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Small Double Limit for an SDE With a Neumann Boundary Condition

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Abstract. *We provide a Large Deviation Principle for the SDE, with a reflecting boundary condition, if $\frac{\delta}{\varepsilon}$ tends to a positive constant $\lambda > 0$ when the two parameters δ(homogenization parameter) and ε(the large deviations parameter) tend to zero with the same speed. To achieve this, we have to come up with estimates to the logarithmic moment generating function.*

Key words: Local Time, Homogenization, Large Deviations,Stochastic Differential Equation, Neumann Boundary Condition.

AMS Subject Classifications: 0H15 , 35K00 , 35B27

1. Introduction

The theory of homogenization tries to understand what equations should be used at a macroscopic level, in order to approximate the behavior of physical phenomena described at a microscopic level by equations with highly oscillatory coefficients. This theory has motivated the development of various notions of weak convergence in analysis, see in particular Tartar [18]. The large deviation principle (LDP, in short) in probability theory concerns the asymptotic behavior of tails following the law of probabilities. It formalizes the heuristic ideas of the concentration of measures and generalizes the notion of convergence in law. The pioneering works on this topic were developed by Donsker and Varadhan in ([6-7]). After that, this theory became an active and important topic in probability theory, and has rightly received considerable attention. For these reasons, interesting works for general theory with important applications have been provided in scientific literature. We can mention, among others, the books of Varadhan [21], Stroock [16] and Dembo and Zeitouni [5]. Let $X^{x,\varepsilon,\delta}$ be a diffusion process started at x which moves in the interior of a domain $\boldsymbol{D} \subseteq \mathbb{R}^d$ according to the differential generator, in (1) below, indexed by two parameters $\varepsilon > 0$ and $\delta > 0$, and on

reaching the smooth boundary $\partial\boldsymbol{D}$ of $\boldsymbol{D}$ is reflected instantaneously in the direction γ, where γ is a smooth vector field directed strictly into the interior of $\boldsymbol{D}$. The combined effects of homogenization and LDP in a behavior solution SDE with Reflecting boundary condition consists in computing the limit as $\varepsilon, \delta \to 0$ of $\varepsilon \log \mathsf{P}\{X^{x,\varepsilon,\delta} \in B\}$, where B is a Borel subset of $C_x([0,T], \overline{\boldsymbol{D}})$, the set of continuous functions on $[0,T]$ with $\overline{\boldsymbol{D}}$-values which take x at zero.
This problem has been studied for diffusion processes in the case of $\boldsymbol{D} = \mathsf{R}^d$ and $\partial\boldsymbol{D} = \emptyset$ by [3]. In [9], the authors generalize [5]'s work and have focused on LDP to SDE depending on two parameters in order to explicit the rate function and as applications to wave-front propagation by comparing the relative rate between the parameters (three regimes). Next [4], we established LDP occurring in Homogenization theory. This result can be seen as a generalization of the coupling of LDP and homogenization (first regime) studied in [9] in the cases of SDE with reflection at the edge. In this work we suppose that the homogenization parameter converges (to zero) faster than the parameter of viscosity LDP. The difficulty encountered in this case is to calculate the logarithm of the moment of order n of the local time at the edge of the domain. The aim of this article is to pursue the same program for Neumann-type boundary condition thanks to some early works on SDE with reflecting condition [4]. Here we consider only the second regime, [9], when the two parameters δ(homogenization parameter) and ε(the large deviations parameters) tend to zero with the same speed. Indeed in this case the logarithm moment generating function of $X^{x,\varepsilon,\delta}$, noted $g^{\varepsilon}_{T,x}$ below,may not have and explicit value limit. Thanks to [2],we overcome this challenge by using the superior and inferior limits of the function $g^{\varepsilon}_{T,x}$.

1.1 **Assumption and definition**

Let $(\Omega, \mathsf{F}, \mathsf{P})$ be a probability space on which a d-dimensional Brownian motion $(B^1, \cdots, B^d)$ is defined $\langle .,. \rangle$ is the standard euclidean inner product on $\mathbb{R}^d$; let $\|.\|$ be the associated norm. Also let $\boldsymbol{T}^d$ be the d-dimensional torus of size 1 and $C^\infty(E,F)$ be the space of absolute continuous mapping from E to F; let $\|.\|_{C^\infty(E,F)}$ be associated supremum norm. Also we define P^d as the collection of all probability measure on $\boldsymbol{T}^d$.We consider a new defined parameter $\delta = \delta_\varepsilon$

Definition 1.1. Let $X^{x,\varepsilon,\delta_\varepsilon}$ be a $\overline{\boldsymbol{D}}$-valued random variable and let $\mathsf{P}_{\varepsilon,\delta_\varepsilon}$ denote its distribution on the Borel subsets of $\overline{\boldsymbol{D}}$, that is, $\mathsf{P}_{\varepsilon,\delta_\varepsilon}(A) = \mathsf{P}(X^{x,\varepsilon,\delta_\varepsilon} \in A)$. The family $\{X^{x,\varepsilon,\delta_\varepsilon}; \varepsilon \geqslant 0\}$ satisfies a *large deviation principle* (LDP) if there exists a lower semicontinuous function $I : \overline{\boldsymbol{D}} \longrightarrow [0, +\infty]$ such that

The paper is organized as follows. In section 2, after this introduction, we recall some adaptations by coupling homogenization and LDP for a reflected diffusion process with periodic coefficients. In section 3, we give the main result on this LDP.

2. Statement of the Problem

Here, we state some analogue adaptations from [17] which we will use in the rest of the paper. Let$\left\{\boldsymbol{D} = (x_1, \ldots, x_d) \in \mathbb{R}^d,\ x_1 > 0\right\}$ and let

$$L_{\varepsilon,\delta_\varepsilon} = \frac{\varepsilon}{2}\sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial^2}{\partial x_i \partial x_j} + \frac{\varepsilon}{\delta_\varepsilon}\sum_{i=1}^{d} b_i\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial}{\partial x_i} + \sum_{i=1}^{d} c_i\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial}{\partial x_i}, \quad x \in \boldsymbol{D}, \tag{1}$$

$$\Gamma_{\delta_\varepsilon} = \sum_{i=1}^{d} \gamma_i\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial}{\partial x_i},\ x \in \boldsymbol{\partial D}, \tag{2}$$

be given (with $\varepsilon > 0$). We also set

$$L_\lambda = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i + \lambda c_i)(x),\ x \in \mathbb{R}^d, \tag{3}$$

and require the following :

(H1) L is uniformly elliptic and the matrix $a(x) = (a_{ij}(x))$ can be factored as $\sigma^t(x)\sigma(x)$.

(H2) The functions $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \to \mathbb{R}^d$, $c : \mathbb{R}^d \to \mathbb{R}^d$ and $\gamma : (\boldsymbol{\partial D} \approx \mathbb{R}^{d-1}) \to \mathbb{R}^d$ are smooth and periodic of period 1 in each variable, and $\gamma_1(x) = 1$.

The differential operator inside $\boldsymbol{D}$ together with the boundary condition $\Gamma_{\delta_\varepsilon} u = 0$ on $\boldsymbol{\partial D}$ determines a unique diffusion process $X^{x,\varepsilon,\delta_\varepsilon}$ in $\overline{\boldsymbol{D}}$ which called $(L_{\varepsilon,\delta_\varepsilon}, \Gamma_{\delta_\varepsilon})$-diffusion, such a process $X^{x,\varepsilon,\delta_\varepsilon}$ with its local time $\phi^{\varepsilon,\delta_\varepsilon}$ on $\boldsymbol{\partial D}$ according to [17] satisfies the following SDE in $\overline{\boldsymbol{D}}$:

$$X_t^{x,\varepsilon,\delta_\varepsilon} - x = \sqrt{\varepsilon}\int_0^t \sigma\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)dB_s + \int_0^t \alpha\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds + \int_0^t \gamma\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)d\phi_s^{\varepsilon,\delta_\varepsilon};$$

the drift α is defined as $\alpha\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) = \frac{\varepsilon}{\delta_\varepsilon} b\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) + c\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)$;

$X_t^{1,x,\varepsilon,\delta_\varepsilon} \geq 0,\ \phi_t^{\varepsilon,\delta_\varepsilon}$ is continuous and increasing, $\phi_0 = 0$, and

$$\int_0^t X_s^{1,x,\varepsilon,\delta_\varepsilon}\, d\phi_s^{\varepsilon,\delta_\varepsilon} = 0, t > 0,$$

where $X^{1,x,\varepsilon,\delta_\varepsilon}$ denotes the first component of $X^{x,\varepsilon,\delta_\varepsilon}$. We recall that $\boldsymbol{D} = \mathbb{R}_+^* \times \mathbb{R}^{d-1}$, so that $X^{x,\varepsilon,\delta}$ lives is $\overline{\boldsymbol{D}}$, that is, $X^{1,x,\varepsilon,\delta_\varepsilon}$ remains non-negative and $\phi^{\varepsilon,\delta_\varepsilon}$ increases when and only when $X^{1,x,\varepsilon,\delta_\varepsilon}$ is zero.

Let $\hat{X}$ denote the unique diffusion with values in the d-dimensional torus T^d, whose generator is L_λ. It is well known that $\hat{X}$ is ergodic. We denote by μ its unique invariant measure. In order for the diffusion $X^{x,\varepsilon,\delta_\varepsilon}$ to satisfy an LDP, we need the following be in force. Now we set

$$\bar{a} := \mathop{inf}_{\phi\in C^\infty(\mathsf{T}^d)} \sup_{\mu\in \mathsf{P}(\mathsf{T}^d)} \int_{\mathsf{T}^d} (I + \nabla\phi)^*(x) a(x)\ (I + \nabla\phi)(x)\ \mu(dx),$$

$$\bar{c} := \inf_{\phi \in C^\infty(\mathsf{T}^d)} \sup_{\mu \in \mathsf{P}(\mathsf{T}^d)} \int_{\mathsf{T}^d} \Big[c + \frac{1}{\lambda}(b + L_\lambda \phi) \Big](x)\ \mu(dx).$$

Then we write $u = \mathsf{H}\varphi$ for the solution u of :

$$\begin{cases} L_\lambda u = 0 \quad \text{in } \boldsymbol{D} \\ u = \varphi \quad \text{on } \partial \boldsymbol{D}. \end{cases} \tag{4}$$

Then H sends functions defined on $\partial\boldsymbol{D}$ to functions defined on $\overline{\boldsymbol{D}}$, while $\Gamma\mathsf{H}$ sends functions defined on $\partial\boldsymbol{D}$ to functions on $\partial\boldsymbol{D}$, where $\Gamma = \sum_{i=1}^d \gamma_i(x)\frac{\partial}{\partial x_i}$. There exists a unique Markov process on $\partial\boldsymbol{D}$ with generator $\Gamma\mathsf{H}$. By the periodicity assumption, this process induce a Markov process on $\boldsymbol{T}^{d-1}$; let $\tilde{\mu}$ be the invariant measure of the induced Markov process. And we set

$$\overline{\gamma} := \int_{\mathsf{T}^{d-1}} (I + \nabla\phi)(x)\gamma(x)\tilde{\mu}(dx).$$

We consider the $\boldsymbol{T}^d$-values pull-back $\hat{X}_t^{\varepsilon,\delta_\varepsilon} := \frac{1}{\delta_\varepsilon} X^{x,\varepsilon,\delta_\varepsilon}_{\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2 t}$ which satisfies the following reflecting SDE :

$$\hat{X}_t^{x,\varepsilon,\delta_\varepsilon} - \frac{x}{\delta_\varepsilon} = \int_0^t \sigma\Big(\hat{X}_s^{x,\varepsilon,\delta_\varepsilon}\Big) dB_s^{\varepsilon,\delta_\varepsilon} + \int_0^t \overline{\alpha}\Big(\hat{X}_s^{x,\varepsilon,\delta_\varepsilon}\Big) ds$$

$$+ \int_0^t \gamma\Big(\hat{X}_s^{x,\varepsilon,\delta_\varepsilon}\Big) d\,\widehat{\phi}_s^{\varepsilon,\delta_\varepsilon};$$

the drift $\overline{\alpha}$ is defined as $\overline{\alpha}\Big(\hat{X}_t^{x,\varepsilon,\delta_\varepsilon}\Big) = \frac{\varepsilon}{\delta_\varepsilon} b\Big(\hat{X}_t^{x,\varepsilon,\delta_\varepsilon}\Big) + \frac{\delta_\varepsilon}{\varepsilon} c\Big(\hat{X}_t^{x,\varepsilon,\delta_\varepsilon}\Big)$;

$\hat{X}_t^{1,x,\varepsilon,\delta_\varepsilon} \geq 0$, $\widehat{\phi}_s^{\varepsilon,\delta_\varepsilon} := \frac{1}{\delta_\varepsilon} X^{x,\varepsilon,\delta_\varepsilon}_{\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2 t}$ is continuous and increasing, and

$$\int_0^t \hat{X}_s^{1,x,\varepsilon,\delta_\varepsilon}\, d\,\widehat{\phi}_s^{\varepsilon,\delta_\varepsilon} = 0;$$

and with the generator inside the interior of $\boldsymbol{D}$ is given by

$$\overline{L}_{\varepsilon,\delta_\varepsilon} := \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i} + \frac{\delta_\varepsilon}{\varepsilon}\sum_{i=1}^d c_i(x)\frac{\partial}{\partial x_i},\ x \in \boldsymbol{T}^d.$$

Let $\mu_{\varepsilon,\delta_\varepsilon}$ be the unique invariant measure of $\hat{X}^{\varepsilon,\delta_\varepsilon}$. Our main hypothesis is

(H3) $\lim_{\varepsilon \to 0} \frac{\boldsymbol{\delta_\varepsilon}}{\boldsymbol{\varepsilon}} = \lambda$.

Then we notice that : $\overline{L}_{\varepsilon,\delta_\varepsilon} \to L_\lambda$ and $\mu_{\varepsilon,\delta_\varepsilon} \to \mu_\lambda$ when ε tends to zero, and according to [9] we may state the lemma that follows.

Lemma 2.1. *Under (H.1), (H.2), (H.3) the* $(\overline{L}_{\varepsilon,\delta_\varepsilon}, \Gamma_{\delta_\varepsilon})$*-diffusion* $\hat{X}^{\varepsilon,\delta_\varepsilon}$ *converges in law to the* (L_0, Γ_0)*-diffusion* X *as* $\varepsilon \downarrow 0$. *Moreover on the space* $C([0,T], \boldsymbol{T}^{2d+1})$ *equipped with the sup-norm topology,*

$$\Big(\hat{X}^{\varepsilon,\delta_\varepsilon}, M^{\hat{X}^{\varepsilon,\delta_\varepsilon}}, \hat{\phi}^{\varepsilon,\delta_\varepsilon}\Big) \to (X, M^X, \phi)$$

where

- $M_t^{\hat{X}^{\varepsilon,\delta_\varepsilon}} = \int_0^t \left(I + \nabla \hat{b}\right)\sigma\left(\hat{X}_s^{\varepsilon,\delta_\varepsilon}\right) dB_s^{\varepsilon,\delta_\varepsilon}$,
- M^X *is the martingale part of X and* ϕ *(resp.* $\hat{\phi}^{\varepsilon,\delta_\varepsilon}$*) is the local time of X (resp.* $\hat{X}^{\varepsilon,\delta_\varepsilon}$*).*

3. Large Deviation Principle

The aim of this section is to prove that the family of $(\overline{L}_{\varepsilon,\delta_\varepsilon}, \Gamma_{\delta_\varepsilon})$-diffusion $\hat{X}^{\varepsilon,\delta_\varepsilon}$ has a large deviations principle for details of the calculation of the logarithm of the local n-order time see [4]. Let $y \in \mathbb{R}^d$, we denote by $\mathbf{1^{st}}(y)$ the transposed d-dimensional vector $(y_1, 0, \dots, 0)$. Let us define, for each $T > 0$ and $x \in \mathbb{R}^d$

$$g^{\varepsilon}_{T,x}(\theta) = \varepsilon \log \mathsf{E}\left[\exp\left(\frac{1}{\varepsilon}\left\langle \theta, X_T^{x,\varepsilon,\delta_\varepsilon}\right\rangle\right)\right] \quad \varepsilon > 0\,,\ \theta \in \mathbb{R}^d.$$

$$\bar{\mathsf{J}}_\lambda(\theta) := \frac{1}{2}\langle \theta, \bar{a}\theta\rangle + \langle \bar{c} + (\det \bar{a})^{-1}\bar{\gamma}_1, \theta\rangle$$

$$g_\lambda^{T,x}(\theta) = \lim_{\varepsilon\to 0} g^{\varepsilon}_{T,x}(\theta) = \lim_{\varepsilon\to 0} \varepsilon \log \mathsf{E}\left[\exp\left(\frac{1}{\varepsilon}\left\langle \theta, X_T^{x,\varepsilon,\delta_\varepsilon}\right\rangle\right)\right]$$

exists uniformly with respect to $x \in \mathbb{R}^d$.

$$g_\lambda^{T,x}(\theta) = \langle x, \theta\rangle + T\bar{\mathsf{J}}(\theta)$$

Theorem 3.1. (Main result) *For all* $T > 0$,*we assume that the hypothesis* $(H.1)$,$(H.2)$ *and* $(H.3)$ *hold true. Then we have*

$$\lim_{\varepsilon\to 0} \inf g^{\varepsilon}_{T,x}(\theta) \geq \langle x, \theta\rangle + T(\bar{\mathsf{J}}_\lambda(\theta) + \beta_1(\theta)),$$
$$\lim_{\varepsilon\to 0} \sup g^{\varepsilon}_{T,x}(\theta) \leq \langle x, \theta\rangle + T(\bar{\mathsf{J}}_\lambda(\theta) + \beta_2(\theta)),$$

where

$$\beta_1(\theta) = \frac{\lambda^2}{2} \inf_{\psi\in C^\infty(\mathbf{T}^d)} \left\{\int_{\mathbf{T}^d} \|\sigma^*(x)\nabla\psi(x)\|\mu_\lambda(x)\right\},$$
$$\beta_2(\theta) = \frac{\lambda^2}{2} \inf_{\psi\in C^\infty(\mathbf{T}^d)} \sup_{\mu\in \mathsf{P}(\mathbf{T}^d)} \left\{\int_{\mathbf{T}^d} \|\sigma^*(x)\nabla\psi(x)\|\mu(x)\right\}.$$

Finally, set

$$I_\lambda^{T,x}(\eta) = \sup_\theta\left\{\langle \eta, \theta\rangle - g_\lambda^{T,x}(\theta)\right\} = \sup_\theta\{\langle \eta - x, \theta\rangle - T\mathsf{J}_\lambda(\theta)\} = T\mathsf{J}\left(\frac{\eta - x}{T}\right),$$

Then for every set T on $B(\mathbb{R}^d)$ *,*

$$\lim_{\varepsilon\to 0}\inf \varepsilon \log \mathsf{P}(X^{x,\varepsilon,\delta_\varepsilon} \in \mathsf{T}) \geq -\inf_{\eta\in \mathring{\boldsymbol{T}}} I_\lambda^{T,x}(\eta).$$

$$\lim_{\varepsilon\to 0}\sup \varepsilon \log \mathsf{P}(X^{x,\varepsilon,\delta_\varepsilon} \in \mathsf{T}) \leq -\inf_{\eta\in \bar{\mathsf{T}}} I_\lambda^{T,x}(\eta).$$

Proof. Thanks to Baxendale & Stroock [2], we characterize the LDP with inequality of deviation on the logarithm moment generating function.
$*$*Step*1 :
Let $\phi \in C^\infty(\overline{\boldsymbol{D}})$ be a bounded, smooth and $\mathbb{R}^d$-values funtions. Let us set $v : \overline{\boldsymbol{D}} \longrightarrow \mathbb{R}^d$ solution of the system (1) such that $Lv = \overline{\gamma} - (I + \nabla\phi)\gamma$. By the Itô formula,we have

$$\hat{X}_t^{\varepsilon,\delta_\varepsilon} = X_t^{x,\varepsilon,\delta_\varepsilon} + \delta\varepsilon\left[\phi\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) - \phi\left(\frac{x}{\delta_\varepsilon}\right)\right] + \delta\varepsilon\left[v\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) - v\left(\frac{x}{\delta_\varepsilon}\right)\right]$$

$$\underbrace{-\frac{\varepsilon}{\delta\varepsilon}\left(\frac{\delta\varepsilon}{\varepsilon} - \lambda\right)\int_0^t \langle\nabla(\phi+v), c\rangle\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds - \sqrt{\varepsilon}\int_0^t \langle\nabla v,\sigma\rangle\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)dBs}_{K_t^{\varepsilon,\delta_\varepsilon}} \tag{5}$$

$$= x + \int_0^t\left[c + \frac{1}{\lambda}(b + L_\lambda\phi)\right]\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds + \overline{\gamma}\int_0^t \mathbf{1}_{\left\{X_s^{x,\varepsilon,\delta_\varepsilon}\in\partial\boldsymbol{D}\right\}}d\phi_S^{\varepsilon,\delta_\varepsilon} + \sqrt{\varepsilon}\, M_t^{\frac{1}{\delta_\varepsilon}X_t^{x,\varepsilon,\delta_\varepsilon}}.$$

Since $\int_{\boldsymbol{T}^{d-1}} \Gamma\varphi(x)\tilde{\mu}(dx) = 0$ then one can easy show that

$$\lim_{\varepsilon\to 0} \mathsf{E}\left\{\max_{0\leq t\leq T}\left\|\delta\varepsilon\left[v\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) - v\left(\frac{x}{\delta_\varepsilon}\right)\right] + K_t^{\varepsilon,\delta_\varepsilon}\right\|\right\} = 0. \tag{6}$$

From (6) and with the fact that all functions and coefficients $\phi, v, \nabla v, \nabla\phi$ are regulars, we deduct that $X^{x,\varepsilon,\delta_\varepsilon}$ and $\hat{X}^{\varepsilon,\delta_\varepsilon}$ are indistinguishable.
$*$*Step*2 :
Fix θ and introduce the logarithm moment generating function $g_t^{\varepsilon,\delta_\varepsilon}$ defined on $\mathbb{R}^d$ with $\mathbb{R}$-values:

$$g_t^{\varepsilon,\delta_\varepsilon}(\theta) = \varepsilon\log\mathsf{E}\left\{\exp\left(\frac{1}{\varepsilon}\left\langle\theta, \hat{X}_t^{\varepsilon,\delta_\varepsilon}\right\rangle\right)\right\}$$
$$= (\theta, x) + \varepsilon\log\mathsf{E}\left\{\exp\left(\frac{1}{\varepsilon}\langle\theta,\ \int_0^t\left[c + \frac{1}{\lambda}(b + L_\lambda\phi)\right]\left(\frac{X_S^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right.\right.$$
$$\left.\left.+\overline{\gamma}\int_0^t \mathbf{1}_{\left\{X_s^{x,\varepsilon,\delta_\varepsilon}\in\partial\boldsymbol{D}\right\}}d\phi_S^{\varepsilon,\delta_\varepsilon} + \sqrt{\varepsilon}\, M_t^{\frac{1}{\delta_\varepsilon}X_t^{x,\varepsilon,\delta_\varepsilon}}\right\rangle\right)\right\}. \tag{7}$$

By Girsanov formula, we have

$$g_t^{\varepsilon,\delta_\varepsilon}(\theta) = \langle\theta, x\rangle + \varepsilon\hat{\mathsf{E}}\left\{\exp\left(\frac{1}{\varepsilon}\left\langle\theta, \int_0^t\left[c + \frac{1}{\lambda}(b + L_\lambda\phi)\right]\left(\frac{X_S^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right\rangle\right.\right.$$
$$\left.\left.+\frac{1}{\varepsilon}\left\langle\theta, \overline{\gamma}\int_0^t \mathbf{1}_{\left\{X_s^{x,\varepsilon,\delta_\varepsilon}\in\partial\boldsymbol{D}\right\}}d\phi_S^{\varepsilon,\delta_\varepsilon}\right\rangle + \frac{1}{2\varepsilon}\left\langle\theta,\ M_t^{\frac{1}{\delta_\varepsilon}X_t^{x,\varepsilon,\delta_\varepsilon}}\theta\right\rangle_t\right)\right\}. \tag{8}$$

where

$$\frac{d\widehat{\mathsf{P}}}{d\,\mathsf{P}} := \exp\left(\frac{1}{\sqrt{\varepsilon}}\left\langle\theta,\ M_t^{\frac{1}{\delta_\varepsilon}X_t}{}^{x,\varepsilon,\delta_\varepsilon}\right\rangle - \frac{1}{2\varepsilon}\left\langle\theta,\left\langle M_t^{\frac{1}{\delta_\varepsilon}X_t}{}^{x,\varepsilon,\delta_\varepsilon}\right\rangle_t\theta\right\rangle\right),$$

with $\left\langle M_t^{\frac{1}{\delta_\varepsilon}X_t}{}^{x,\varepsilon,\delta_\varepsilon}\right\rangle_t$ denoting the quadratic variation of $M_t^{\frac{1}{\delta_\varepsilon}X_t}{}^{x,\varepsilon,\delta_\varepsilon}$.

∗ *Step 3*:

Set

$$\Im(z,\theta) := \left\langle\theta,\left[c+\frac{1}{\lambda}(b+L_\lambda\phi)\right](z)\right\rangle + \frac{1}{2}\left\langle\theta,\left\langle M^X_{\phi(z)}\right\rangle\theta\right\rangle,\ z\in\overline{\boldsymbol{D}},$$

and consider the solution of the following Poisson equation, with zero integral against the measure μ_λ : $L\psi = \overline{\Im} - \Im$, with

$$\overline{\Im}(\theta) := \int_{\boldsymbol{T}^d}\Im(x,\theta)\mu_\lambda(dx), \tag{9}$$

such that $\int_{\boldsymbol{T}^d}\psi(x)\mu(dx) = 0$, and

$$\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2\left[\psi\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)-\psi\left(\frac{x}{\delta_\varepsilon}\right)\right] = \beta_t^{\varepsilon,\delta_\varepsilon} + \int_0^t L_\lambda\psi\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds, \tag{10}$$

where

$$\beta_t^{\varepsilon,\delta_\varepsilon} := \left[\left(\frac{\delta_\varepsilon}{\varepsilon}-\lambda\right)\int_0^t\langle\nabla\psi,c\rangle\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds + \frac{\delta_\varepsilon}{\varepsilon}\int_0^t\langle\nabla\psi,\gamma\rangle\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)\mathbf{1}_{\left\{X_s^{x,\varepsilon,\delta_\varepsilon}\in\partial\boldsymbol{D}\right\}}d\phi_s^{\varepsilon,\delta_\varepsilon}\right.$$
$$\left.+\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\int_0^t\langle\nabla\psi,\sigma\rangle\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)dB_s\right]. \tag{11}$$

From (8) we have, taking into account to (10) and (11),

$$g_t^{\varepsilon,\delta_\varepsilon}(\theta) = \langle x,\theta\rangle + t\langle\overline{\Im},\theta\rangle + \varepsilon\log\hat{\mathsf{E}}\left\{\exp\left(\frac{1}{\varepsilon}\left\langle\theta,\overline{\gamma}\int_0^t\mathbf{1}_{\left\{X_s^{x,\varepsilon,\delta_\varepsilon}\in\partial\boldsymbol{D}\right\}}d\phi_s^{\varepsilon,\delta_\varepsilon}+\beta_t^{\varepsilon,\delta_\varepsilon}\right\rangle\right.\right.$$
.
$$\left.\left.-\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left[\psi\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)-\psi\left(\frac{x}{\delta_\varepsilon}\right)\right]\right)\right\}. \tag{12}$$

Since $\int_0^t X_s^{1,x,\varepsilon,\delta_\varepsilon}d\phi_s^{\varepsilon,\delta_\varepsilon} = 0$, then the measures $d\phi_t^{\varepsilon,\delta_\varepsilon}$ is *a.s.* carried by the set $\left\{t : X_t^{1,x,\varepsilon,\delta_\varepsilon} = 0\right\}$, *i.e.* this set is the support of the local times. From this we have

$$g_t^{\varepsilon,\delta_\varepsilon}(\theta) = \langle x,\theta\rangle + t\langle\overline{\mathfrak{I}},\theta\rangle + \varepsilon\log\hat{\mathsf{E}}\left\{\exp\left(\frac{1}{\varepsilon}\left\langle\theta,\overline{\gamma}\int_0^{\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2 t}\mathbf{1}_{\left\{X_s^{1,x,\varepsilon,\delta_\varepsilon}=0\right\}}d\phi_S^{\varepsilon,\delta_\varepsilon}\right\rangle\right)\right\}$$
$$+\varepsilon\log\hat{\mathsf{E}}\left\{\exp\left(\frac{1}{\varepsilon}\left\langle\theta,\beta_t^{\varepsilon,\delta_\varepsilon}\right\rangle - \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left[\psi\left(\hat{X}_t^{\varepsilon,\delta_\varepsilon}\right)-\psi\left(\frac{x}{\delta_\varepsilon}\right)\right]\right)\right\}. \quad (13)$$

By a change of a variable we get

$$g_t^{\varepsilon,\delta_\varepsilon}(\theta) = \langle x,\theta\rangle + t\langle\overline{\mathfrak{I}},\theta\rangle + \varepsilon\log\hat{\mathsf{E}}\left\{\exp\left(\frac{1}{\varepsilon}\left\langle\theta,\overline{\gamma}\int_0^t\mathbf{1}_{\left\{X_s^{1,x,\varepsilon,\delta_\varepsilon}=0\right\}}d\hat{\phi}_S^{\varepsilon,\delta_\varepsilon}\right\rangle\right)\right\}$$
$$+\varepsilon\log\hat{\mathsf{E}}\left\{\exp\left(\frac{1}{\varepsilon}\left\langle\theta,\beta_t^{\varepsilon,\delta_\varepsilon}\right\rangle - \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left[\psi\left(\hat{X}_t^{\varepsilon,\delta_\varepsilon}\right)-\psi\left(\frac{x}{\delta_\varepsilon}\right)\right]\right)\right\}. \quad (14)$$

$*$ *Step 4*:
First, we prove a result on the tail asymptotic of the local time ϕ (see [4], [17]). Hence, for ε small enough we observe that

$$\varepsilon\log\hat{\mathsf{E}}\left\{\exp\left(\frac{1}{\varepsilon}\left\langle\theta,\overline{\gamma}\int_0^t\mathbf{1}_{\{X_s^1=0\}}d\phi s\right\rangle\right)\right\} = \varepsilon\log\hat{\mathsf{E}}\left\{\exp\left(\frac{1}{\varepsilon}\langle\theta_1,\overline{\gamma}_1\rangle\phi_t\right)\right\}$$
$$= \frac{t\langle\theta_1,\overline{\gamma}_1\rangle}{\det(\overline{a})} + o(1). \quad (15)$$

Second, one can easily show there exists $K > 0$ such that

$$\left(\frac{\delta_\varepsilon}{\varepsilon}-\lambda\right)\left\|\int_0^t\langle\nabla\psi,c\rangle\left(\hat{X}_S^{x,\varepsilon,\delta_\varepsilon}\right)ds\right\| + \left\|\frac{\delta_\varepsilon}{\varepsilon}\int_0^t\langle\nabla\psi,\gamma\rangle\left(\hat{X}_S^{x,\varepsilon,\delta_\varepsilon}\right)d\hat{\phi}_S^{\varepsilon,\delta_\varepsilon}\right\|$$
$$+\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left\|\psi\left(\hat{X}_t^{\varepsilon,\delta_\varepsilon}\right)-\psi\left(\frac{x}{\delta_\varepsilon}\right)\right\| \le \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2 K. \quad (16)$$

Third, we also have that

$$\varepsilon\left\|\log\hat{\mathsf{E}}\left\{\exp\frac{1}{\varepsilon}\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\int_0^t\langle\nabla\psi,\sigma\rangle\left(\hat{X}_S^{x,\varepsilon,\delta_\varepsilon}\right)dB_S^{\varepsilon,\delta_\varepsilon}\right)\right\}\right\|$$
$$\le\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\frac{t}{2}\|\sigma\nabla\psi\|_{C^\infty(\boldsymbol{T})}. \quad (17)$$

From (15), (16), (17) and in view of (14) we get

$$t\beta_1 + \bar{\mathsf{J}}_\lambda(\theta) \le \lim_{\varepsilon\to 0} g^{\delta_\varepsilon}(\theta) \le \bar{\mathsf{J}}_\lambda(\theta) + t\beta_2 .$$

These estimates provide the justification for the final assertion of this theorem. ■

We also have the following tail condition.

Remark 3.1. For any fixed $T > 0$, $x \in \boldsymbol{D}$ and $\alpha \in (0,\frac{1}{2})$,

$$\limsup_{\beta\to\infty}\ \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\log\mathsf{P}\left\{\left\|X_s^{x,\varepsilon,\delta_\varepsilon}\right\|_{C^\alpha([0,T],\boldsymbol{D})}\ge\beta\right\} = -\infty,$$

where $C^{\alpha}([0,T],\boldsymbol{D})$ is the space of Hölder continuous functions of exponent α from $[0,T]$ to $\boldsymbol{D}$.

4. Conclusion

We have used the general expression of a function for a family of stochastic equations, with a first-order approximation at the edge of the domain, to generalize the work of [9] with respect to regime 2. Since we are interested in the process trajectory with probability measures, such results may have implications, for both random optimization and statistics, when studying trajectories for density estimates or regression problems [10], [19].

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