{\beta}{4}\mathbf{E}\int_t^T e^{\beta A(r)} a^2(r)|\bar{Y}_r|^2 dr$$
$$\left(\frac{4}{\beta} + \alpha\right)\mathbf{E}\left(\int_t^T e^{\beta A(r)}|\bar{Z}_r|^2 dr + \int_t^T\int_E e^{\beta A(r)}|\bar{U}_r(e)|^2 \imath(de)dr\right). \tag{20}$$

By Lemma 3.4, and taking β large enough, there exists a nonnegative constant C_1 such that

$$\mathbf{E}\int_t^T e^{\beta A(r)} a^2(r)|\bar{Y}_r|^2 dr + \mathbf{E}\int_t^T e^{\beta A(r)}|\bar{Z}_r|^2 dr + \mathbf{E}\int_t^T\int_E e^{\beta A(r)}|\bar{U}_r(e)|^2\lambda(de)dr$$
$$\leq C_1\ \mathbf{E}\int_t^T e^{\beta A(r)} a^2(r)\phi(|\bar{Y}_r|^2)dr. \tag{21}$$

By (19), (21) and Burkholder-Davis-Gundy inequality, there exists a nonnegative constant C_2 such that

$$
\begin{aligned}
\mathbf{E}\left[\sup_{t\le s\le T} e^{\beta A(s)}|\bar{Y}_s|^2\right] &+ \mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)|\bar{Y}_r|^2dr + \mathbf{E}\int_t^T e^{\beta A(r)}|\bar{Z}_r|^2dr \\
&+ \mathbf{E}\int_t^T\int_E e^{\beta A(r)}|\bar{U}_r(e)|^2\ell(de)dr \\
&\le C_2\ \mathbf{E}\int_t^T e^{\beta A(r)}a^2(r)\phi(|\bar{Y}_r|^2)dr \\
&\le C_2\ \int_t^T a^2(r)\phi\left(\mathbf{E}\sup_{r\le s\le T} e^{\beta A(s)}|\bar{Y}_s|^2\right)dr.
\end{aligned}
$$

Then by Bihari inequality, we obtain $\bar{Y}=0$, $\bar{Z}=0$ and $\bar{U}=0$, a.s. Here the *uniqueness* proof completes. ■

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