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Large Deviations for One-Dimensional Perturbed Stochastic Differential Equations

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Abstract. *We prove a large deviation principle for the solutions of perturbed stochastic differential equations (SDEs) incorporating a past extremal process. On one hand,we consider the presence of the double perturbation given by the past maximum and the past minimum process and establish a large deviation principle by checking a uniform Freidlin-Wentzell estimates. On the other hand, we give, through a contraction principle, a large deviation principle for SDEs having the maximum process in the drift.*

Key words: Large Deviations, Perturbed SDEs, Uniform Freidlin-Wentzell Estimates, Skorokhod's Lemma.

AMS Subject Classifications: 60F10, 60H10

1. Introduction

After the earlier work of Le Gall and Yor [17, 18], the theory of perturbed SDEs has a significant headway. Several authors contributed to obtain existence and uniqueness in weakening the assumptions required or by improving the considered equation (see among others Carmona et al. [6, 7], Perman and Werner [21], Chaumont and Doney [8], Doney and Zhang [12], Davis [10] and Chaumont et al. [9]). In the same idea Belfadli et al. [2] proved existence and pathwise uniqueness results for four different types of SDEs perturbed by the past maximum process and / or the past minimum process and / or the local time process. A doubly perturbed Brownian motion behaves as a Brownian motion between its minimum and maximum, and is perturbed at its extrema. Henceforth, the perturbed SDEs admit many fields of applications such as finance, physics, biology, mechanics, engineering.

Otherwise, the application of the large deviation principle (LDP) to SDEs was first studied by Freidlin and Wentzell [15]. Many authors contributed to develop the original work in [15] (Doss and Priouret [13], Millet et al. [19]) and obtained a LDP for finite dimensional problems by using the uniform Freidlin-Wentzell estimates. It should be noted that Gao and Jiang [16] extended the work of [15] to stochastic differential equations driven by G-Brownian motion

(G-SDEs). The authors used discrete time approximation to establish LDP for G-SDEs. Their proof avoids the stopping time technique and the Girsanov transformation. Their main tool is exponential moment estimates.
Moreover, it is worth noting that several authors have established a LDP for a different class of stochastic differential equations and many of the original assumptions made in [15] have been significantly relaxed [4, 25, 26]. For example, Budhiraja et al. [4] used an approach based on variational representations introduced in [5] to establish LDP for infinite dimensional stochastic partial differential equation driven by a Poisson random measure. A benefit of this approach is the fact that in infinite dimensional model, the solution may have little spatial regularity, and then the classical approximation of Freidlin-Wentzell become intractable. In two dimensions, Wang [25] proved a LDP for 2D stochastic Navier-Stokes equations with Lévy noises by using an appropriate weak convergence method to overcome the difficulty due to the jump. An advantage of the weak convergence approach, introduced by Dupuis and Ellis [14], is that one can show large deviations properties under weaker conditions than the usual proofs based on Freidlin-Wentzell approximations. Also it works well for infinite dimensional stochastic dynamical systems.

Some works in the literature motivated our present study. Mohammed and Zhang [20] established a LDP for small noise perturbed family of stochastic systems with memory. Further, Bo and Zhang [3] considered the LDP for perturbed reflected diffusion processes, through checking uniform Freidlin-Wentzell estimates. The key in their paper is the estimates concerning the perturbed term. In our present paper the news reside on one hand in the presence of the double perturbation and on the other hand on the drift of the maximum process.

The rest of the paper is organized as follows. In Section 2, we propose some useful notions and results in large deviation principle. We prove our main results in Section 3 and Section 4 is devoted to the conclusion and some possible extensions.

2. Large Deviation Principle

Consider $(\Omega, \mathcal{F}, \mathsf{P})$ a probability space and $(\mathcal{F}_t)_{0\le t\le T}$ a filtration of $\mathcal{F}$ satisfying the conditions of right continuity and $\mathsf{P}-$ completeness. Let $(X^\varepsilon)_{\varepsilon>0}$ be a family of random variables defined on this space and taking values in E a complete separable metric space (Polish space).

Definition 2.1 (Rate function). A function $I : E \to [0,+\infty]$ is called a rate function if I is lower semicontinuous. A rate function I is called good rate function if for each $\gamma \in [0,\infty)$, the level set $K_\gamma = \{x \in E : I(x) \le \gamma\}$ is compact in E.

Definition 2.2 (Large Deviation Principle). Let I be a rate function on E. We say that the family $(X^\varepsilon)_{\varepsilon>0}$ satisfies the large deviation principle with the rate function I if the following condition holds:
1. For each closed subset F of E

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathsf{P}(X^\varepsilon \in F) \le -\inf_{x\in F} I(x).$$

2. For each open subset O of E

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathsf{P}(X^\varepsilon \in O) \geq -\inf_{x\in O} I(x).$$

The following result by Schilder enhances a LDP for a family of probability measure induced by standard Brownian motion.

Theorem 2.1 [11, Theorem 5.2.3] (Schilder). *Let $(\mu^\varepsilon)_{\varepsilon>0}$ be the family of probability measures induced by $W^\varepsilon(.) = \sqrt{\varepsilon}\, W(.)$ on C_0 where $W(.)$ denotes the Brownian motion in R. Then $(\mu^\varepsilon)_{\varepsilon>0}$ satisfies on C_0 a LDP with good rate function*

$$\tilde{I}(h) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{h}(s)|^2 ds, & \text{if } h \in H_1, \\ +\infty, & \text{otherwise.} \end{cases}$$

H_1 denotes the space of all absolutely continuous functions with square integrable derivatives equipped with the norm $\|h\|_{\mathsf{H}_1} = \left(\int_0^1 |\dot{h}(s)|^2 ds\right)^{\frac{1}{2}}$ and C_0 is the space of continuous functions $f : [0,1] \to \mathbb{R}$ such that $f(0) = 0$ equipped with the supremum norm.

In the sequel (Section 3) the next result will plays an important role. It states how a LDP on one space can be transferred to a large deviation principle on another space via a continuous function.

Theorem 2.2 [11, Theorem 4.2.1] (Contraction principle). *Let X and Y be Haussdorff topological spaces and let $(\mu_\varepsilon)_{\varepsilon>0}$ be a family of probability measures on X that satisfies the large deviation principle with the rate function $I : X \to [0,+\infty]$. Let $F : X \to Y$ be a continuous function. Then, $(\mu_\varepsilon \circ F^{-1})_{\varepsilon>0}$ satisfies the LDP with the rate function*

$$\tilde{I}(y) = \inf\left\{ I(x) \,:\, x \in X,\, F(x) = y \right\}.$$

Our method in the Subsection 3.1 will be based on a classical result due to Azencott [1] (see Theorem III 2.4), which can be stated as a proposition that follows.

Proposition 2.1 [3, Proposition 1.1]. *Let $(E_i, d_i)\,(i = 1,2)$ be two polish spaces and $\Phi_i^\varepsilon : \Omega \to E_i,\ \varepsilon > 0$ be two families of random variables. Assume that*

1. $(\Phi_1^\varepsilon, \varepsilon > 0)$ satisfies a LDP with the rate function $I_1 : E_1 \to [0,+\infty]$.

2. There exists a map $K : \{I_1 < +\infty\} \to E_2$ such that for every $a < \infty$, $K : \{I_1 \leq a\} \to E_2$ is continuous.

3. For any $R, \delta, a > 0$, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that, for any $h \in E_1$ satisfying $I_1(h) \leq a$ and $\varepsilon \leq \varepsilon_0$,

$$\mathsf{P}\Big(d_2(\Phi_2^\varepsilon, K(h)) \geq \delta,\ d_1(\Phi_1^\varepsilon, h) \leq \rho\Big) \leq \exp\Big(-\frac{R}{\varepsilon}\Big). \tag{1}$$

Then, $(\Phi_2^\varepsilon, \varepsilon > 0)$ satisfies a LDP with the rate function

$$I_2(g) = \inf\{I_1(h) \,:\, K(h) = g\}.$$

The inequality (1) is also known as the uniform Freidlin-Wenzell estimate.

3. Perturbed Stochastic Differential Equations

Let Ω be the set of continuous functions from $\mathbb{R}^+$ into $\mathbb{R}$, P the Wiener measure on Ω, $(W_t)_{t\geq 0}$ the process of coordinate maps from Ω into $\mathbb{R}$, $\mathcal{F} = \sigma\{W_t, t \geq 0\}$, $(\mathcal{F}_t)_{t\geq 0}$ the completion of the natural filtration of W with the P-null sets of $\mathcal{F}$. Therefore $(W_t)_{t\geq 0}$ is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$.
Consider the following equations

$$X_t = \xi + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds + \alpha \sup_{0\leq s\leq t} X_s + \beta \inf_{0\leq s\leq t} X_s, \tag{2}$$

$$Y_t = y + W_t + \int_0^t v(Y_s, \sup_{0\leq u\leq s} Y_u) ds, \tag{3}$$

where $\alpha, \beta \in \mathbb{R}$.

We set the following hypothesis on α, β and on the functions b, σ, v:

(**H1**) $\alpha < 1, \quad \beta < 1, \quad \frac{|\alpha\beta|}{(1-\alpha)(1-\beta)} < 1.$

(**H2**) For all $t \geq 0$ and $x, x' \in \mathbb{R}$, there exists a constant $L > 0$ such that:

$$|\sigma(t,x) - \sigma(t,x')| \leq L|x - x'| \qquad \text{and} \qquad |b(t,x) - b(t,x')| \leq L|x - x'|.$$

(**H3**) The random variable ξ is such that $\mathsf{E}(|\xi|^2) < \infty$.

(**H4**) b and σ are measurable and bounded functions on $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$.

(**H5**) v is measurable and bounded function on $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

(**H6**) For each $x \in \mathbb{R}$, $y \mapsto v(x,y)$ is strictly increasing on the set $\{y \in \mathbb{R} : y \geq x\}$.

(**H7**) For each $x, y, x', y' \in \mathbb{R}$, there exists a constant $L > 0$ such that

$$|v(x,y) - v(x',y)| \leq L|x - x'| \qquad |v(x,y) - v(x,y')| \leq L|y - y'|.$$

Under the conditions (**H1**) – (**H3**), Belfadli et al. [2] proved existence of a unique continuous $(\mathcal{F}_t)_{t\geq 0}$ adapted process $(X_t, t \geq 0)$ of the equation (2). Also they established, under (**H5**) and (**H6**), that equation (3) has a strong solution which is pathwise unique.

From now on consider the following perturbed SDEs:

$$X_t^\varepsilon = x + \sqrt{\varepsilon} \int_0^t \sigma(s, X_s^\varepsilon) dW_s + \int_0^t b(s, X_s^\varepsilon) ds + \alpha \sup_{0\leq s\leq t} X_s^\varepsilon + \beta \inf_{0\leq s\leq t} X_s^\varepsilon, \quad t \in [0,1]. \tag{4}$$

$$Y_t^\varepsilon = y + \sqrt{\varepsilon}\, W_t + \int_0^t v(Y_s^\varepsilon, \sup_{0\leq u\leq s} Y_u^\varepsilon) ds, \quad t \in [0,1]. \tag{5}$$

The aim of this paper is to prove a large deviation principle for the law of X^ε (and Y^ε) on the space of continuous functions equipped with uniform topology. Our study generalizes the results of Bo and Zhang [3] on the large deviations for perturbed diffusion processes. The news in this paper reside on one hand in the presence of the double perturbation $(\sup_{0\leq s\leq t} X_s^\varepsilon, \inf_{0\leq s\leq t} X_s^\varepsilon)$ and on the other hand on the drift involving the maximum process.

3.1. **LDP for a peturbed diffusion process**

In this section we will establish a large deviation principle for the law of X^ε solution of the equation (4).
Let H denote the Cameron-Martin space, i.e

$$\mathsf{H} = \left\{h : [0,1] \to \mathbb{R};\ h(t) = \int_0^t \dot{h}(s)ds, \int_0^1 |\dot{h}(s)|^2 ds < \infty\right\},$$

equipped with the norm $\|h\|_{\mathsf{H}} = \left(\int_0^1 |\dot{h}(s)|^2 ds\right)^{\frac{1}{2}}$ and $\{\mu^\varepsilon, \varepsilon > 0\}$ be the probability measure induced by $X^\varepsilon_.$ on $\mathsf{C}_x([0,1],\mathbb{R})$, the space of continuous functions $f : [0,1] \to \mathbb{R}$ such that $f(0) = x$, equipped with the supremum norm topology. For $h \in \mathsf{C}_0([0,1],\mathbb{R})$, define $\tilde{I} : \mathsf{C}_0([0,1],\mathbb{R}) \to [0,+\infty]$ by

$$\tilde{I}(h) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{h}(s)|^2 ds, & \text{if } h \in \mathsf{H}, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

The well-know Schilder theorem states that the law $\tilde{\mu}_\varepsilon$ of $\{\sqrt{\varepsilon}\, W_t, t \in [0,1]\}$ satisfies a LDP on $\mathsf{C}_0([0,1],\mathbb{R})$ with the rate function $\tilde{I}(.)$.
Let $F(h)$ be the unique solution of the following deterministic perturbed equation: for all $h \in \mathsf{H}$ and $t \in [0,1]$,

$$F(h)(t) = x + \int_0^t \sigma(s,F(h)(s))\dot{h}(s)ds + \int_0^t b(s,F(h)(s))ds + \alpha \sup_{0\le s\le t} F(h)(s) + \beta \inf_{0\le s\le t} F(h)(s). \quad (6)$$

The existence of the unique solution $F(h)$ is obtained by Picard iteration and deduced on Belfadli et al. [2, Theorem 5.1].

The main result in this section now follows.

Theorem 3.1. *The family* $\{\mu_\varepsilon, \varepsilon > 0\}$ *verifies a LDP with the rate function:*

$$I_x(g) = \inf_{\{h\in\mathsf{H}:\ F(h)=g\}} \tilde{I}(h), \qquad \text{for } g \in \mathsf{C}_x([0,1],\mathbb{R}),$$

where the inf *over the empty set is taken to be* ∞*.*

Before proving this theorem we need the results that follow.

Lemma 3.1 [24, Lemma 2.19]. *Let* $Z_t = \int_s^t C(u)dW_u + \int_s^t D(u)du$ *be an Itô process, where* $0 \le s < t < \infty$ *and* $C, D : [0,\infty) \times \Omega \to R$ *are* $(F_t)_{t\ge 0}$ *progressively measurable random processes. If* $|C(.)| \le M_1$ *and* $|D(.)| \le M_2$*, then for* $T > s$ *and* $R > 0$ *satisfying* $R > M_2(T - s)$*, we have*

$$\mathsf{P}\left(\sup_{s\le t\le T}|Z_t|\ge R\right)\le \exp\left(-\frac{(R-M_2(T-s))^2}{2M_1^2(T-s)}\right).$$

Proposition 3.1. *For all $a>0,\ R>0,\ \delta>0$ there exist $\rho>0,\ \varepsilon_0>0$ such that*

$$\mathsf{P}\left(\sup_{0\le t\le 1}|X_t^\varepsilon - F(h)(t)|\ge \delta,\ \ \sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t - h(t)|\le \rho\right)\le \exp\left(-\frac{R}{\varepsilon}\right), \tag{7}$$

for all $\varepsilon\le\varepsilon_0$, all $h\in C_0([0,1],\mathbb{R})$ satisfying $\tilde{I}(h)\le a$.

Proof. Let $\widehat{X}_t^\varepsilon = \sup_{0\le s\le t} X_s^\varepsilon$ and $\widetilde{X}_t^\varepsilon = \inf_{0\le s\le t} X_s^\varepsilon$. By Skorokhod's lemma (see Revuz and Yor [22]) one can easily see that:

$$\begin{cases} (1-\alpha)\widehat{X}_t^\varepsilon = \sup_{0\le s\le t}\left(x+\sqrt{\varepsilon}\int_0^s \sigma(u,X_u^\varepsilon)dW_u + \int_0^s b(u,X_u^\varepsilon)du + \beta\widetilde{X}_s^\varepsilon\right) \\ (\beta-1)\widetilde{X}_t^\varepsilon = \sup_{0\le s\le t}\left(-x-\sqrt{\varepsilon}\int_0^s \sigma(u,X_u^\varepsilon)dW_u - \int_0^s b(u,X_u^\varepsilon)du - \alpha\widehat{X}_s^\varepsilon\right) \end{cases} \tag{8}$$

Combining the equations (4) and (6), we deduce on one hand that:

$$\sup_{0\le s\le t}|X_s^\varepsilon - F(h)(s)| \le a^\varepsilon(t)+b^\varepsilon(t)+|\alpha|\sup_{0\le s\le t}\left|\widehat{X}_s^\varepsilon - \widehat{F(h)}(s)\right| + |\beta|\sup_{0\le s\le t}\left|\widetilde{X}_s^\varepsilon-\widetilde{F(h)}(s)\right|, \tag{9}$$

on the other hand, by (8) we have:

$$\begin{cases} \sup_{0\le s\le t}\left|\widehat{X}_s^\varepsilon - \widehat{F(h)}(s)\right| \le \frac{1}{1-\alpha}(a^\varepsilon(t)+b^\varepsilon(t)) + \frac{|\beta|}{1-\alpha}\sup_{0\le s\le t}\left|\widetilde{X}_s^\varepsilon - \widetilde{F(h)}(s)\right| \\ \sup_{0\le s\le t}\left|\widetilde{X}_s^\varepsilon - \widetilde{F(h)}(s)\right| \le \frac{1}{1-\beta}(a^\varepsilon(t)+b^\varepsilon(t)) + \frac{|\alpha|}{1-\beta}\sup_{0\le s\le t}\left|\widehat{X}_s^\varepsilon - \widehat{F(h)}(s)\right| \end{cases}, \tag{10}$$

where $\widehat{F(h)}(t) = \sup_{0\le s\le t}F(h)(s)$, $\widetilde{F(h)}(t) = \inf_{0\le s\le t}F(h)(s)$,
$a^\varepsilon(t) = \sup_{0\le s\le t}|\int_0^u \sqrt{\varepsilon}\,\sigma(s,X_s^\varepsilon)dW_s - \sigma(s,F(h)(s))\dot{h}(s)ds|$
and $b^\varepsilon(t) = \int_0^t |b(s,X_S^\varepsilon) - b(s,F(h)(s))|ds$, for $t\in[0,1]$.
Thus

$$\left(1-\frac{|\alpha\beta|}{(1-\alpha)(1-\beta)}\right)\sup_{0\le s\le t}\left|\widehat{X}_s^\varepsilon - \widehat{F(h)}(s)\right| \le \frac{1}{1-\alpha}\left(1+\frac{|\beta|}{1-\beta}\right)(a^\varepsilon(t)+b^\varepsilon(t)). \tag{11}$$

Combining (9), (10) and (11) we have,

$$\sup_{0\le s\le t}|X_S^\varepsilon - F(h)(s)| \le \left(1+\frac{|\beta|}{1-\beta}\right)(a^\varepsilon(t)+b^\varepsilon(t)) \times\left(1+\frac{|\alpha|(1-\beta)}{(1-\alpha)(1-\beta)-|\alpha\beta|} + \frac{|\alpha\beta|}{(1-\alpha)(1-\beta)-|\alpha\beta|}\right). \tag{12}$$

In the following we set

$$c(\alpha,\beta) = \left(1+\frac{|\beta|}{1-\beta}\right)\left(1+\frac{|\alpha|(1-\beta)+|\alpha\beta|}{(1-\alpha)(1-\beta)-|\alpha\beta|}\right).$$

We can dominate a^ε by,

$$a^\varepsilon(t) \le \sup_{0\le u\le t}\left|\int_0^u \sigma(s,X_s^\varepsilon)(\sqrt{\varepsilon}\,dW_s - \dot{h}(s)ds)\right| + \sup_{0\le u\le t}\left|\int_0^u (\sigma(s,X_s^\varepsilon)-\sigma(s,F(h)(s)))\dot{h}(s)ds\right|.$$

Putting together and by the condition (**H2**) on b and σ, the inequality (12) becomes

$$\begin{aligned}\sup_{0\le s\le t}|X_s^\varepsilon - F(h)(s)| \le c(\alpha,\beta)\sup_{0\le s\le t}\left|\int_0^s \sigma(u,X_u^\varepsilon)(\sqrt{\varepsilon}\,dW_u - \dot{h}(u)du)\right| \\ + c(\alpha,\beta)\int_0^t \sup_{0\le u\le s}|X_u^\varepsilon - F(h)(u)|(1+|\dot{h}(s)|)ds.\end{aligned}$$

By the Gronwall lemma, this yields that

$$\sup_{0\le t\le 1}|X_t^\varepsilon - F(h)(t)| \le c(\alpha,\beta)\sup_{0\le t\le 1}\left|\int_0^t \sigma(s,X_s^\varepsilon)(\sqrt{\varepsilon}\,dW_s - \dot{h}(s)ds)\right| \times \exp\left(c(\alpha,\beta)\int_0^1 (1+|\dot{h}(s)|)ds\right).$$

Hence, by Hölder inequality, we obtain

$$\sup_{0\le t\le 1}|X_t^\varepsilon - F(h)(t)| \le c\sup_{0\le t\le 1}\left|\int_0^t \sigma(s,X_s^\varepsilon)(\sqrt{\varepsilon}\,dW_s - \dot{h}(s)ds)\right|,$$

where the constant is

$$c = c(\alpha,\beta)\exp[c(\alpha,\beta)L(1+\|h\|_{\mathsf{H}})].$$

Thus to prove (7), it suffices to prove that, for any $R,\delta,a>0$, there exist $\rho>0$ and $\varepsilon_0>0$ such that, for any $h\in \mathsf{C}_0([0,1],\mathbb{R})$ satisfying $\tilde{I}(h)\le a$ and $\varepsilon\le\varepsilon_0$,

$$\mathsf{P}\left(\sup_{0\le t\le 1}\left|\int_0^t \sigma(s,X_s^\varepsilon)(\sqrt{\varepsilon}\,dW_s - \dot{h}(s)ds)\right| \ge \delta,\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t - h(t)|\le\rho\right) \le \exp\left(-\frac{R}{\varepsilon}\right). \tag{13}$$

For $\varepsilon>0$, define a probability measure P^ε on Ω by

$$d\mathsf{P}^\varepsilon = Z_\varepsilon d\mathsf{P} = \exp\left(\frac{1}{\sqrt{\varepsilon}}\int_0^1 \dot{h}(s)dW_s - \frac{1}{2\varepsilon}\int_0^1|\dot{h}(s)|^2ds\right)d\mathsf{P}.$$

Then, by Girsanov theorem, $\{W_t^\varepsilon = W_t - \frac{1}{\sqrt{\varepsilon}}h(t),\ t\in[0,1]\}$ is a standard Brownian motion on $(\Omega,\mathcal{F},\mathsf{P}^\varepsilon)$. If we let

$$\begin{aligned}A^\varepsilon &= \{\sup_{0\le t\le 1}\left|\int_0^t \sigma(s,X_s^\varepsilon)(\sqrt{\varepsilon}\,dW_s - \dot{h}(s)ds)\right| \ge \delta,\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t - h(t)|\le\rho\} \\ &= \{\sup_{0\le t\le 1}\left|\int_0^t \sigma(s,X_s^\varepsilon)\sqrt{\varepsilon}\,dW_s^\varepsilon\right| \ge \delta,\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t^\varepsilon|\le\rho\},\end{aligned}$$

then by the Hölder inequality,

$$\mathsf{P}(A^\varepsilon) = \int_\Omega Z_\varepsilon^{-1}\mathbf{1}_{A^\varepsilon}(w)\mathsf{P}^\varepsilon(dw) \le \left(\int_\Omega Z_\varepsilon^{-2}(w)\mathsf{P}^\varepsilon(dw)\right)^{\frac{1}{2}}(\mathsf{P}^\varepsilon(A^\varepsilon))^{\frac{1}{2}}.$$

Note that $\{W_t^\varepsilon,\ t\in[0,1]\}$ is a standard Brownian motion under P^ε, then

$$\begin{aligned}\int_\Omega Z_\varepsilon^{-2}(w)\mathsf{P}^\varepsilon(dw) &= \mathsf{E}_{\mathsf{P}^\varepsilon}\left[\exp\left(-\frac{2}{\sqrt{\varepsilon}}\int_0^1 \dot{h}(s)dW_s + \frac{1}{\varepsilon}\int_0^1|\dot{h}(s)|^2ds\right)\right]\\ &= \mathsf{E}_{\mathsf{P}^\varepsilon}\left[\exp\left(-\frac{2}{\sqrt{\varepsilon}}\int_0^1 \dot{h}(s)dW_s^\varepsilon - \frac{1}{\varepsilon}\int_0^1|\dot{h}(s)|^2ds\right)\right]\\ &= \mathsf{E}_{\mathsf{P}^\varepsilon}\left[\exp\left(-\frac{2}{\sqrt{\varepsilon}}\int_0^1 \dot{h}(s)dW_s^\varepsilon - \frac{2}{\varepsilon}\int_0^1|\dot{h}(s)|^2ds\right)\right]\\ &\quad\times\exp\left(\frac{1}{\varepsilon}\int_0^1|\dot{h}(s)|^2ds\right)\\ &= \exp\left(\frac{1}{\varepsilon}\|h\|_{\mathsf{H}}^2\right).\end{aligned}$$

Therefore, if $\tilde{I}(h)\le a$, then

$$\mathsf{P}(A^\varepsilon) \le \exp(\frac{a}{\varepsilon})(\mathsf{P}^\varepsilon(A^\varepsilon))^{\frac{1}{2}}. \tag{14}$$

Note that under the probability measure P^ε, X^ε satisfies the following stochastic differential equation:

$$U_t^\varepsilon = x + \sqrt{\varepsilon}\int_0^t \sigma(s,U_s^\varepsilon)dW_s + \int_0^t(b(s,U_s^\varepsilon)+\sigma(s,U_s^\varepsilon)\dot{h}(s))ds + \alpha\sup_{0\le s\le t}U_s^\varepsilon + \beta\inf_{0\le s\le t}U_s^\varepsilon.$$

Therefore,

$$\begin{aligned}\mathsf{P}^\varepsilon(A^\varepsilon) &= \mathsf{P}^\varepsilon\left(\sup_{0\le t\le 1}\left|\int_0^t\sigma(s,X_s^\varepsilon)\sqrt{\varepsilon}\,dW_s^\varepsilon\right|\ge\delta,\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t^\varepsilon|\le\rho\right)\\ &= \mathsf{P}\left(\sup_{0\le t\le 1}\left|\int_0^t\sigma(s,U_s^\varepsilon)\sqrt{\varepsilon}\,dW_s\right|\ge\delta,\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t|\le\rho\right).\end{aligned}$$

Thus, in view of (14), the proof of (13) is reduced to show that, for any $R,\delta,a>0$, there exist $\rho>0$ and $\varepsilon_0>0$ such that, for any $h\in\mathsf{C}_0([0,1],\mathbb{R})$ satisfying $\tilde{I}(h)\le a$ and $\varepsilon\le\varepsilon_0$,

$$\mathsf{P}\left(\sup_{0\le t\le 1}\left|\int_0^t\sigma(s,U_s^\varepsilon)\sqrt{\varepsilon}\,dW_s\right|\ge\delta,\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t|\le\rho\right)\le\exp\left(-\frac{R}{\varepsilon}\right). \tag{15}$$

For $n\in\mathsf{N}^*$ fixed, set $t_k=\frac{k}{n}$ for $k\in\{0,1,2,\dots,n\}$. Define

$$U_t^{\varepsilon,n} = U_{t_k}^\varepsilon,\quad \text{if } t_k\le t<t_{k+1},\ k\in\{0,1,2,\dots,n-1\}.$$

Then, for $\delta_1>0$,

$$\begin{aligned}
A^\varepsilon \subset\ & \{\sup_{0\le t\le 1}\left|\int_0^t (\sigma(s,U_s^\varepsilon)-\sigma(s,U_s^{\varepsilon,n}))\sqrt{\varepsilon}\,dW_s\right| \ge \frac{\delta}{2},\ \sup_{0\le t\le 1}|U_t^\varepsilon - U_t^{\varepsilon,n}|\le \delta_1\} \\
& \cup\ \{\sup_{0\le t\le 1}|U_t^\varepsilon - U_t^{\varepsilon,n}|> \delta_1\} \\
& \cup\ \{\sup_{0\le t\le 1}\left|\int_0^t \sigma(s,U_s^{\varepsilon,n})\sqrt{\varepsilon}\,dW_s\right| \ge \frac{\delta}{2},\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t|\le \rho\} \\
:= \quad & B^\varepsilon \cup C^\varepsilon \cup D^\varepsilon.
\end{aligned}$$

On the set $\{\sup_{0\le t\le 1} |U_t^\varepsilon - U_t^{\varepsilon,n}|\le \delta_1\}$,

$$\sup_{0\le s\le 1}\varepsilon|\sigma(s,U_s^\varepsilon)-\sigma(s,U_s^{\varepsilon,n})|^2 \le L^2\varepsilon\delta_1^2.$$

By Lemma 3.1, it follows that

$$\mathsf{P}(B^\varepsilon) \le \exp\Big(-\frac{\delta^2}{8L^2\varepsilon\delta_1^2}\Big) \le \exp\Big(-\frac{R}{\varepsilon}\Big), \tag{16}$$

if $\delta_1 \le \delta/2L\sqrt{2R}$.

On the other hand, on the set $\{\sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t|\le \rho\}$, for any $t\in[0,1]$,

$$\begin{aligned}
\left|\int_0^t \sigma(s,U_s^{\varepsilon,n})\sqrt{\varepsilon}\,dW_s\right| &= \sqrt{\varepsilon}\left|\sum_{j=0}^{n-1}\sigma(t_j,U_{t_j}^\varepsilon)(W_{t_{j+1}\wedge t}-W_{t_j\wedge t})\right| \\
&\le 2\sup_{0\le t\le 1}|\sqrt{\varepsilon}\,W_t| \sum_{j\in\{i\in\{0,\dots,n-1\}:t_i\le t\}} |\sigma(t_j,U_{t_j}^\varepsilon)| \\
&\le 2nM\rho,
\end{aligned}$$

where $M>0$ is a common bound of b and σ. Therefore, if $\rho<\delta/4nM$, then

$$D^\varepsilon = \emptyset.$$

To treat C^ε, we note that for $t\in[t_k,t_{k+1}),\ k\in\{0,1,\dots,n-1\}$,

$$\begin{aligned}
|U_t^\varepsilon - U_t^{\varepsilon,n}| \le & \left|\int_{t_k}^t \sigma(s,U_s^\varepsilon)\sqrt{\varepsilon}\,dW_s + \int_{t_k}^t (b(s,U_s^\varepsilon)+\sigma(s,U_s^\varepsilon)\dot{h}(s))ds\right| \\
& + \alpha\left|\sup_{0\le s\le t}U_s^\varepsilon - \sup_{0\le s\le t_k}U_s^\varepsilon\right| + \beta\left|\inf_{0\le s\le t}U_s^\varepsilon - \inf_{0\le s\le t_k}U_s^\varepsilon\right|.
\end{aligned}$$

Similar calculations to the estimate of $\sup_{0\le t\le 1}|X_t^\varepsilon - F(h)(t)|$ (where U_t^ε plays the role of X_t^ε and $U_t^{\varepsilon,n}$ the role of $F(h)(t)$) give that:

$$\sup_{t_k\le t<t_{k+1}}|U_t^\varepsilon - U_t^{\varepsilon,n}|\le c(\alpha,\beta)\sup_{t_k\le t<t_{k+1}}\left|\int_{t_k}^t \sigma(s,U_s^\varepsilon)\sqrt{\varepsilon}\,dW_s + \int_{t_k}^t (b(s,U_s^\varepsilon)+\sigma(s,U_s^\varepsilon)\dot{h}(s))ds\right|. \tag{17}$$

Since

$$\sup_{t_k\le s<t_{k+1}} \varepsilon|\sigma(s,U_s^\varepsilon)|^2 \le \varepsilon M^2, \quad \sup_{t_k\le s<t_{k+1}} |b(s,U_s^\varepsilon)|\le M$$

and

$$\sup_{t_k\le t<t_{k+1}} \left|\int_{t_k}^t \sigma(s,U_s^\varepsilon)\dot{h}(s)ds\right| \le M\sqrt{\frac{2a}{n}},$$

let $n_1 := [2Mc(\alpha,\beta)/\delta_1]+1$ and $n_2 := [8c(\alpha,\beta)^2M^2a/\delta_1^2]+1$, it follow from (17) and Lemma 3.1 that for $n \ge n_1 \vee n_2$,

$$\begin{aligned} \mathsf{P}\Big(\sup_{t_k\le t<t_{k+1}} |U_t^\varepsilon - U_t^{\varepsilon,n}|> \delta_1\Big) &\le \mathsf{P}\Big(\sup_{t_k\le t<t_{k+1}} \Big|\int_{t_k}^t \sigma(s,U_s^\varepsilon)\sqrt{\varepsilon}\,dW_s + \int_{t_k}^t b(s,U_s^\varepsilon)ds\Big| > \frac{\delta_1}{2c(\alpha,\beta)}\Big) \\ &\quad + \mathsf{P}\Big(\sup_{t_k\le t<t_{k+1}} \Big|\int_{t_k}^t \sigma(s,U_s^\varepsilon)\dot{h}(s)ds\Big| > \frac{\delta_1}{2c(\alpha,\beta)}\Big) \\ &\le \exp\Big(-n\frac{\delta_1^2}{8c(\alpha,\beta)^2M^2\varepsilon}\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathsf{P}(C^\varepsilon) &= \mathsf{P}\Big(\sup_{k\in\{0,1,\dots,n-1\}} \sup_{t_k\le t<t_{k+1}} |U_t^\varepsilon - U_t^{\varepsilon,n}|> \delta_1\Big) \\ &\le \sum_{k=0}^{n-1} \mathsf{P}\Big(\sup_{t_k\le t<t_{k+1}} |U_t^\varepsilon - U_t^{\varepsilon,n}|> \delta_1\Big) \\ &\le n\exp\Big(-n\frac{\delta_1^2}{8c(\alpha,\beta)^2M^2\varepsilon}\Big). \end{aligned}$$

Now, given $R>0$, choose first $\delta_1>0$ such that (16) holds. Then, choose n large enough so that

$$n\exp\Big(-n\frac{\delta_1^2}{8c(\alpha,\beta)^2M^2\varepsilon}\Big) \le \exp\Big(-\frac{R}{\varepsilon}\Big).$$

Finally, for the fixed $n\in\mathsf{N}$, there exists $\rho>0$ such that $D^\varepsilon=\emptyset$. Combining above inequalities prove (15) and consequently Proposition 3.1 is verified. ■

Proof of Theorem3.1. The proof will follow from Proposition 2.1 applied to

$$E_1 = \mathsf{C}_0([0,1],\mathbb{R}),\ E_2 = \mathsf{C}_x([0,1],\mathbb{R}),\ \Phi_1^\varepsilon(.) = \sqrt{\varepsilon}\,W_.,\ \Phi_2^\varepsilon(.) = X_.^\varepsilon,\ I_1=\tilde{I},$$
$$I_2 = I_x,\ K(.) = F(.).$$

The Schilder's theorem and Proposition 3.1 give respectively the proof of points 1. and 3. of Proposition 2.1. The main point to check is 2. of Proposition 2.1. It means to prove that the map $F:\mathsf{H}\to\mathsf{C}_x([0,1],\mathbb{R})$ is continuous on $\{h:\tilde{I}(h)\le a\}$ for $a\in[0,\infty)$. Let $h_n,h\in\{f\in\mathsf{C}_0([0,1],\mathbb{R}):\tilde{I}(f)\le a\}$ with $h_n\to h$. Since $\{h:\tilde{I}(h)\le a\}$ is weakly compact in H, we conclude that h_n also weakly converges to h in H.

Using the same calculations as at the beginning of the proof, we show that there exists a constant $C > 0$ dependent on $\alpha, \beta,$ such that:

$$\sup_{0\le s\le t}|F(h_n)(s) - F(h)(s)| \le C\sup_{0\le s\le t}\left|\int_0^s \left\{\sigma(u,F(h_n)(u))\dot{h}_n(u) - \sigma(u,F(h)(u))\dot{h}(u)\right\}du\right|$$
$$+ C\sup_{0\le s\le t}\int_0^s |b(u,F(h_n)(u)) - b(u,F(h)(u))|du.$$

By standard estimates (the constant C can change from line to line), the right hand side is less than

$$C\xi_n(t) + C\int_0^t \sup_{0\le u\le s}|F(h_n)(u) - F(h)(u)|(1+|\dot{h}_n(s)|)ds,$$

where $\xi_n(t) = \sup_{0\le s\le t}\left|\int_0^s \sigma(u,F(h)(u))(\dot{h}_n(u) - \dot{h}(u))du\right|$. Applying the Gronwall's lemma,

$$\sup_{0\le t\le 1}|F(h_n)(t) - F(h)(t)| \le C\xi_n(1)\times\exp\left(C\int_0^1(1+|\dot{h}_n(s)|)ds\right)$$
$$\le C\exp\left(C(1+\sqrt{2a})\right)\xi_n(1).$$

Since $h_n \to h$ weakly in H, for every $s \in [0,1]$,

$$\int_0^s \sigma(u,F(h)(u))(\dot{h}_n(u) - \dot{h}(u))du \to 0, \quad \text{as } n\to\infty.$$

On the other hand, it is easy to see that there exists a constant $C_a > 0$ such that

$$\left|\int_s^t \sigma(u,F(h)(u))(\dot{h}_n(u) - \dot{h}(u))du\right| \le C_a|t-s|^{1/2}.$$

Therefore, we conclude that $\xi_n(1) \to 0,$ as $n \to \infty$. Consequently,

$$\sup_{0\le t\le 1}|F(h_n)(t) - F(h)(t)| \to 0, \text{ as } n\to\infty,$$

which proves the continuity of the mapping F and consequently Theorem 3.1. ■

3.2. **LDP for SDE involving maximum process in the drift**

In this section we will establish a LDP for the solution of the stochastic differential equation (5).

Let the deterministic map $F : \mathsf{C}_0([0,1],\mathbb{R}) \longrightarrow \mathsf{C}_y([0,1],\mathbb{R})$ defined by $g = F(h)$ where g is the unique continuous solution of

$$g(t) = y + h(t) + \int_0^t v(g(s), \sup_{0\le u\le s} g(u))ds. \tag{18}$$

The existence and uniqueness of the solution of this equation follow from Belfadli et al. [2, Theorem 4.1].

We claim what follows.

Theorem 3.2. *The family* $(Y^\varepsilon)_{\varepsilon>0}$ *verifies a LDP in* $C_y([0,1],\mathbb{R})$ *with the good rate function*

$$I_y(g) = \begin{cases} \frac{1}{2}\int_0^1 |\dot g(t) - v(g(t), \sup_{0\le u\le t} g(u))|^2 dt, & g \in \mathsf{H}_y \\ \infty, & g \notin \mathsf{H}_y, \end{cases} \tag{19}$$

where H_y *is the set of functions g in H such that* $g(0) = y$.

Proof. Here we apply the contraction principle (Theorem 2.2). For this we show that F is continuous on $\mathsf{C}_0([0,1],\mathbb{R})$. Indeed for $g_1 = F(h_1)$ and $g_2 = F(h_2)$, then by (18)

$$|g_1(t) - g_2(t)| \le |h_1(t) - h_2(t)| + \int_0^t |v(g_1(s), \sup_{0\le u\le s} g_1(u)) - v(g_2(s), \sup_{0\le u\le s} g_2(u))| ds.$$

Let $\eta(t) = \int_0^t |v(g_1(s), \sup_{0\le u\le s} g_1(u)) - v(g_2(s), \sup_{0\le u\le s} g_2(u))| ds$. We have

$$\begin{aligned}\eta(t) \le & \int_0^t |v(g_1(s), \sup_{0\le u\le s} g_1(u)) - v(g_1(s), \sup_{0\le u\le s} g_2(u))| ds \\ & + \int_0^t |v(g_1(s), \sup_{0\le u\le s} g_2(u)) - v(g_2(s), \sup_{0\le u\le s} g_2(u))| ds.\end{aligned}$$

Exploiting this and the condition (**H7**), we deduce that

$$|g_1(t) - g_2(t)| \le |h_1(t) - h_2(t)| + \int_0^t L|\sup_{0\le u\le s} g_1(u) - \sup_{0\le u\le s} g_2(u)| ds + \int_0^t L|g_1(s) - g_2(s)| ds.$$

So the Gronwall lemma and standard majorations give that

$$\sup_{0\le t\le 1} |g_1(t) - g_2(t)| \le \sup_{0\le t\le 1} |h_1(t) - h_2(t)| \times \exp(2L).$$

If the functions $h_1,\ h_2 \in \mathsf{C}_0([0,1],\mathbb{R})$ are such that $\|h_1 - h_2\|_\infty \le \delta$. Then

$$\sup_{0\le t\le 1} |g_1(t) - g_2(t)| \le \delta \exp(2L)$$

and the continuity of F is established. Combining Schilder theorem and the contraction principle, it follows that $(Y^\varepsilon)_{\varepsilon>0}$ satisfies, in $\mathsf{C}_y([0,1],\mathbb{R})$, a LDP with the good rate function

$$I_y(g) = \inf_{\{h\in H:\, F(h)=g\}} \tilde I(h).$$

Thus by (**H6**) and (**H7**),

$$|\dot g(t)| \le L\int_0^t |\dot g(s)| ds + L|g(0)| + |v(0, \sup_{0\le t\le 1} g(t))| + |\dot h(t)|,$$

and consequently, by Gronwall's lemma and condition (**H5**), $h \in \mathsf{H}$ implies that

$g = F(h) \in \mathrm{H}_y$ as well, which completes the proof. ■

4. Conclusion

In this paper, we have extended the results of Bo and Zhang, in [13], to establish the results of large deviations for the solutions of perturbed SDEs involving the past extrema process. By checking the uniform Freidlin-Wentzell estimates and using the contraction principle, these results of large deviations have been obtained. These classical method give as good approximations of large deviations as the weak convergence method or the variational representation in finite dimensional model driven by a Brownian motion.

We conclude by giving some possible extensions. Let G be an absolutely continuous and increasing function such that $0 < G'(x) < 1$ and $G(0) = 0$. We can develop the same arguments as previously to show that the following SDE:

$$X_t^\varepsilon = \sqrt{\varepsilon}\int_0^t \sigma(s, X_s^\varepsilon)dW_s + \int_0^t b(s, X_s^\varepsilon)ds + G\left(\sup_{0\le s\le t} X_s^\varepsilon\right), \qquad t \in [0,1]$$

verifies a LDP. For this, it suffices to note that there exists a constant $0 < \alpha < 1$ such that the following estimate:

$$|X_t^\varepsilon - F(h)(t)| \le \left|\int_0^t \sigma(s, X_s^\varepsilon)(\sqrt{\varepsilon}\,dW_s - \dot{h}(s)ds)\right| + L\int_0^t |X_s^\varepsilon - F(h)(s)|(1 + |\dot{h}(s)|)ds$$
$$+ \alpha|\sup_{0\le s\le t} X_s^\varepsilon - \sup_{0\le s\le t} F(h)(s)|,$$

holds true, where $\alpha = \text{ess sup } G'(x_0)$, x_0 is between $\sup_{0\le s\le t} X_s^\varepsilon$ and $\sup_{0\le s\le t} F(h)(s)$ and the determinist map F is defined by $g = F(h)$, where g is the solution of the following equation :

$$g(t) = x + \int_0^t \sigma(s, g(s))\dot{h}(s)ds + \int_0^t b(s, g(s))ds + G\left(\sup_{0\le s\le t} g(s)\right).$$

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