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Existence Results for Stochastic Integrodifferential Equations With Nonlocal Conditions

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Abstract. *The objective of this paper is to study existence results of mild solutions for a class of stochastic integrodifferential equations with nonlocal conditions and stochastic impulsive integrodifferential equations with nonlocal conditions in Hilbert spaces. Appropriate conditions for the existence of mild solutions are derived by means of stochastic analysis theory, resolvent operator theory in the sense of Grimmer and Leray-Schauder nonlinear alternative. The theorems displayed within this paper extend some similar results in this direction. An illustration to demonstrate the viability of the key findings of ours is given.*

Key words: Stochastic Integrodifferential Equations, Impulsive Equations, Nonlocal Conditions.

AMS Subject Classifications: 60H15, 60H20, 34K45

1. Introduction

Integrodifferential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of physics, chemistry, mechanics, engineering and economics. One can see Corduneanu [4], Lakshmikantham and Rao [15], Sobczyk [21] and references therein.

In the context of a growing stochastic version, the deterministic version of integrodifferential systems was examined extensively (see Dhage and Graef [5], Hernández [7], Liang and Xiao [16], Mohsen, el-Gamel [18]). For a lot of investigations recently, the generalized Cauchy problems involving nonlocal and/or impulsive conditions have been extensively studied in differential equations and dynamical systems. Nonlocal conditions are known to make a much better description of real models than classical initial ones, e.g. the condition

$$u(s) + \sum_{i=1}^{M} c_i u(\tau_i + s) = \phi(s),$$

allows taking additional measurements instead of solely initial datum.

Byszewski's work [14] provides the first result, as well as the physical significance for nonlocal issues. It then generated increased interest in many nonlocal issues regarding differential equations. For some remarkable solvability results, we quote here the works in [7, 8, 9, 12, 17, 22, 24]. The theorems about existence, uniqueness and stability of solutions for (stochastic) differential equations with nonlocal conditions have been investigated by many authors, one can see Balachandran, Park, and Chanderasekran [1], Balasubramaniam and Park [2], Balasubramaniam, Park, and Vincent Antony Kumar [3], Deng [6], Liang and Xiao [16] and references therein.

To the best of the authors knowledge, there are limited works by considering stochastic integrodifferential equations and nonlocal conditions with resolvent operator. Motivated by the above works we will make the first attempt to study such systems in this paper.The first goal of this paper is studying the existence results of mild solutions for the following stochastic integrodifferential equations with nonlocal conditions:

$$\begin{cases} du(t) = \left[Au(t) + \int_0^t \Upsilon(t-s)u(s)ds + f(t, u(t), \int_0^t h_1(t,s,u(s))ds) \right] dt \\ \qquad + g(t, u(t), \int_0^t h_2(t,s,u(s))ds) dW(t), \; t \in [0,T] = I, \\ u(0) + b(u) = u_0, \end{cases} \tag{1}$$

in a real separable Hilbert space H with inner product $(.,.)$ and norm $\| \cdot \|_{\mathsf{H}}$, where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on H with domain $D(A)$. Here Υ is a closed linear operator on H with domain $D(\Upsilon) \supset D(A)$ which is independent of $t.f,\; g,\; b$ and $h_i, i = 1,\; 2$ are given functions to be specified latter, u_0 is an F_0-measurable random variable with finite second moment.

Impulsive effects are common phenomena due to instantaneous perturbations at certain moment, such phenomena are described by impulsive differential equation which have been used efficiently in modeling many practical problems that arise in the fields of engineering, physics, and science as well. So the theory of impulsive differential equations is also attracting much attention in recent years [23].

Evidently, generalized Cauchy issues with impulsive effects and nonlocal conditions play a crucial part in describing many real life issues.The next goal of this paper is investigating the existence of mild solutions for impulsive stochastic integrodifferential equations with nonlocal conditions of the form:

$$\begin{cases} du(t) = \left[Au(t) + \int_0^t \Upsilon(t-s)u(s)ds + f(t, u(t), \int_0^t h_1(t,s,u(s))ds) \right] dt \\ \qquad + g(t, u(t), \int_0^t h_2(t,s,u(s))ds) dW(t), \; t \in [0,T] = I, \; t \neq t_p, \\ u(0) + b(u) = u_0, \\ \Delta u(t_p) = \mathsf{I}_p(u(t_p)), \; p = 0,1,2\ldots,m, \end{cases} \tag{2}$$

in a real separable Hilbert space H with inner product $(.,.)$ and norm $\|\cdot\|_{\mathsf{H}}$, where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on H with domain $D(A)$. Here Υ is a closed linear operator on H with domain $D(\Upsilon) \supset D(A)$ which is independent of $t.f$, g, b and $h_i, i = 1, 2$ are given functions to be specified latter, u_0 is an F_0-measurable random variable with finite second moment.The impulsive times t_p satisfy $0 = t_0 < t_1 < t_2 <\ldots< t_m < T$. By $u(t_{p^+})$ and $u(t_{p^-})$ we mean the right and left limit of u at $t = t_p$ respectively, $\Delta u(t_p) = u(t_{p^+}) - u(t_{p^-})$ represents the jump in the state u at time t_p, $\mathsf{I}_p \in C(\mathsf{H},\mathsf{H})$ $(p = 1,2,\cdots,m)$ are bounded functions which determine the size of the jump, where, $C(\mathsf{H},\mathsf{H})$ denotes the space of all continuous functions mapping H into H.

The main contributions of this paper are summarized as follows: in this work, a general classes stochastic integrodifferential equations with nonlocal conditions are considered.Then, using methods of functional analysis, a set of sufficient conditions are proposed ensuring existence results for mild solutions. The results are established with the use of the resolvent operator approach. Our paper expands the usefulness of stochastic integrodifferential equations, since the literature shows results for existence for such equations in the case of semigroup only.

The paper is organized as follows. In Section 2, we recall some basic definitions and preliminary facts, which will be used throughout this paper. In Section 3, we study of existence of mild solutions for systems (1) and (2). In Section 4, we provide an example, which shows the results obtained in a type of stochastic integrodifferential equations with nonlocal conditions.

2. Preliminaries

Throughout this paper, $(\mathsf{H}, \|\cdot\|_{\mathsf{H}},(.,.))$ and $(\mathsf{K}, \|\cdot\|_{\mathsf{K}},(.,.))$ denote two real separable Hilbert spaces. We denote by $\mathsf{L}(\mathsf{K},\mathsf{H})$ the space of all bounded linear operators from K to H equipped with the usual operator norm $\|\cdot\|$. In this article, we use the symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces involved when no confusion possibility arises.

Let $(\Omega,\mathsf{F},\{\mathsf{F}_t\}_{t\geq 0},\mathsf{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and F_0 contains all P-null sets of F. Let $\{\beta_k(t)\}_{k\geq 1}$ be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega,\mathsf{F},\{\mathsf{F}_t\}_{t\geq 0},\mathsf{P})$. Set

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \beta_k(t) e_k,\ \ t \geq 0,$$

where $\lambda_k \geq 0$, $(k = 1,2,3,\ldots)$ are nonnegative real numbers and $\{e_k\}(k = 1,2,3,\ldots)$ is a complete and orthonormal basis in K. Let $Q \in \mathsf{L}(\mathsf{K},\mathsf{H})$ be an operator defined by $Qe_k = \lambda_k e_k$ with a finite trace $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k$. Then, the above K-valued stochastic process $W(t)$ is called a Q-Wiener process.

Let $\varphi \in \mathsf{L}(\mathsf{K},\mathsf{H})$ and define

$$\|\varphi\|^2_{\mathsf{L}_2^0} = Tr(\varphi Q \varphi^\star) = \left\{ \sum_{k=1}^{\infty} \left\| \sqrt{\lambda_k}\, \varphi e_k \right\|^2 \right\}.$$

If $\|\varphi\|^2_{\mathsf{L}_2^0} < \infty$, then φ is called a Q-Hilbert-Schmidt operator, where L_2^0 denote the space of all Q-Hilbert-Schmidt operators $\varphi : \mathsf{K} \to \mathsf{H}$.

We denote by $L^2(\Omega, \mathsf{H})$, the Banach space of all strongly-measurable, square-integrable H-valued random variables equipped with the norm $\|u\|_{L^2} = (\mathsf{E}\|u(t,\omega)\|^2_{\mathsf{H}})^{1/2}$. An important subspace of $L^2(\Omega, \mathsf{H})$ is given by

$$L_0^2(\Omega, \mathsf{H}) = \left\{ u \in L^2(\Omega, \mathsf{H}),\ u \text{ is } \mathsf{F}_0 - \text{measurable} \right\}$$

Let $C\big(I\ ;\ \mathsf{H}\big)$ be the space of all continuous almost surely functions from the interval I into H and $S\big(I\ ;\ \mathsf{H}\big)$ the space of all continuous function at $t \neq t_p$, $u(t_p^-) = u(t_p)$, and $u(t_p^+)$ exists, $p = 1, 2, \ldots, m$.
Also let $C\big(I\ ;\ L^2(\Omega, \mathsf{H})\big)$ the Banach space of all continuous mappings from I into $L^2(\Omega, \mathsf{H})$ satisfying $\sup_{t \in I} \mathsf{E}\|u(t)\|^2_{\mathsf{H}} < +\infty$. We denote by B, the space of all F_t-adapted, measurable processes $u \in C\big(I\ ;\ L^2(\Omega, \mathsf{H})\big))$ endowed with the norm

$$\|u\|_{\mathsf{B}} = (\sup_{s \in I} \mathsf{E}\|u(s,\omega)\|^2_{\mathsf{H}})^{1/2}.$$

It is obvious that $(\mathsf{B}, \|.\|_{\mathsf{B}})$ is a Banach space.

In order to investigate the system (2), we introduce the following space $S\big(I\ ;\ L^2(\Omega, \mathsf{H})\big) = \{u : I \to L^2(\Omega, \mathsf{H}), u$ is continuous $t \neq t_p$, $u(t_p^-) = u(t_p), u(t_p^+)$ exists and $\sup_{s \in I} \mathsf{E}\|u(s)\|^2_{\mathsf{H}} < \infty\}$.
Let S be the space of all F_t -adapted, stochastic and measurable processes $u \in S\big(I\ ;\ L^2(\Omega, \mathsf{H}\big)$ endowed with the norm $\|u\|_{\mathsf{S}} = (\sup_{s \in I} \mathsf{E}\|u(s)\|^2_{\mathsf{H}})^{1/2}$, it is clear that $(\mathsf{S}, \|.\|_{\mathsf{S}})$ is a Banach space.

For the following, we put $t_{p+1} = T$ and for $u \in \mathsf{S}$, we denote by $\tilde{u}_p \in S([t_p, t_{p+1}], L^2(\Omega, \mathsf{H}))$, $p = 0, 1, \ldots, m$, the function given by

$$\tilde{u}_p(t) = \begin{cases} u(t),\ t \in (t_p, t_{p+1}], \quad u(t_p^+)\ ,\ t = t_p. \\ \\ u(t_p^+)\ ,\ t = t_p. \end{cases}$$

Moreover, for $K \subset \mathsf{S}$, we denote by $\tilde{K}_p = \{\tilde{u}_p : u \in K\}$, $p = 0, 1, \cdots, m$.

Lemma 2.1. [10] *A set $K \subset S$ is relatively compact in S if and only if, the set $\tilde{K}_p$ is relatively compact in $C\big([t_p, t_{p+1}]\ ;\ L^2(\Omega, \mathsf{H})\big)$, for every $p = 0, 1, \cdots, m$.*

Lemma 2.2. [11] *Let E be a closed and convex subset of a Banach space B. Assume that U is a relatively open subset of E with $0 \in U$ and $\Phi : \bar{\mathsf{U}} \longrightarrow E$ is a compact map, then :*
- *Φ has a fixed point in $\bar{\mathsf{U}}$, or*
- *there is a point $x \in \partial U$ and $\lambda \in]0, 1[$ with $x \in \lambda\Phi(x)$.*

In the sequel we introduce resolvent operators that will be used to develop the main results

of this work
Throughout this work, H is a Banach space, A and $\Upsilon(t)$ are closed linear operators on H. Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_Y := |Ay| + |y| \quad \text{for} \quad y \in Y.$$

The notations $C([0,+\infty);Y), \mathsf{B}(Y,\mathsf{H})$ stand for the space of all continuous functions from $[0,+\infty)$ into Y, the set of all bounded linear operators from Y into H, respectively. We consider the following Cauchy problem

$$\begin{cases} u'(t) = Au(t) + \int_0^t \Upsilon(t-s)u(s)ds \quad \text{for} \quad t \geq 0, \\ u(0) = u_0 \in \mathsf{H}. \end{cases} \tag{3}$$

Definiton 2.1. [19] A resolvent operator for Eq.(3) is a bounded linear operator valued function $R(t) \in \mathsf{L}(\mathsf{H})$ for $t \geq 0$, satisfying the following properties:
(i) $R(0) = I$ and $|R(t)| \leq Ne^{\beta t}$ for some constants N and β.
(ii) For each $x \in \mathsf{H}$, $R(t)x$ is strongly continuous for $t \geq 0$.
(iii) $R(t) \in \mathsf{L}(Y)$ for $t \geq 0$. For $x \in Y$, $R(\bullet)x \in C^1([0,+\infty);\mathrm{H}) \cap C([0,+\infty);Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t \Upsilon(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)\Upsilon(s)xds \quad \text{for} \quad t \geq 0. \end{aligned}$$

We impose the following assumptions on the considered system:
($\mathbf{E_1}$) A is the infinitesimal generator of a strongly continuous semigroup on H.
($\mathbf{E_2}$) For all $t \geq 0$, $\Upsilon(t)$ is a closed linear operator from $\mathrm{D}(A)$ to H, and $\Upsilon(t) \in \mathsf{B}(Y,\mathsf{H})$. For any $y \in Y$, the map $t \to \Upsilon(t)y$ is bounded, differentiable and the derivative $t \to \Upsilon'(t)y$ is bounded and uniformly continuous on R^+.

Theorem 2.1. [19; Theorem3.7] *Assume that* $(\mathbf{E}_1) - (\mathbf{E}_2)$ *hold. Then, there exists a unique resolvent operator for the Cauchy problem (3).*

Lemma 2.3. [13] *Let the conditions* $(\mathbf{E}_1)$ *et* $(\mathbf{E}_2)$ *be satisfied.Then there exists a constant* $\alpha(T)$ *such that*

$$\|R(t+\epsilon) - R(\epsilon)R(t)\| \leq \alpha\ \epsilon\ , \forall\ 0 \leq \epsilon \leq t \leq T.$$

Theorem 2.2.[20; Theorem 2] *Let A be the infinitesimal generator of a* C_0*-semigroup* $(T(t))_{t\geq 0}$ *and* $(\Upsilon(t))_{t\geq 0}$ *satisfy* (E_2)*. Then the resolvent* $(R(t))_{t\geq 0}$ *for (3) is compact for* $t > 0$ *if only if* $(T(t))_{t\geq 0}$ *is compact for* $t > 0$.

3. Main Results

3.1. **Existence of mild solutions of system (1)**

The system (1) is taken into account in this subsection. We present first the concept of mild solutions.

Definiton 3.1. An H- valued stochastic process $\{u(t),\ t \in [0,T]\}$ is said to be a mild solution of system (1) if
1. $u_0, b \in L^2(\Omega, \mathsf{H})$;
2. $u(t)$ is F_t - adapted and satisfies the following stochastic integral equation

$$u(t) = R(t)[u_0 - b(u)] + \int_0^t R(t-s) f\Big(s, u(s), \int_0^s h_1(s,r,u(r))dr\Big)ds$$

$$+ \int_0^t R(t-s) g\Big(s, u(s), \int_0^s h_2(s,r,u(r))dr\Big)dW(s)$$

for a.e. $t \in I$.
The following assumptions are needed to demonstrate the existence of mild solution.
We impose the following assumptions on the considered system:
($\mathbf{E_3}$) The resolvent operator $R(t)$ associated to (3) is compact and there exists a positive constant M such that $\|R(t)\| \leq M, \forall\ t \in I$.
We impose the following assumptions on the considered system:
($\mathbf{E_4}$) For each $t \in I$, $f(t,.) : \mathsf{H} \times \mathsf{H} \to \mathsf{H}$, $g(t,.) : \mathsf{H} \times \mathsf{H} \to \mathsf{L}(\mathsf{K},\mathsf{H})$ are continuous and for each $(u,v) \in \mathsf{H} \times \mathsf{H}$, $f(.,u,v) : I \to \mathsf{H}$ and $g(.,u,v) : I \to \mathsf{L}(\mathsf{K},\mathsf{H})$ are measurables. $h_i : I \times I \times \mathsf{H} \to \mathsf{H}$, $i = 1,2$ are continuous. Moreover there exists a constant $L > 0$ such that for all $u_i, v_i, i = 1,2$, $x,y \in \mathsf{H}$ and $t \in I$,

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\|^2 \vee \|g(t,u_1,v_1) - g(t,u_2,v_2)\|^2 \leq L[\|u_1 - u_2\|^2_{\mathsf{H}} + \|v_1 - v_2\|^2_{\mathsf{H}}].$$

$$\|\int_0^t h_i(t,s,x) - h_i(t,s,y)ds\|^2_{\mathsf{H}} \leq L\|x-y\|^2_{\mathsf{H}},\ i = 1,2.$$

($\mathbf{E_5}$) (i)- $b : C(I,\mathsf{H}) \to L^2_0(\Omega,\mathsf{H})$ is continuous and there exists a $\delta \in]0,T[$ such that $b(\psi) = b(\varrho)$ for any $\psi = \varrho$ on $[\delta, T]$;
(ii)- there is a continuous nondecreasing function $\gamma : [0,+\infty[\to [0,+\infty[$ such that

$$\mathsf{E}\|b(u)\|^2_{\mathsf{H}} \leq \gamma(\|u\|^2_{\mathsf{B}}).$$

In addition,

$$\sup_{s\in I}\|f(s,0,0)\|_{\mathsf{H}} \vee \sup_{s\in I}\|g(s,0,0)\|_{\mathsf{L}^0_2} \vee \sup_{(t,s)\in I^2}\|h_i(t,s,0)\|_{\mathsf{H}} \leq K,\ i = 1,2.$$

Theorem 3.1. *Assume that assumptions* (E_1) - (E_5) *hold . Then the system* (1) *has at least one solution provided that there exists a constant* $N^* > 0$ *such that*

$$\frac{N^*}{\big[6M^2\big[\mathsf{E}\|u_0\|^2_H + \gamma(N^*)\big] + k_0\big]e^{\beta_0 T}} > 1, \tag{4}$$

where

$$k_0 = 6M^2TK^2(2LT^2+1)(T + trQ)$$

and

$$\beta_0 = 6M^2L(1+2L)(T+trQ).$$

Proof. On the space B we introduce the equivalent norm $\| \ . \ \|_{\bar{B}}$ defined by

$$\|x\|_{\bar{B}} = \sup_{t\in I} e^{-pt}E\|x(t)\|_H^2$$

where p is a positive constant such that

$$\frac{2M^2L(1+L)(T+trQ)}{p} < 1. \tag{5}$$

It is easy to check that $\bar{B} := (B, \|\cdot\|_{\bar{B}})$ is a Banach space. Let $x \in B$ fixed. For $t \in I$, $u \in \bar{B}$, we consider the operator $F_x : B \to \bar{B}$ by

$$(F_xu)(t) = R(t)[u_0 - b(x)] + \int_0^t R(t-s)f\left(s, u(s), \int_0^s h_1(s,r,u(r))dr\right)ds$$
$$+ \int_0^t R(t-s)g\left(s, u(s), \int_0^s h_2(s,r,u(r))dr\right)dW(s). \tag{6}$$

We strive to show that F_x is contractive on $\bar{B}$. Let $u, v \in \bar{B}$, we have

$$\begin{aligned} e^{-pt}E\|(F_xu)(t) - (F_xv)(t)\|_H^2 &\le 2e^{-pt}E\Big\| \int_0^t R(t-s)\Big[f\Big(s,u(s),\int_0^s h_1(s,r,u(r))dr\Big) \\ &\quad - f\Big(s,v(s),\int_0^s h_1(s,r,v(r))dr\Big)\Big]ds\Big\|_H^2 \\ &\quad + 2e^{-pt}E\Big\| \int_0^t R(t-s)\Big[g\Big(s,u(s),\int_0^s h_2(s,r,u(r))dr\Big) \\ &\quad - g\Big(s,v(s),\int_0^s h_2(s,r,v(r))dr\Big)\Big]dW(s)\Big\|_H^2 \\ &= J_1 + J_2. \end{aligned}$$

From assumptions $(\mathbf{E_1})$ - $(\mathbf{E_4})$ it follows that

$$\begin{aligned} J_1 &\le 2TM^2e^{-pt}L\int_0^t E\Big[\|u(s)-v(s)\|_H^2 + \Big\|\int_0^s (h_1(s,r,u(r)) - h_1(s,r,v(r)))dr\Big\|_H^2\Big]ds \\ &\le 2TM^2L\ e^{-pt}\int_0^t (1+L)E\|u(s)-v(s)\|_H^2 ds, \end{aligned}$$

similarly

$$J_2 \leq 2\ M^2\ \mathrm{e}^{-pt}\ trQ\ L\int_0^t \mathsf{E}\Big[\ \|u(s)-v(s)\|_{\mathsf{H}}^2 + \Big\| \int_0^s \Big(h_2(s,r,u(r))dr - \int_0^s h_2(s,r,v(r)\Big)dr\Big\|_{\mathsf{H}}^2\Big]ds$$
$$\leq 2\ M^2 L\ \mathrm{e}^{-pt}\ trQ\int_0^t (1+L)\mathsf{E}\|u(s)-v(s)\|_{\mathsf{H}}^2 ds.$$

Consequently

$$\mathrm{e}^{-pt}\mathsf{E}\|(F_x u)(t)-(F_x v)(t)\|_{\mathsf{B}} \leq 2LM^2(1+L)(T+trQ) \times \int_0^t \mathrm{e}^{-p(t-s)}\mathrm{e}^{-ps}\mathsf{E}\|u(s)-v(s)\|_{\mathsf{H}}^2 ds$$
$$\leq \frac{2M^2\ L\ (1+L)(T+trQ)}{p}\ \sup_{s\in I}\ \mathrm{e}^{-ps}\mathsf{E}\ \|u(s)-v(s)\|_{\mathsf{H}}^2. \tag{7}$$

By Banach fixed point theorem, it follows that there exists a unique fixed point $u_x \in \bar{\mathsf{B}}$ which solves (6).
Based on this fact, for some $\delta \in (0,T]$, we set

$$\bar{x}(t) = \begin{cases} x(t)\ ,\ t \in\ ]\delta, T], \\ x(\delta),\ t \in\ [0,\delta]. \end{cases} \tag{8}$$

From (6), it follows that

$$u_{\bar{x}}(t) = R(t)[u_0 - b(\bar{x})] + \int_0^t R(t-s)f\Big(s, u_{\bar{x}}(s), \int_0^s h_1(s,r,u_{\bar{x}}(r))dr\Big)ds + \int_0^t R(t-s)g\Big(s, u_{\bar{x}}(s), \int_0^s h_2(s,r,u_{\bar{x}}(r))dr\Big)dW(s). \tag{9}$$

Let $B_\delta := C\big([\delta,T];\ L^2(\Omega,\mathsf{H})\big)$. Define a map $\Gamma : \mathsf{B}_\delta \to \mathsf{B}_\delta$ by

$$(\Gamma x)(t) = u_{\bar{x}}(t), \quad t \in [\delta, T].$$

We will show that the operator Γ satisfies all conditions of Lemma 2.2.The proof will be divided into the following steps.

Step 1 . Γ maps bounded sets into bounded sets in B_δ. Let $r > 0$ and

$$\mathsf{F}_r(\delta) := \{x \in \mathsf{B}_\delta, \quad \sup_{t\in[\delta,T]} \mathsf{E}\|x(t)\|_{\mathsf{H}}^2 \leq r\}.$$

It is obvious that $\mathsf{F}_r(\delta)$ is a bounded closed convex set in B_δ.
Let $x \in \mathsf{F}_r(\delta)$ and $t \in I$. Recalling the definition of Γ, by, assumptions $(\mathbf{E_1})$ - $(\mathbf{E_4})$and $(\mathbf{E_5})(ii)$ together with Hölder's inequality we have

$$\begin{aligned}
\mathsf{E}\|u_{\bar{x}}(t)\|_{\mathsf{H}}^2 &\le 3\mathsf{E}\,\|R(t)[u_0 - b(\bar{x})]\,\|_{\mathsf{H}}^2 \\
&\quad + 3\mathsf{E}\left\|\int_0^t R(t-s)f\left(s, u_{\bar{x}}(s), \int_0^s h_1(s,r,u_{\bar{x}}(r))dr\right)ds\right\|_{\mathsf{H}}^2 \\
&\quad + 3\mathsf{E}\left\|\int_0^t R(t-s)g\left(s, u_{\bar{x}}(s), \int_0^s h_2(s,r,u_{\bar{x}}(r))dr\right)dW(s)\right\|_{\mathsf{H}}^2, \\
&\le 6M^2[\mathsf{E}\|u_0\|_{\mathsf{H}}^2 + \gamma(r)] \\
&\quad + 3M^2T\int_0^t \mathsf{E}\left\|f\left(s, u_{\bar{x}}(s), \int_0^s h_1(s,r,u_{\bar{x}}(r))dr\right) - f(s,0,0) + f(s,0,0)\right\|_{\mathsf{H}}^2 ds \\
&\quad + 3M^2 trQ \int_0^t \mathsf{E}\left\|g\left(s, u_{\bar{x}}(s), \int_0^s h_2(s,r,u_{\bar{x}}(r))dr\right) - g(s,0,0) + g(s,0,0)\right\|_{\mathsf{L}_2^0}^2 ds \\
&\le 6M^2[\mathsf{E}\|u_0\|_{\mathsf{H}}^2 + \gamma(r)] + 6\;M^2\left(T\int_0^t \|f(s,0,0)\|_{\mathsf{H}}^2 ds + trQ\int_0^t \|g(s,0,0)\|_{\mathsf{L}_2^0}^2 ds\right) \\
&\quad + 6\;T\;M^2\int_0^t\left(L\;\mathsf{E}\left[\|u_{\bar{x}}(s)\|_{\mathsf{H}}^2 + \left\|\int_0^s h_1(s,r,u_{\bar{x}}(r))dr\right\|_{\mathsf{H}}^2\right]\right)ds \\
&\quad + 6\;trQ\;M^2\int_0^t\left(L\;\mathsf{E}\left[\|u_{\bar{x}}(s)\|_{\mathsf{H}}^2 + \left\|\int_0^s h_2(s,r,u_{\bar{x}}(r))dr\right\|_{\mathsf{H}}^2\right]\right)ds.
\end{aligned}$$

In view of assumption $(\mathbf{E_4})$ we have

$$\begin{aligned}
\mathsf{E}\|u_{\bar{x}}(t)\|_{\mathsf{H}}^2 &\le 6M^2\;L[E\|u_0\|_{\mathsf{H}}^2 + \gamma(r)] + 6M^2\;L\;(T + trQ)\int_0^t \mathsf{E}\|u_{\bar{x}}(s)\|_{\mathsf{H}}^2 ds \\
&\quad + 6\;T\;M^2\;L\int_0^t \mathsf{E}\left\|\int_0^s [h_1(s,r,u_{\bar{x}}(r))\; - h_1(s,r,0) + h_1(s,r,0)]dr\right\|_{\mathsf{H}}^2 ds \\
&\quad + 6\;trQ\;M^2\;L\int_0^t \mathsf{E}\left\|\int_0^s [h_2(s,r,u_{\bar{x}}(r)dr) - h_2(s,r,0) + h_2(s,r,0)]dr\right\|_{\mathsf{H}}^2 ds \\
&\quad + 6M^2\;T(trQK^2 + TK^2) \\
&\le \Gamma_0 + \int_0^t \Delta_0 \mathsf{E}\|u_{\bar{x}}(s)\|_{\mathsf{H}}^2 ds,
\end{aligned}$$

where

$$\Gamma_0 = 6M^2[\mathsf{E}\|u_0\|_{\mathsf{H}}^2 + \gamma(r)] + 6M^2K^2T\;(T + trQ)(2T^2L + 1)$$

and

$$\Delta_0 = 6M^2\;L(T + trQ)(1 + 2L).$$

Applying Gronwall's inequality it follows that

$$\mathsf{E}\,\|(\Gamma x)(t)\,\|_{\mathsf{H}}^2 = \mathsf{E}\|u_{\bar{x}}(t)\|_{\mathsf{H}}^2 \le \Gamma_0\;e^{\Delta_0 T}.$$

Step 2. Γ maps bounded sets into equicontinuous sets of B. Let $\delta \le t_1 < t_2 \le T$ and $x \in \mathsf{F}_r(\delta)$, in view of Hölder's inequality together with assumption $(\mathbf{E_4})$ we have

$$
\begin{aligned}
&\mathsf{E}\|(\Gamma x)(t_2)-(\Gamma x)(t_1)\|_{\mathsf{H}}^2\\
&\quad\le 3\mathsf{E}\|[R(t_2)-R(t_1)](u_0-b(\bar{x}))\|_{\mathsf{H}}^2\\
&\qquad+6T\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|^2\mathsf{E}\left\|f\left(s,u_{\bar{x}}(s),\int_0^s h_1(s,r,u_{\bar{x}}(s))dr\right)\right\|_{\mathsf{H}}^2 ds\\
&\qquad+6M^2(t_2-t_1)\int_{t_1}^{t_2}\mathsf{E}\left\|f\left(s,u_{\bar{x}}(s),\int_0^s h_1(s,r,u_{\bar{x}}(s))dr\right)\right\|_{\mathsf{H}}^2 ds\\
&\qquad+6M^2trQ\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|^2\mathsf{E}\left\|g\left(s,u_{\bar{x}}(s),\int_0^s h_2(s,r,u_{\bar{x}}(s))dr\right)\right\|_{\mathsf{L}_2^0}^2 ds\\
&\qquad+6M^2trQ\int_{t_1}^{t_2}\mathsf{E}\left\|g\left(s,u_{\bar{x}}(s),\int_0^s h_2(s,r,u_{\bar{x}}(s))dr\right)\right\|_{\mathsf{L}_2^0}^2 ds. \qquad (10)
\end{aligned}
$$

Now from the conditions on $f,:\ g$ and $h_i, i=1,2$, we get

$$
\begin{aligned}
&\mathsf{E}\left\|f\left(s,u_{\bar{x}}(s),\int_0^s h_1(s,r,u_{\bar{x}}(r))dr\right)\right\|_{\mathsf{H}}^2\\
&\quad=\mathsf{E}\left\|f\left(s,u_{\bar{x}}(s),\int_0^s h_1(s,r,u_{\bar{x}}(r))dr\right)-f(s,0,0)+f(s,0,0)\right\|_{\mathsf{H}}^2\\
&\quad\le 2L\left[\mathsf{E}\|u_{\bar{x}}(s)\|_{\mathsf{H}}^2+\mathsf{E}\left\|\int_0^s h_1(s,r,u_{\bar{x}}(r))dr\right\|_{\mathsf{H}}^2\right]+2\mathsf{E}\|f(s,0,0)\|_{\mathsf{H}}^2\\
&\quad\le 2\ L\left[\mathsf{E}\|u_{\bar{x}}(s)\|_{\mathsf{H}}^2+2\ L\ \mathsf{E}\|u_{\bar{x}}(s)\|_{\mathsf{H}}^2+2T^2K^2\right]+2K^2\\
&\quad\le 2\ L(1+2L)\|u_{\bar{x}}\|_{\mathsf{B}}^2+4\ L\ T^2K^2+2K^2=l. \qquad (11)
\end{aligned}
$$

Similarly

$$
\mathsf{E}\left\|g\left(s,u_{\bar{x}}(s),\int_0^s h_2(s,r,u_{\bar{x}}(r))dr\right)\right\|_{\mathsf{L}_2^0}^2\le 2\,L\,(1+2L)\|u_{\bar{x}}\|_{\mathsf{B}}^2+4\,L\,T^2K^2+2K^2=l. \quad (12)
$$

Hence,

$$
\begin{aligned}
\mathsf{E}\|(\Gamma x)(t_2)-(\Gamma x)(t_1)\|_{\mathsf{H}}^2 \le\ & 3\mathsf{E}\|[R(t_2)-R(t_1)](u_0-b(\bar{x}))\|_{\mathsf{H}}^2\\
&+6lT\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|^2ds+6M^2l(t_2-t_1)^2\\
&+6l\ trQ\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|^2ds+6M^2l(t_2-t_1)trQ.
\end{aligned}
$$

The right hand side of (13) tends to 0independently of $x$as $t_1\to t_2$, since the compactness of $R(t),\ t>0$implies the continuity in the uniform operator topology. Thus $\{\Gamma x(t), x\in \mathsf{F}_r(\delta)\}$is equicontinuous.

Step 3 . For each $t\in[\delta,T]$, the set $\{\Gamma x(t), x\in \mathsf{F}_r(\delta)\}$is precompact in H.
Let $t\in[\delta,T]$be fixed and $0<\epsilon<t$. For $x\in \mathsf{F}_r(\delta)$, we define the operators

$$(\Gamma_\epsilon^* x)(t) = R(t)[u_0 - b(\bar{x})] + R(\epsilon)\int_0^{t-\epsilon} R(t-s-\epsilon) f\Big(s, u_{\bar{x}}(s), \int_0^s h_1(s,r,u_{\bar{x}}(r))dr\Big)ds$$
$$+ R(\epsilon)\int_0^{t-\epsilon} R(t-s-\epsilon) g\Big(s, u_{\bar{x}}(s), \int_0^s h_2(s,r,u_{\bar{x}}(r))dr\Big)dW(s),$$

and

$$(\tilde{\Gamma}_\epsilon^* x)(t) = R(t)[u_0 - b(\bar{x})] + \int_0^{t-\epsilon} R(t-s) f\Big(s, u_{\bar{x}}(s), \int_0^s h_1(s,r,u_{\bar{x}}(r))dr\Big)ds$$
$$+ \int_0^{t-\epsilon} R(t-s) g\Big(s, u_{\bar{x}}(s), \int_0^s h_2(s,r,u_{\bar{x}}(r))dr\Big)dW(s).$$

By the compactness of the operator $R(t)$, the set $V_\epsilon^*(t) = \{(\Gamma_\epsilon^* x)(t) \ , x \ \in \mathsf{F}_r(\delta)\}$ is relatively compact in H for every $0 \ < \epsilon \ < t$. Moreover, also by Lemma 2.3, Holder's inequality and Burkholder-Davis-Gundy inequality for each $x \ \in \mathsf{F}_r(\delta)$, we obtain

$$\mathsf{E}\|(\Gamma_\epsilon^* x)(t) - \big(\tilde{\Gamma}_\epsilon^* x\big)(t)\|_{\mathsf{H}}^2 \le 2\alpha^2 \ \epsilon^2 \ T\int_0^{t-\epsilon} \mathsf{E}\Big\| f\Big(s, u_{\bar{x}}(s), \int_0^s h_1(s,r,u_{\bar{x}}(r))dr\Big)\Big\|_{\mathsf{H}}^2 ds$$
$$+ 2\alpha^2 \ \epsilon^2 \ trQ\int_0^{t-\epsilon} \mathsf{E}\Big\| g\Big(s, u_{\bar{x}}(s), \int_0^s h_2(s,r,u_{\bar{x}}(r))dr\Big)\Big\|_{\mathsf{L}_2^0}^2 ds$$
$$\to \ 0 \text{ as } \epsilon \to \ 0.$$

So the set $V_\epsilon^*(t) = \{\tilde{\Gamma}_\epsilon^* x(t) \ , x \ \in \mathsf{F}_r(\delta)\}$ is precompact in H by using the total boundedness. Applying the idea again, we obtain

$$\mathsf{E}\|(\Gamma x)(t) - \big(\tilde{\Gamma}_\epsilon^* x\big)(t)\|_{\mathsf{H}}^2 \le 2T\int_{t-\epsilon}^{t} \mathsf{E}\Big\| R(t-s) f\Big(s, u_{\bar{x}}(s), \int_0^s h_1(s,r,u_{\bar{x}}(r))dr\Big)\Big\|_{\mathsf{H}}^2 ds$$
$$+ 2trQ\int_{t-\epsilon}^{t} \mathsf{E}\Big\| R(t-s) g\Big(s, u_{\bar{x}}(s), \int_0^s h_2(s,r,u_{\bar{x}}(r))dr\Big)\Big\|_{\mathsf{L}_2^0}^2 ds$$
$$\to \ 0 \text{ as } \ \epsilon \to \ 0,$$

and there are precompact sets arbitrarily closed to the set $V(t) = \{\Gamma x(t) \ , x \ \in \mathsf{F}_r(\delta)\}$. Thus, the set $V(t) = \{\Gamma x(t), \ x \ \in \mathsf{F}_r(\delta)\}$ is precompact in H.

Step 4 . Existence of an open subset $\mathsf{U} \subset \mathsf{B}_\delta$ with $y \notin \lambda\Gamma y$ for $\lambda \in (0,1)$ and $y \in \partial\mathsf{U}$. Let $\lambda \in \ ]0,1[$ and $y \in \mathsf{B}_\delta$ be a possible solution of $y = \lambda(\Gamma y)$.Then, for $t \in \ ]0,T]$ we have $\lambda(\Gamma y)(t) = y(t) = \lambda u_{\bar{y}}(t)$
and

$$\mathsf{E}\|y(t)\|_{\mathsf{H}}^2 \le E\Big\|u_{\bar{y}}(t)\Big\|_{\mathsf{H}}^2 . \tag{13}$$

Moreover, from (11), (12) and assumption $(\mathbf{E_5})$, we get

$$
\begin{aligned}
\mathsf{E}\|u_{\bar y}(t)\|^2_{\mathsf{H}} \le\ & 6\Big[\mathsf{E}\|u_0\|^2_{\mathsf{H}} + \gamma(\|\bar y\|_{\mathsf{B}})\Big] \\
& + 3M^2T\int_0^t\Big[2\ L\ (1+L)\mathsf{E}\|u_{\bar y}(s)\|^2_{\mathsf{H}} + 4\ L\ T^2K^2 + 2K^2\Big]ds \\
& + 3M^2trQ\int_0^t\Big[2\ L\ (1+L)\mathsf{E}\|u_{\bar y}(s)\|^2_{\mathsf{H}} + 4\ L\ T^2K^2 + 2K^2\Big]ds \\
\le\ & \gamma_0 + \int_0^t \beta_0\mathsf{E}\|u_{\bar y}(s)\|^2_{\mathsf{H}}ds,
\end{aligned}
$$

where

$$\gamma_0 = 6M^2[\mathsf{E}\|u_0\|^2_{\mathsf{H}} + \gamma(r)] + 6M^2TK^2 \quad (2\ L\ T^2+1)(T+trQ)$$

and

$$\beta_0 = 6M^2_{;}L\ (1+2L)(T+trQ).$$

By Gronwall's inequality and (13), we get

$$\mathsf{E}\|y(t)\|^2_{\mathsf{H}} \le \mathsf{E}\|u_{\bar y}(t)\|^2_{\mathsf{H}} \le \gamma_0\, \mathrm{e}^{\beta_0 T}.$$

Set

$$\mathsf{U} = \{y \in \mathsf{B}_\delta, \quad \sup_{t\in[\delta,T]} \mathsf{E}\|y(t)\|^2_{\mathsf{H}} < N^*\},$$

U is an open subset of F_r, $r \ge N^*$ and from (9), there is no $y \in \partial\mathsf{U}$ such that $y \in \lambda(\Gamma y)(t)$ for $\lambda \in]0,1[$.

From the arguments in step 1 to step 4 together with Arzela - Ascoli theorem, it suffices to show that $\Gamma : \bar{\mathsf{U}} \to \mathsf{B}_\delta$ is a compact operator. According to the Lemma 2.2, Γ has a fixed point $\bar y \in \bar{\mathsf{U}}$. Let $v = u_{\bar y}$, we have from (6)

$$
\begin{aligned}
v(t) = R(t)[u_0 - b(\bar y)] &+ \int_0^t R(t-s)f\Big(s, v(s), \int_0^s h_1(s,r,v(r))dr\Big)ds \\
&+ \int_0^t R(t-s)g\Big(s, v(s), \int_0^s h_2(s,r,v(r))dr\Big)dW(s). \qquad (14)
\end{aligned}
$$

Using the definition of Γ, we get
$v(t) = u_{\bar y}(t) = (\Gamma y)(t) = \bar y(t), \ \forall t \in [\delta, T]$.
This concludes, together with (14) and assumption($\mathbf{E_4}$), that $v(t)$ is a mild solution of system (1).This completes the proof. ■

3.1. **Existence of mild solutions of system** (**2**)

We first present the mild solution for system (2).

Definition 3.2. A stochastic process $u \in S(I, \mathsf{H})$ is called a mild solution of system (2) if

1. $u_0, b \in L^2(\Omega, \mathsf{H})$;
2. $u(t)$ is F_t - adapted and satisfies the following integral equation

$$u(t) = R(t)[u_0 - b(u)] + \int_0^t R(t-s)f\Big(s, u(s), \int_0^s h_1(s,r,u(r))dr\Big)ds + \int_0^t R(t-s)g\Big(s, u(s), \int_0^s h_2(s,r,u(r))dr\Big)dW(s) + \sum_{0<t_p<t} R(t-t_p)\mathsf{I}_p(u(t_p))$$

for a.e.$t \in I$.

Before giving our second result, the following assumptions are listed.

(E_6) $b : S(I,\mathsf{H}) \to L^2(\Omega,\mathsf{H})$ is continuous and there exists a $\delta \in (0,t_1)$ such that $b(\varphi) = b(\phi)$ for any $\varphi, \phi \in S(I,\mathsf{H})$ with $\varphi = \phi$ on $[\delta, T]$. Furthermore, there is a continuous nondecreasing function $\Phi : [0,+\infty[\to [0,+\infty[$ such that

$$\mathsf{E}\|b(u)\|^2_{\mathsf{H}} \le \Phi(\|u\|^2_{\mathsf{S}}), u \in S(I,\mathsf{H}).$$

(E_7) $\mathsf{I}_p : S(I,\mathsf{H}) \to \mathsf{H}$, $p = 1,2,\dots,m$, are compact operators and there exists continuous nondecreasing functions $\Xi_p : [0,+\infty[\to (0,+\infty)$, $p = 1,2,\cdots,m$ such that

$$\mathsf{E}\|\mathsf{I}_p(u)\|^2_{\mathsf{H}} \le \Xi_p(\|u\|^2_{\mathsf{S}}), \quad u \in S(I,\mathsf{H}).$$

Theorem 3.2. *Assume that assumptions* (E_1) - (E_4)*,* (E_6) - (E_7) *hold.Then the system* (2) *has at least one solution provided that there exists a constant* $S^* > 0$ *such that*

$$\frac{S^*}{\Big[8M^2[\mathsf{E}\|u_0\|^2_{\mathsf{H}} + \gamma(S^*)] + 4mM^2\sum_{p=1}^m \Xi_p(S^*) + \gamma_1\Big]e^{\beta_1 T}} > 1, \tag{15}$$

where
$\gamma_1 = 8M^2TK^2(2LT^2 + 1)(T + trQ)$
and
$\beta_1 = 8M^2(1 + 2L)L(T + trQ)$.

Proof. By analogy to the previous proof, we introduce on the space S the equivalent norm $\|\ .\ \|_{\bar{\mathsf{S}}}$ defined by

$$\|u\|^2_{\bar{\mathsf{S}}} := \sup_{t\in I} e^{-pt}\mathsf{E}\|u(t)\|^2_{\mathsf{H}},$$

where p is a positive constant defined in (5). It is easy to check that $(\mathsf{S}, \|\ .\ \|_{\bar{\mathsf{S}}})$ is a Banach space. Let $x \in \mathsf{S}$ be fixed. For $t \in I$, $u \in \bar{\mathsf{S}}$ we define the operator $\mathsf{O}_x : \bar{\mathsf{S}} \to \bar{\mathsf{S}}$ by

$$(\mathsf{O}_x u)(t) = R(t)[u_0 - b(x)] + \int_0^t R(t-s)f\Big(s, u(s), \int_0^s h_1(s,r,u(r))dr\Big)ds + \int_0^t R(t-s)g\Big(s, u(s), \int_0^s h_2(s,r,u(r))dr\Big)dW(s) + \sum_{0<t_p<t} R(t-t_p)\mathsf{I}_p(x(t_p)). \tag{16}$$

In what follows we aim to show that O_x is contractive on $\bar{\mathsf{S}}$. Let $u, v \in \bar{\mathsf{S}}$, then from assumptions $(\mathbf{E_3})-(\mathbf{E_4})$, we have

$$\begin{aligned} e^{-pt}\mathsf{E}\|(\mathsf{O}_x u)(t)-(\mathsf{O}_x v)(t)\|_{\bar{\mathsf{S}}} &\le 2TM^2 e^{-pt}\int_0^t L\ (1+L)\mathsf{E}\|u(s)-v(s)\|^2_{\mathsf{H}}ds \\ &\quad + 2trQM^2\ e^{-pt}\int_0^t L\ (1+L)e^{-ps}E\|u(s)-v(s)\|^2_{\mathsf{H}}ds \\ &\le 2M^2\ L(1+L)(T+trQ)\int_0^t e^{-p(t-s)}\sup_{s\in I} e^{-ps}\mathsf{E}\|u(s)-v(s)\|^2_{\mathsf{H}}ds. \\ &\le \frac{2M^2\ L(1+L)(T+trQ)}{p}\sup_{s\in I} e^{-ps}\mathsf{E}\|u(s)-v(s)\|^2_{\mathsf{H}}. \end{aligned}$$

By Banach fixed point theorem, it follows that O_x has is a unique fixed point $u_x \in \bar{\mathsf{S}}$ that solves Eq.(16).
Consider the function $\bar{x}$ defined in (8), we have

$$\begin{aligned} u_{\bar{x}}(t) &= R(t)[u_0 - b(\bar{x})] + \int_0^t R(t-s)f\Big(s, u_{\bar{x}}(s), \int_0^s h_1(s,r,u_{\bar{x}}(r))dr\Big)ds \\ &\quad + \int_0^t R(t-s)g\Big(s, u_{\bar{x}}(s), \int_0^s h_2(s,r,u_{\bar{x}}(r))dr\Big)dW(s) + \sum_{0<t_p<t} R(t-t_p)\mathsf{I}_p(x(t_p)). \end{aligned} \tag{17}$$

Consider a map $\Gamma_1 : \mathsf{S}_\delta \to \mathsf{S}_\delta$ defined by

$$(\Gamma_1 x)(t) = u_{\bar{x}}(t)\ ,\ \forall\ t \in\ [\delta, T],$$

where $\mathsf{S}_\delta = S\big([\delta,T]\ ;\ L^2(\Omega),\mathsf{H}\big))$. For $r_1 > 0$, let

$$\mathsf{F}_r(\delta) = \{u \in \mathsf{S}_\delta\ :\ \sup_{t\,\in\,[\delta,T]} \mathsf{E}\|x(t)\|^2_{\mathsf{H}} \le r_1\}.$$

then for each $r_1 > 0, \mathsf{F}_r(\delta)$ is a bounded closed convex set in S_δ. In what follows, by similar arguments as steps 1-4 of Theorem 3.1, it is easy to see that the system (2) has at least one mild solution.This completes the proof. ∎

4. Illustrative Example

In this section, we present an example to illustrate our results.
Consider the following model:

$$\begin{aligned} &\frac{\partial}{\partial t}x(t,\xi) = \frac{\partial^2}{\partial x^2}x(t,\xi) + \int_0^t \beta(t-s)x(s,\xi)ds + \Big[a_1(t)\sin x(t,\xi) + \int_0^t a_2(t-s)e^{-x(t,\,\xi)}ds\Big] \\ &\qquad + \Big[a_3(t)\sin x(t,\xi) + \int_0^t a_4(t-s)e^{-x(t,\xi)}ds\Big]W(t),\ 0 \le \xi \le \pi,\ t \in I, \\ &x(t,0) = x(t,\pi) = 0,\ t \in I, \\ &x(0,\xi) + \sum_{k=0}^{n}\int_0^t g_i(\tau)x(\tau,\xi)d\tau = r(\xi), \end{aligned} \tag{18}$$

where $0 < t_1 < t_2 < \cdots t_k \leq T$ are given, $\beta : \mathsf{R}^+ \to \mathsf{R}$ is continuous, $W(t)$ denotes a standard one-dimensional Wiener process in H defined on a probability space $(\Omega, \mathsf{F}, \mathsf{P})$, $a_i, g_i \in L^2([0,T])$, $r \in L^2([0,\pi])$. Let $\mathsf{H} = \mathsf{K} = L^2([0,\pi])$ with a norm $\|.\|$ and inner product $(.,.)$.

Consider the operator $Ax = \dfrac{\partial^2 x}{\partial \xi^2}$ with the domain

$$D(A) = \left\{x \in \mathsf{H}, : x,\ x' \text{ are absolutely continuous },\ x'' \in \mathsf{H} \text{ and } x(0) = x(\pi) = 0\right\}$$

It is well-known that A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on H, which is compact for $t > 0$ and given by $T(t)v = -\sum_{n=1}^{\infty} e^{-n^2 t} < v, e_n > e_n$, $v \in D(A)$, where $e_n := \sqrt{\frac{2}{\pi}} \sin(nx), (n = 1,2,3,\ldots)$ is the orthonormal set of eigenvectors of A, and $T(t)v = -\sum_{n=1}^{\infty} e^{-n^2 t} < v, e_n > e_n$, $v \in \mathsf{H}$.
Let $\Upsilon : D(A) \subset \mathsf{H} \to \mathsf{H}$ be the operator defined by $\Upsilon(t)(z) = \beta(t)Az$ for $t \geq 0$ and $z \in D(A)$.
We define the operators $f : [0,T] \times \mathsf{H} \times \mathsf{H} \to \mathsf{H}$ and $g : [0,T] \times \mathsf{H} \times \mathsf{H} \to \mathsf{L}_0^2$ by

$$f(t, x(t,\xi), \int_0^t h_1(t,s,x(s,\xi))ds = a_1(t) \sin x(t,\xi) + \int_0^t a_2(t-s)e^{-x(t,\xi)}ds,$$

$$g(t, x(t,\xi), \int_0^t h_2(t,s,x(s,\xi))ds = a_3(t) \sin x(t,\xi) + \int_0^t a_4(t-s)e^{-x(t,\xi)}ds,$$

$$b(x(\xi)) = \sum_{k=1}^{n} \int_0^{t_i} g_i(\tau)x(\tau,\xi)d\tau.$$

Then Eq.(18)can be written in the following abstract form

$$\begin{cases} du(t) = \left[Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u(t), \int_0^t h_1(t,s,u(s)ds)) \right]dt \\ \qquad + g(t,u(t), \int_0^t h_2(t,s,u(s)ds))dW(t) \ \ t \in [0,T] = I, \\ u(0) + b(u) = u_0. \end{cases} \tag{19}$$

Moreover, if β is bounded and C^1 function such that β' is bounded and uniformly continuous, then $(\mathbf{E_1})$ and $(\mathbf{E_2})$ are satisfied and hence, by Theorem 2.1 and Lemma 2.3, Eq. (18) has a resolvent operator $(R(t))_{t\geq 0}$ on H which is compact. From the definitions of f, g and b, it is easy to check that the conditions $(\mathbf{E_3})$ - $(\mathbf{E_5})$ are satisfied. By choosing t_i in way that (4) is satisfied, it follows from Theorem 3.1 that Eq.(18) has a mild solution.
By the same method, we can study the existence of mild solutions to Eq.(18) with impulsive effects.

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