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Factors Involving Almost Increasing Sequences in Absolute Riesz Summability

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Abstract. *Absolute summability of infinite series has been a subject of profound interest in theory and application, particularly in bounded input bounded output (BIBO) stability. In this note we report on a generalization of the* $|\overline{N}, p_n; \delta; \gamma|_q$ *summability to this same summability with pertaining factors involving almost increasing sequences.*

Key words: Riesz Mean, Absolute Summability, $|\overline{N}, p_n; \delta; \gamma|_q$ Summability, Hölder's Inequality.

AMS Subject Classifications: 26D15, 40F05, 40D15, 40G05

1. Notation & Introduction

A sequence (μ_n) is of bounded variation, $(\mu_n) \in BV$, if

$$\sum_{n=1}^{\infty} |\Delta\mu_n| = \sum_{n=1}^{\infty} |\mu_n - \mu_{n-1}| < \infty,$$

and a positive sequence (g_n) is said to be (s.t.b.) almost increasing, if $\exists$ a +ive increasing (h_n) and there are 2 +ive constants, M and N such that

$$M\, h_n \leqslant g_n \leqslant N\, h_n.$$

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Let $\sum a_n$ be an infinite series whose partial sum's, $s_n = \sum_{k=0}^{n} a_k$, sequence is (s_n) with the mean of (s_n), denoted by u_n, is

$$u_n = \sum_{k=0}^{\infty} u_{nk} s_k. \tag{1}$$

Definition 1.1. An infinite series $\sum a_n$ is absolutely summable, if $\lim_{n\to\infty} u_n = s$ and

$$\sum_{n=0}^{\infty} |u_n - u_{n-1}| < \infty. \tag{2}$$

Definition 1. 2. [1] Let the r^{th} Cesàro mean of (na_n) be denoted by t_r. If

$$\sum_{r=1}^{\infty} (1/r)|t_r|^q < \infty, \tag{3}$$

then $\sum a_n$ is s.t.b. $|C,1|_q$ summable where $q \geq 1$.

Definition 1. 3. [2] Let $\{p_s\}$ be of +ive numbers and

$$P_s = \sum_{r=0}^{s} p_r \to \infty,\ s \to \infty, (P_{-s} = p_{-s} = 0, s \geqslant 1). \tag{4}$$

If σ_s defines the $(\overline{N}, p_s)$ mean i.e.

$$\sigma_s = \frac{1}{P_s} \sum_{k=0}^{s} p_k s_k,\ P_s \neq 0,\ s \in N, \tag{5}$$

& $\lim_{s\to\infty} \sigma_s = m$, then $\sum a_s$ is s.t.b. $(\overline{N}, p_s)$ summable generated by (p_s).

Furthermore, if (σ_s) is of bounded variation with index $q \geqslant 1$, i.e.

$$\sum_{s=1}^{\infty} \left(\frac{P_s}{p_s}\right)^{q-1} |\sigma_s - \sigma_{s-1}|^q < \infty, \tag{6}$$

then $\sum a_s$ is s.t.b. $|\overline{N}, p_s|_q$ summable.

Definition 1. 4. [3] If

$$\Delta\sigma_{s-1} = -\frac{p_s}{P_s P_{s-1}} \sum_{\upsilon=1}^{s} P_{\upsilon-1} a_\upsilon,\ s \geqslant 1. \tag{7}$$

and

$$\sum_{s=1}^{\infty} \left(\frac{P_s}{p_s}\right)^{\delta q + q-1} |\Delta\sigma_{s-1}|^q < \infty, \tag{8}$$

then $\sum a_s$ is $|\overline{N}, p_s; \delta|_q$ summable, while if

$$\sum_{s=1}^{\infty} \left(\frac{P_s}{p_s}\right)^{\gamma(\delta q + q-1)} |\Delta\sigma_{s-1}|^q < \infty, \tag{9}$$

then it is summable $|\overline{N}, p_s; \delta; \gamma|_q$, where $q \geqslant 1$, $\delta \geqslant 0$ and γ is a real number.

Bor generalized in [4]-[5] a theorem dealing with factors affecting the absolute Riesz summability and established two theorems using more general conditions for the infinite series. Later works [6-7] of Bor, on weighted arithmetic mean summability, contained many practical applications. Sonker and Munjal reformulated in [8]-[[12] the absolute Riesz summability theory by using minimum conditions on the pertaining factors. In this note, we extended a result of Sonker and Munjal, in [13], to develop a generalized absolute Riesz summability for weaker pertaining factors, involving almost increasing sequences.

2. Earlier Related Work

Sonker and Munjal [13] have proved the following result using $|\overline{N}, p_n; \delta; \gamma|_q$ summability.

Theorem 2. 1. [13] *Let (χ_n) be an almost increasing and the (β_n) and (λ_n) are such that satisfying the conditions:*

$$|\Delta\lambda_n| \leqslant \beta_n, \tag{10}$$

$$\beta_n \to 0 \text{ as } n \to \infty, \tag{11}$$

$$\sum_{n=1}^{\infty} n\,|\Delta\beta_n|\,\chi_n \leqslant \infty, \tag{12}$$

$$|\lambda_n|\chi_n = O(1) \text{ as } n \to \infty, \tag{13}$$

$$\sum_{n=\upsilon+1}^{\infty} \frac{1}{P_{n-1}}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta q + q-1)-q} = O\left\{\frac{1}{P_\upsilon}\left(\frac{P_\upsilon}{p_\upsilon}\right)^{1-q+\gamma(\delta q + q-1)}\right\}, \tag{14}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q + q-1)-q} |t_n|^q = O(\chi_m), \tag{15}$$

$$\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1)\,, \tag{16}$$

and

$$\sum_{n=1}^{m} \frac{1}{n}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)-q}|t_n|^q = O(\chi_m), \text{ as } m\to\infty, \tag{17}$$

then $\sum a_n\lambda_n$ *is* $|\bar{N},p_n;\delta;\gamma|_q$ *summable.*

Another result, to be needed later, is the lemma to follow.

Lemma 2. 1. [14] *Let* $\{\chi_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be as in the known result, satisfy the condition (12). Then*

$$n\beta_n\,\chi_n = O(1) \text{ as } n\to\infty, \tag{18}$$

$$\sum_{n=1}^{\infty}\beta_n\chi_n < \infty. \tag{19}$$

3. Main Result

The main result of this note is a proof of the earlier result of [14] under a weaker condition on the n-th Cesàro means t_n. Accordingly, this result states as follows.

Theorem 3. 1. *Let the sequences* (χ_n), (β_n) *and* (λ_n) *be such that the conditions (10-17) hold with the condition (15) and (17) replaced respectively by:*

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)-q}\frac{|t_n|^q}{\chi_n^{q-1}} = O(\chi_m), \tag{20}$$

and

$$\sum_{n=1}^{m}\frac{1}{n}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)-q}\frac{|t_n|^q}{\chi_n^{q-1}} = O(\chi_m), \text{ as } m\to\infty. \tag{21}$$

Then $\sum a_n\lambda_n$ *is* $|\bar{N},p_n,\delta;\gamma|_q$ *summable.*

Proof. Let Y_n be the $(\bar{N},p_n)$ mean of $\sum a_n\lambda_n$. Then, we have

$$Y_n = \frac{1}{P_n}\sum_{\upsilon=0}^{n}p_\upsilon\sum_{i=0}^{\upsilon}a_i\lambda_i = \frac{1}{P_n}\sum_{\upsilon=0}^{n}(P_n-P_{\upsilon-1})\,a_\upsilon\lambda_\upsilon. \tag{22}$$

Then, for $n\geq 1$

$$\Delta Y_n = Y_n - Y_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{\upsilon=1}^{n}P_{\upsilon-1}a_\upsilon\lambda_\upsilon = \frac{p_n}{P_n P_{n-1}}\sum_{\upsilon=1}^{n}\frac{P_{\upsilon-1}\lambda_\upsilon}{\upsilon}\upsilon\, a_\upsilon.$$

$$= \frac{n+1}{n P_n} p_n t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v}$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \Delta\lambda_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v}$$

$$= Y_{(n,1)} + Y_{(n,2)} + Y_{(n,3)} + Y_{(n,4)}. \tag{23}$$

To complete the proof, it is enough to prove

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q + q - 1)} |\bar{\Delta} Y_n|^q < \infty. \tag{24}$$

Using Minkowski's inequality,

$$|Y_{(n,1)} + Y_{(n,2)} + Y_{(n,3)} + Y_{(n,4)}|^q \leqslant 4^q(|Y_{(n,1)}|^q + |Y_{(n,2)}|^q + |Y_{(n,3)}|^q + |Y_{(n,4)}|^q),$$

then equation (24) reduces to

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q + q - 1)} |Y_{(n,r)}|^q < \infty \ \text{ for } \ r = 1, 2, 3, 4. \tag{25}$$

Now the left hand side (L.H.S.) of equation (25) is:

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q + q - 1)} |Y_{(n,1)}|^q = \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q + q - 1)} \left| \frac{n+1}{n P_n} p_n t_n \lambda_n \right|^q$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q + q - 1) - q} \frac{|t_n|^q}{\chi_n^{q-1}} |\lambda_n|$$

$$= O(1) |\lambda_m| \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q + q - 1) - q} \frac{|t_n|^q}{\chi_n^{q-1}}$$

$$+ O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\gamma(\delta q + q - 1) - q} \frac{|t_v|^q}{\chi_v^{q-1}}$$

$$= O(1)|\lambda_m|\chi_m + O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| \chi_n$$

$$= O(1) \text{ as } m \to \infty, \tag{26}$$

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)} |Y_{(n,2)}|^q = O(1) \sum_{n=2}^{m+1} \frac{1}{P_{n-1}} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)-q}$$

$$\times \sum_{\upsilon=1}^{n-1} p_\upsilon |t_\upsilon|^q |\lambda_\upsilon| \left(\frac{1}{P_{n-1}} \sum_{\upsilon=1}^{n-1} p_\upsilon\right)^{q-1}$$

$$= O(1) \sum_{\upsilon=1}^{m} p_\upsilon |t_\upsilon|^q |\lambda_\upsilon|$$

$$\times \sum_{n=\upsilon+1}^{m+1} \frac{1}{P_{n-1}} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)-q}$$

$$= O(1) \sum_{\upsilon=1}^{m} p_\upsilon \frac{|t_\upsilon|^q}{\chi_\upsilon^{q-1}} |\lambda_\upsilon| \frac{1}{P_\upsilon}$$

$$\times \left(\frac{P_\upsilon}{p_\upsilon}\right)^{1-q+\gamma(\delta q+q-1)}$$

$$= O(1) |\lambda_m| \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)} \frac{|t_n|^q}{\chi_n^{q-1}}$$

$$+ O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{\upsilon=1}^{n} \left(\frac{P_\upsilon}{p_\upsilon}\right)^{\gamma(\delta q+q-1)-q} \frac{|t_\upsilon|^q}{\chi_\upsilon^{q-1}}$$

$$= O(1)|\lambda_m|\chi_m + O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n|\chi_n$$

$$= O(1) \text{ as } m \to \infty, \tag{27}$$

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)} |Y_{(n,3)}|^q = O(1) \sum_{n=2}^{m+1} \frac{1}{P_{n-1}} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)-q}$$

$$\times \sum_{\upsilon=1}^{n-1} P_\upsilon |t_\upsilon|^q \beta_\upsilon \left(\frac{1}{P_{n-1}} \sum_{\upsilon=1}^{n-1} P_\upsilon \beta_\upsilon\right)^{q-1}$$

$$= O(1) \sum_{\upsilon=1}^{m} P_\upsilon \beta_\upsilon |t_\upsilon|^q$$

$$\times \sum_{n=\upsilon+1}^{m+1} \frac{1}{P_{n-1}} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)-q}$$

$$= O(1) \sum_{\upsilon=1}^{m} P_\upsilon \frac{|t_\upsilon|^q}{\chi_\upsilon^{q-1}} \beta_\upsilon \frac{1}{p_\upsilon}$$

$$\times \left(\frac{P_\upsilon}{p_\upsilon}\right)^{1-q+\gamma(\delta q+q-1)}$$

$$= O(1) \sum_{\upsilon=1}^{m} \upsilon\beta_\upsilon \frac{1}{\upsilon} \left(\frac{P_\upsilon}{p_\upsilon}\right)^{1-q+\gamma(\delta q+q-1)-1} \frac{|t_\upsilon|^q}{\chi_\upsilon^{q-1}}$$

$$= O(1)\, m\beta_m \sum_{\upsilon=1}^{m} \frac{1}{\upsilon} \left(\frac{P_\upsilon}{p_\upsilon}\right)^{1-q+\gamma(\delta q+q-1)-1} \frac{|t_\upsilon|^q}{\chi_\upsilon^{q-1}}$$

$$= O(1) \sum_{\upsilon=1}^{m-1} \Delta(\upsilon\beta_\upsilon)$$

$$\times \sum_{i=1}^{\upsilon} \frac{1}{i} \left(\frac{P_i}{p_i}\right)^{1-q+\gamma(\delta q+q-1)-1} \frac{|t_i|^q}{\chi_i^{q-1}}$$

$$= O(1) m\beta_m \chi_m + O(1) \sum_{\upsilon=1}^{m-1} |\Delta(\upsilon\beta_\upsilon)| \chi_\upsilon$$

$$= O(1) m\beta_m \chi_m + O(1) \sum_{\upsilon=1}^{m-1} |\Delta\beta_\upsilon| \upsilon\chi_\upsilon$$

$$+ O(1) \sum_{\upsilon=1}^{m-1} \beta_\upsilon \chi_\upsilon$$

$$= O(1) \ as \ m \to \infty. \tag{28}$$

and

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)} |Y_{(n,4)}|^q = \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)}$$

$$\times \left| \frac{p_n}{P_n P_{n-1}} \sum_{\upsilon=1}^{n-1} P_\upsilon t_\upsilon \lambda_{\upsilon+1} \frac{1}{\upsilon} \right|^q$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{P_{n-1}} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)-q}$$

$$\times \sum_{\upsilon=1}^{n-1} P_\upsilon \frac{|\lambda_{\upsilon+1}|}{\upsilon} |t_\upsilon|^q \left(\frac{1}{P_{n-1}} \sum_{\upsilon=1}^{n-1} P_\upsilon \frac{|\lambda_{\upsilon+1}|}{\upsilon} \right)^{q-1}$$

$$= O(1) \sum_{\upsilon=1}^{m} P_\upsilon \frac{|\lambda_{\upsilon+1}|}{\upsilon} |t_\upsilon|^q$$

$$\times \sum_{n=\upsilon+1}^{m+1} \frac{1}{P_{n-1}} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta q+q-1)-q}$$

$$= O(1) \sum_{\upsilon=1}^{m} P_\upsilon \frac{|\lambda_{\upsilon+1}|}{\upsilon} \frac{|t_\upsilon|^q}{\chi_\upsilon^{q-1}} \frac{1}{P_\upsilon}$$

$$\times \left(\frac{P_\upsilon}{p_\upsilon}\right)^{1-q+\gamma(\delta q+q-1)}$$

$$= O(1) |\lambda_{\upsilon+1}| \sum_{\upsilon=1}^{m} \frac{1}{\upsilon} \left(\frac{P_\upsilon}{p_\upsilon}\right)^{1-q+\gamma(\delta q+q-1)-1} \frac{|t_\upsilon|^q}{\chi_\upsilon^{q-1}}$$

$$+ O(1) \sum_{\upsilon=1}^{m-1} \Delta |\lambda_{\upsilon+1}|$$

$$\times \sum_{i=1}^{\upsilon} \frac{1}{i} \left(\frac{P_i}{p_i}\right)^{1-q+\gamma(\delta q+q-1)-1} \frac{|t_i|^q}{\chi_i^{q-1}}$$

$$= O(1)|\lambda_{m+1}|\chi_m + O(1) \sum_{\upsilon=1}^{m-1} \beta_{\upsilon+1} \chi_{\upsilon+1}$$

$$= O(1) \text{ as } m \to \infty. \tag{29}$$

Collecting(22)- (24), demonstrates that the validity of (25). Here the proof ends. ■

From this main result, we can prove some previous results as well as new results in the form of corollaries.

Corollary 3. 1. *If in the main theorem we take* $\gamma = 1$, $\delta = 0$ *and* $q = 1$, *then* $\sum a_n\lambda_n$ *is* $|\overline{N}, p_n|$ *summable.*

Corollary 3. 2. *If in the main theorem we take* $\gamma = 1$ *and* $\delta = 0$, *then we are directed towards the new result :* $\sum a_n\lambda_n$ *is* $|\overline{N}, p_n|_q$ *summable.*

4. Conclusion

In this note, we have extended an earlier result of Sonker and Munjal, in [13], to develop a generalized absolute Riesz summability for weaker pertaining factors, involving almost increasing sequences. The reported corollaries illustrate the consistency of our main result with classical absolute Riesz summability of infinite series.

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