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On Closed-Form Solutions to Integro-Differential Equations

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Abstract. *This paper presents an iterative technique based on homotopy analysis method for solving system of Volterra integro-differential equations. The technique provides us series solutions to the problems which are combined with the diagonal Padé approximants and Laplace transform to obtain closed-form solutions. The technique is effectively applied on system of linear and nonlinear Volterra integro-differential equations which eventually yield closed-form solutions of the problems and this technique is also extended to boundary value problem for the integro-differential equation related to Blasius problem. An interesting comparison of the present solution is made with solutions of other methods and it is observed that the results are in excellent agreement with other methods in the literature.*

Key words: Homotopy Analysis Method, Padé Approximants, Laplace Transforms, Blasius Equation, Integro-Differential Equation.

AMS Subject Classifications: 45D05, 45G10, 65D99, 65L05

1. Introduction

Integro-differential equations have been a subject of interest for mathematicians and researchers due to their applications in the mathematical modeling of different physical phenomena. These equations usually arise in electromagnetic, plasma physics, elasticity, fluid dynamics, oscillation theory, polymer rheology, chemical kinetics, bio-mechanics, control theory, etc. The theorems on existence and uniqueness of solution for boundary value problems of ordinary differential equations and integro-differential equation can be found in [2, 3]. Despite the usefulness of integro-differential equations in modeling many real world problems in science and engineering, the so-called closed-form/analytical solutions in most instances are difficult to achieve. For this reason, several numerical and analytical methods for the solutions of integro-differential equations have been studied and proposed by researchers as detailed in

[6-11, 14, 22, 27, 29, 32]. Other methods that have been used for solving integro-differential equations include variational iteration method (VIM) [16], [30], homotopy perturbation method (HPM) [17], [31], spline functions expansion method (SPEM) [18], [23], Adomian decomposition method (ADM) [21], homotopy analysis method (HAM) [28], etc.

The homotopy analysis method has been applied to solve a large class of linear and nonlinear differential equations. This is an analytical technique which uses the concept of homotopy from topology to generate a convergent series solution for mathematical problems. The technique is superior to the traditional perturbation methods in that it leads to convergent series solutions of strongly nonlinear problems, independent of any small or large physical parameter. The method was first introduced in 1992 by Liao [19, 20] and several modifications of the method have been reported in the literatures with different names such as spectral homotopy analysis method (SHAM) [24], predictor homotopy analysis method (PHAM) [1], tau homotopy analysis method (THAM) [15], multi-stage homotopy analysis method (MHAM) [13], improved spectral homotopy analysis method (ISHAM) [25], etc.

The purpose of this paper therefore, is to apply homotopy analysis method coupled with Padé approximants to obtain closed-form solutions of the following system of integro-differential equations

$$u^{(p)}(x) = f_j(x, u_1(x), u_2(x), \cdots u_n(x)) + \int_0^x K_j(t, u_1(t), u_2(t), \cdots u_n(t))dt, \tag{1}$$

where $j = 1, \cdots, n$ and $u^{(p)}(x)$ indicates the pth derivative of $u(x)$. The functions $f_j(x, u_1(x), u_2(x), \cdots u_n(x))$ and $u_1(x), u_2(x), \cdots u_n(x)$ are given real valued and unknown functions, respectively.

The remainder of this paper is organized as follows: In Section 2, we give a brief description of homotopy analysis method and Padé approximants. Our proposed method of solution is presented in Section 3. Numerical illustrations and extension of the method to Blasius equation are presented in Section 4. Finally, concluding remark is given in Section 5.

2. Basic Features of Homotopy Analysis and Padé Approximation Methods

In this section, we present some basic ideas of the homotopy analysis method and Padé approximants which are very useful in the next section of the paper.

2.1. **Basic idea of the homotopy analysis method**

The homotopy analysis method was proposed by means of homotopy which is a fundamental concept of topology. It is dependent on four factors, namely, initial approximation u_0, auxiliary linear operator L, a non-zero auxiliary function $\mathsf{H}(x)$ and a nonzero convergence-control parameter, $\hbar$. Within the framework of homotopy analysis method, we have great freedom to choose all these parameters but their choices must conform to some prescribed rules or conditions, (see [20]).

Consider a nonlinear ordinary differential equation of the form

$$\mathsf{N}[u(x)] = 0, \quad x \in \Omega, \tag{2}$$

where N is a nonlinear operator and $u(x)$ is an unknown function.

The zeroth order deformation equation is computed by setting

$$(1-q)\mathsf{L}[\phi(x;q)-u_0(x)] = q\hbar \mathsf{H}(x)\mathsf{N}[\phi(x;q)], \quad q \in [0,1], \tag{3}$$

where $\phi(x;q)$ is an unknown mapping function, q and $\hbar$ are respectively embedding and convergence-control parameters, while $\mathsf{H}(x), \mathsf{L}, u_0(x)$ are auxiliary function, auxiliary linear operator and initial guess to the solution $u(x)$ respectively.
At $q = 0$ and $q = 1$, we have

$$\phi(x;0) = u_0(x), \qquad \phi(x;1) = u(x). \tag{4}$$

As the embedding parameter $q \in [0,1]$ increases from 0 to 1, the solution $u(x;q)$ of the zeroth order deforms from initial guess $u_0(x)$ to the exact solution $u(x)$. Expanding $u(x;q)$ using Taylor's series with respect to the embedding parameter q yields

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)q^m, \tag{5}$$

where

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x;q)}{\partial q^m}\Bigg|_{q=0}. \tag{6}$$

If the convergence-control parameter, $\hbar$, the auxiliary function, $\mathsf{H}(x)$, the initial guess, $u_0(x)$, and the auxiliary linear operator, L, are properly chosen such that the series (5) converges at $q = 1$, then (5) becomes

$$\phi(x;1) = u_0(x) + \sum_{m=1}^{\infty} u_m(x), \tag{7}$$

and using the fact that $\phi(x;1) = u(x)$, we have

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \tag{8}$$

To obtain higher order deformation equations, we differentiate the zeroth order deformation equation (3)m times with respect to the embedding parameter q, evaluate at $q = 0$ and finally dividing by $m!$, we obtain the so-called *mth* order deformation equation

$$\mathsf{L}[u_m(x) - \chi_m u_{m-1}(x)] = \hbar \mathsf{H}(x) R_m \vec{u}_{m-1}(x), \tag{9}$$

where

$$R_m \vec{u}_{m-1}(x) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathsf{N}[\phi(x;q)]}{\partial q^m}\Bigg|_{q=0}, \tag{10}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1. \end{cases} \tag{11}$$

The solution $u_m(x)$ of higher-order deformation equation can be easily solved by means of computation software such as Maple, Mathematical, Matlab, etc.

2.2. **Padé approximent**

The Padé approximant to an analytic function

$$u(x) = \sum_{n=0}^{\infty} u_n x^n, \qquad 0 \le x \le T, \tag{12}$$

of order $[L/M]$ denoted by $[L/M]_u(x)$ is defined by

$$[L/M]_u(x) = \frac{p_0+p_1x+\cdots+p_Lx^L}{1+q_1x+\cdots+q_M x^M}, \tag{13}$$

where the numerator and denominator have no common factors and we assume $q_0 = 1$. The polynomials $p_0 + p_1x + \cdots + p_Lx^L$ and $1 + q_1x + \cdots + q_Mx^M$ are constructed in such a way that $u(x)$ and $[L/M]_u(x)$ also agree at $x = 0$ and their derivatives up to $L + M$ agree at $x = 0$. That is,

$$u(x) - [L/M]_u(x) = O\left(x^{L+M+1}\right). \tag{14}$$

Thus from (14), we have

$$u(x)\left(\sum_{i=1}^{M} q_i x^i\right) - \left(\sum_{i=0}^{L} p_i x^i\right) = O\left(x^{L+M+1}\right). \tag{15}$$

Therefore, we obtain the following system of equations from (15):

$$\begin{cases} u_L q_1 + \cdots + u_{L+M+1} q_M = -u_{L+1}, \\ u_{L+1} q_1 + \cdots + u_{L+M+2} q_M = -u_{L+2}, \\ \vdots \\ u_{L+M1} q_1 + \cdots + u_L q_M = -u_{L+M} \end{cases} \tag{16}$$

and

$$\begin{cases} p_0 = u_0, \\ p_1 = u_1 + u_0 q_1, \\ \vdots \\ p_L = u_L + u_{L-1} q_1 + \cdots u_0 q_L. \end{cases} \tag{17}$$

The unknown constants $q_i,\ 1 \le i \le M$ and $p_i,\ 1 \le i \le L$ can be determined from (16) and (17), respectively. For more details on Padé approximants, see [4].

3. Description of Homotopy Analysis-Based Iterative Method (HABIM)

For the sake of clarity, we rewrite (1) in the form

$$
\begin{aligned}
u_1^{(p)}(x) &= f_1(x, u_1(x), u_2(x), \cdots, u_n(x)) + \int_0^x K_1(t, u_1(t), u_2(t), \cdots, u_n(t))dt, \\
u_2^{(p)}(x) &= f_2(x, u_1(x), u_2(x), \cdots, u_n(x)) + \int_0^x K_2(t, u_1(t), u_2(t), \cdots, u_n(t))dt, \\
&\vdots \\
u_n^{(p)}(x) &= f_n(x, u_1(x), u_2(x), \cdots, u_n(x)) + \int_0^x K_n(t, u_1(t), u_2(t), \cdots, u_n(t))dt,
\end{aligned} \tag{18}
$$

where K_j's are linear/nonlinear functions of $x, u_1(x), u_2(x), \cdots u_n(x)$ and $u_j^{(p)}$ is the derivative of u_j with order p, subject to the initial conditions

$$u_j^{(k)} = c_k^j \qquad 1 \le j \le n, \quad 1 \le k < p. \tag{19}$$

For convenience, we rewrite (18) as

$$u^{(p)}(x) = \mathbf{N}_j(x, u_1(\quad)(x), u_2(x), \cdots, u_n(x)), \qquad j = 1, 2, \cdots, n, \tag{20}$$

with the zero artificial initial conditions (19), see [12].
Integrating both sides of (20) p times, we obtain

$$u(x) = \int_0^x \int_0^x \cdots \int_0^x \mathbf{N}_j(x, u_1(x), u_2(x), \cdots, u_n(x))\, \mathrm{dxdx}\cdots\mathrm{dx} + \sum_{i=0}^{p-1} \frac{a_{i,j}\, x^i}{p!}. \tag{21}$$

The zeroth-order deformation equation of (21) is given by

$$(1-q)\left(\phi_j(x, q, \hbar) - \sum_{i=0}^{p-1} \frac{a_{i,j}\, x^i}{p!}\right) = \mathsf{H}(x)\, q\hbar\left(u_j(x) - \int_0^x \int_0^x \cdots \right.$$
$$\left. \int_0^x \mathbf{N}_j(x, u_1(x), u_2(x), \cdots, u_n(x))\mathrm{d}x\mathrm{d}x\cdots\mathrm{d}x - \sum_{i=0}^{p-1} \frac{a_{i,j}\, x^i}{p!}\right). \tag{22}$$

For $q = 0$ and $q = 1$, we have

$$\phi_j(x, 0, \hbar) = \sum_{i=0}^{p-1} \frac{a_{i,j}\, x^i}{p!}, \quad \phi_j(x, 1, \hbar) = u_j(x). \tag{23}$$

Considering Maclaurin series of $\phi(x; q, \hbar_j)$ corresponding to q, we have

$$\phi_j(x; q, \hbar_j) = \phi_j(x; 0, \hbar_j) + \sum_{n=1}^{\infty} u_{j,m}(x) q^m. \tag{24}$$

We respectively define the auxiliary function, linear and nonlinear operators as

$$\mathsf{H}_j(x) = 1, \quad \mathsf{L}_j[\phi(x; q)] = \phi_j(x; q), \tag{25}$$

and

$$\mathsf{N}_j[\phi(x;q)] = \phi_j(x) - \int_0^x \int_0^x \cdots \int_0^x \mathbf{N}_j(x, u_1(x), u_2(x), \cdots, u_n(x)) \mathrm{d}x\mathrm{d}x \cdots \cdots \mathrm{d}x - \sum_{i=0}^{p-1} \frac{a_{i,j}\, x^i}{p!}. \quad (26)$$

Thus we obtain the mth-order deformation equation as

$$\mathsf{L}_j[u_{j,m}(x) - \chi u_{j,m-1}(x)] = \hbar \mathsf{R}_j(\vec{u}_{j,m-1}), \quad (27)$$

where

$$\mathsf{R}_j(u_{j,m-1}) = u_{j,m-1}(x) - \int_0^x \int_0^x \cdots \int_0^x \mathbf{N}_{j,m-1}\big(x, u_1(x), u_2(x), \cdots, u_n(x)\big) \mathrm{dxdx} \cdots \mathrm{dx} - \big(1 - \chi_{j,m}\big)\left(\sum_{i=0}^{p-1} \frac{a_{i,j}\, x^i}{p!}\right). \quad (28)$$

The solution of the mth-order deformation equation for $m \geq 1$ becomes

$$u_{j,0}(x) = \sum_{i=0}^{p-1} \frac{a_{i,j}\, x^i}{p!}, \quad (29)$$

and

$$u_{j,m}(x) = \chi u_{j,m-1}(x) + \hbar_j \mathsf{R}_j(u_{j,m-1}). \quad (30)$$

In this way, it is easy to obtain $u_j(x)$ for $m \geq 1$, at mth-order and finally get the series solution of (1) as

$$u_j(x) = u_{j,0}(x) + \sum_{m=1}^{\infty} u_{j,m}(x). \quad (31)$$

The unknown constants of integration in the series solution (31) can be determined by applying the conditions (19).

Thus, the following algorithm summarizes the process of obtaining the closed form of the exact solution of (1):

i Solve (1) by the homotopy analysis method as explained above

ii Truncate the obtained series (31) and determine the unknown constants by imposing the initial conditions (19)

iii Apply Laplace transform to the truncated series

iv Replace s by $\frac{1}{X}$

v Apply Padé approximation to the resulting series

vi Replace x by $\frac{1}{S}$

vii Finally, take the inverse Laplace transform of the Padé approximation.

Remark: It should be noted that the convergence-control parameter, $\hbar_j$, is taken to be -1, i.e., $(\hbar_j = -1)$.

4. Numerical Examples

In this section, some examples are investigated to show the efficiency and reliability of the method.

Example 4.1. We first consider the following system of Volterra integro-differential equation [12]

$$u'x = 2 + e^x - 3e^{2x} + e^{3x} + \int_0^x (6v(t) - 3w(t))\mathrm{d}t,$$
$$v'x = e^x + 2e^{2x} - e^{3x} + \int_0^x (3w(t) - u(t))\mathrm{d}t,$$
$$w'x = -e^x + e^{2x} + 3e^{3x} + \int_0^x (u(t) - 2v(t))\mathrm{d}t, \quad (32)$$

subject to initial conditions

$$u(0) = 1, \quad v(0) = 1, \quad w(0) = 1. \quad (33)$$

The exact solutions are $u(x) = e^x$, $v(x) = e^{2x}$, and $w(x) = e^{3x}$.
Integrating both sides of (32), we obtain

$$u(x) - u(0) = \int_0^x (2 - e^s - 3e^{2s} + e^{3s})ds + \int_0^x \int_0^x (6v(t) - 3w(t))\mathrm{d}t\mathrm{d}x + a_0,$$
$$v(x) - v(0) = \int_0^x (e^s + 2e^{2s} - e^{3s})ds + \int_0^x \int_0^x (3w(t) - u(t))\mathrm{d}t\mathrm{d}x + b_0,$$
$$w(x) - w(0) = \int_0^x (-e^s + e^{2s} + 3e^{3s})ds + \int_0^x \int_0^x (u(t) - 2v(t))\mathrm{d}t\mathrm{d}x + c_0.$$

which yields

$$u(x) = u(0) + \int_0^x (2 - e^s - 3e^{2s} + e^{3s})ds + \int_0^x \int_0^x (6v(t) - 3w(t))\mathrm{d}t\mathrm{d}x + a_0,$$
$$v(x) = v(0) + \int_0^x (e^s + 2e^{2s} - e^{3s})ds + \int_0^x \int_0^x (3w(t) - u(t))\mathrm{d}t\mathrm{d}x + b_0,$$
$$w(x) = w(0) + \int_0^x (-e^s + e^{2s} + 3e^{3s})ds + \int_0^x \int_0^x (u(t) - 2v(t))\mathrm{d}t\mathrm{d}x + c_0.$$

simplifying further yields

$$u(x) = \frac{1}{6} + 2x + e^x - \frac{3}{2}e^{2x} + \frac{1}{3}e^{3x} + \int_0^x \int_0^x (6v(t) - 3w(t))\mathrm{d}t\mathrm{d}x + a_0,$$
$$v(x) = -\frac{5}{3} + e^x + e^{2x} - \frac{1}{3}e^{3x} + \int_0^x \int_0^x (3w(t) - u(t))\mathrm{d}t\mathrm{d}x + b_0,$$
$$w(x) = -\frac{1}{2} - e^x + \frac{1}{2}e^{2x} + e^{3x} + \int_0^x \int_0^x (u(t) - 2v(t))\mathrm{d}t\mathrm{d}x + c_0. \quad (34)$$

We define the nonlinear operators as follows:

$$\begin{aligned}
\mathsf{N}_1[\phi_1(x;q),\phi_2(x;q),\phi_3(x;q)] &= \phi_1(x;q) - \frac{1}{6} - 2x - e^x + \frac{3}{2}e^{2x} - \frac{1}{3}e^{3x} \\
&\quad - \int_0^x\int_0^x(6\phi_2(t;q) - 3\phi_3(t;q))\mathrm{d}t\mathrm{d}x - a_0, \\
\mathsf{N}_2[\phi_1(x;q),\phi_2(x;q),\phi_3(x;q)] &= \phi_2(x;q) + \frac{5}{3} - e^x - e^{2x} + \frac{1}{3}e^{3x} \\
&\quad - \int_0^x\int_0^x(3\phi_3(t;q) - \phi_1(t;q))\mathrm{d}t\mathrm{d}x - b_0, \\
\mathsf{N}_3[\phi_1(x;q),\phi_2(x;q),\phi_3(x;q)] &= \phi_3(x;q) + \frac{1}{2} + e^x - \frac{1}{2}e^{2x} - e^{3x} \\
&\quad - \int_0^x\int_0^x(\phi_1(t;q) - 2\phi_2(t;q))\mathrm{d}t\mathrm{d}x - c_0. \qquad (35)
\end{aligned}$$

By homotopy analysis method, we construct the zeroth-order deformation equations as

$$\begin{aligned}
&(1-q)\ (\phi_1(x,q) - a_0) = \mathsf{H}_1(\mathrm{x})\mathrm{q}\hbar_1\mathsf{N}_1[\phi_1(x;q),\phi_2(x;q),\phi_3(x;q)], \\
&(1-q)\ (\phi_2(x,q) - b_0) = \mathsf{H}_2(\mathrm{x})\mathrm{q}\hbar_2\mathsf{N}_2[\phi_1(x;q),\phi_2(x;q),\phi_3(x;q)], \\
&(1-q)\ (\phi_3(x,q) - b_0) = \mathsf{H}_3(\mathrm{x})\mathrm{q}\hbar_3\mathsf{N}_3[\phi_1(x;q),\phi_2(x;q),\phi_3(x;q)], \qquad (36)
\end{aligned}$$

Thus, we have the solutions of mth-order ($m \geq 1$) deformation equations given by

$$\begin{aligned}
um(x) &= \chi m u_{m-1} + \hbar_1\mathsf{H}_1(x)\mathsf{R}_{1,m}(u_{m-1}), \\
vm(x) &= \chi m v_{m-1} + \hbar_2\mathsf{H}_2(x)\mathsf{R}_{2,m}(\vec{v}_{m-1}), \\
wm(x) &= \chi m w_{m-1} + \hbar_3\mathsf{H}_3(x)\mathsf{R}_{3,m}(\vec{w}_{m-1}), \qquad (37)
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{R}_{1,m}(\vec{u}_{m-1}) &= u_{m-1}(x) - (1-\chi m)\left(\frac{1}{6} + 2x + e^x - \frac{3}{2}e^{2x} + \frac{1}{3}e^{3x}\right) \\
&\quad - \int_0^x\int_0^x(6v_{m-1}(t) - 3w_{m-1}(t))\mathrm{dtdx}-(1-\chi m)\mathrm{a}_0, \\
\mathsf{R}_{2,m}(\vec{v}_{m-1}) &= v_{m-1}(x) - (1-\chi m)\left(-\frac{5}{3} + e^x + e^{2x} - \frac{1}{3}e^{3x}\right) \\
&\quad - \int_0^x\int_0^x(3w_{m-1}(t) - u_{m-1}(t))\mathrm{dtdx}\text{-}(1-\chi m)\mathrm{b}_0, \\
\mathsf{R}_{2,m}(\vec{w}_{m-1}) &= v_{m-1}(x) - (1-\chi m)\left(-\frac{1}{2} - e^x + \frac{1}{2}e^{2x} + e^{3x}\right) \\
&\quad - \int_0^x\int_0^x(u_{m-1}(t) - 2v_{m-1}(t))\mathrm{dtdx}\text{-}(1-\chi m)\mathrm{c}_0. \qquad (38)
\end{aligned}$$

Then for $\mathsf{H}_1(x) = \mathsf{H}_2(x) = \mathsf{H}_3(x) = 1$, $\hbar_1 = \hbar_2 = \hbar_3 = -1$, and by using equations (37) and (38) for $m = 1, 2, 3, \cdots$, we obtain the following iterates respectively

$$\begin{aligned}
u_1(x) &= \frac{1}{6} + 2x + e^x - \frac{3}{2}e^{2x} + \frac{1}{3}e^{3x} + 3x^2b_0 - \frac{3}{2}x^2c_0, \\
v_1(x) &= -\frac{5}{3} + e^x + e^{2x} - \frac{1}{3}e^{3x} + \frac{3}{2}x^2c_0 - \frac{1}{2}x^2a_0, \\
w_1(x) &= -\frac{1}{2} - e^x + \frac{1}{2}e^{2x} + e^{3x} + \frac{1}{2}x^2a_0 - x^2b_0. \qquad (39)
\end{aligned}$$

$$u_2(x) = -\frac{689}{72} - \frac{115}{12}x + 9e^x + \frac{9}{8}e^{2x} - \frac{5}{9}e^{3x} - \frac{17}{4}x^2 + \frac{3}{4}x^4c_0 - \frac{3}{8}x^4a_0 + \frac{1}{4}x^4b_0,$$
$$v_2(x) = \frac{319}{108} - 4e^x + \frac{3}{4}e^{2x} + \frac{8}{27}e^{3x} + \frac{29}{18}x - \frac{5}{6}x^2 + \frac{1}{8}x^4a_0 - \frac{1}{2}x^4b_0 - \frac{1}{3}x^3 + \frac{1}{8}x^4c_0,$$
$$w_2(x) = \frac{127}{72} - e^x - \frac{7}{8}e^{2x} + \frac{1}{9}e^{3x} + \frac{29}{12}x + \frac{7}{4}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4b_0 - \frac{3}{8}x^4c_0 + \frac{1}{12}x^4a_0.$$
$$u_3(x) = \frac{49399}{2592} + \frac{7325}{432}x - 21e^x + \frac{57}{32}e^{2x} + \frac{13}{81}e^{3x} + \frac{895}{144}x^2 + \frac{29}{72}x^3$$
$$- \frac{41}{48}x^4 + \frac{1}{60}x^6a_0 - \frac{1}{8}x^6b_0 - \frac{3}{20}x^5 + \frac{1}{16}x^6c_0, \tag{40}$$
$$v_3(x) = \frac{2933}{216}x - 12e^x + \frac{535}{72}x^2 - \frac{15}{16}e^{2x} + \frac{8}{81}e^{3x} + \frac{101}{36}x^3 + \frac{19}{24}x^4$$
$$+ \frac{1}{20}x^5 + \frac{1}{48}x^6a_0 + \frac{1}{60}x^6b_0 - \frac{1}{16}x^6c_0 + \frac{16639}{1296},$$
$$w_3(x) = -\frac{21293}{1296}x + 17e^x - \frac{3343}{432}x^2 - \frac{3}{32}e^{2x} - \frac{31}{243}e^{3x} - \frac{461}{216}x^3 - \frac{31}{144}x^4$$
$$+ \frac{1}{30}x^5 - \frac{1}{48}x^6a_0 + \frac{1}{24}x^6b_0 + \frac{1}{60}x^6c_0 - \frac{130471}{7776}. \tag{41}$$
$\vdots$

In view of the above components and by applying the given conditions (33) on the series solutions of the first two iterations, we respectively obtain the following series solutions for $u(x)$, $v(x)$, and $w(x)$:

$$u(x) \cong U(x) = -\frac{605}{72} - \frac{91}{12}x + 10e^x - \frac{3}{8}e^{2x} - \frac{2}{9}e^{3x} - \frac{11}{4}x^2 + \frac{5}{8}x^4,$$
$$v(x) \cong V(x) = \frac{247}{108} - 3e^x + \frac{7}{4}e^{2x} - \frac{1}{27}e^{3x} + \frac{1}{6}x^2 + \frac{29}{18}x - \frac{1}{4}x^4 - \frac{1}{3}x^3,$$
$$w(x) \cong W(x) = \frac{163}{72} - 2e^x - \frac{3}{8}e^{2x} + \frac{10}{9}e^{3x} + \frac{5}{4}x^2 + \frac{29}{12}x + \frac{1}{3}x^3 - \frac{1}{24}x^4. \tag{42}$$

After the implementation of steps (*iii*) and (*iv*) on (42), we approximate the series solutions obtained by their [4/2] Padé approximants and thereafter, we apply steps (*vi*) and (*vii*) to obtain the true solutions

$$u(x) = e^x, \qquad v(x) = e^{2x}, \qquad w(x) = e^{3x}.$$

Example 4.2. We also consider the following system of Volterra Integro-differential equation [5]:

$$u''(x) = 1 - \frac{1}{3}x^3 - \frac{1}{2}(v'^2 + \frac{1}{2}\int_0^x \left(u^2(t) + v^2(t)\right)\mathrm{d}t,$$
$$v''^2 = -1 + x^2 - xu(x) + \frac{1}{4}\int_0^x \left(u^2(t) - v^2(t)\right)\mathrm{d}t, \tag{43}$$

with initial conditions

$$u(0) = 1, \quad u'(0) = 2, \quad v(0) = -1, \quad v'(0) = 0. \tag{44}$$

The exact solutions are $u(x) = x + e^x$ and $v(x) = x - e^x$.

We follow the same procedure as explained in Example 1 to obtain the zeroth- and high-order deformation equations. Thus, we have

$$
\begin{aligned}
u_0 &= a_0 x + a_1, \\
v_0 &= b_0 x + b_1, \\
u m(x) &= \chi m u_{m-1} + \hbar_1 \mathrm{H}_1(x) \mathrm{R}_{1,m}(\vec{u}_{m-1}), \\
v m(x) &= \chi m v_{m-1} + \hbar_2 \mathrm{H}_2(x) \mathrm{R}_{2,m}(\vec{v}_{m-1}),
\end{aligned} \tag{45}
$$

where

$$
\begin{aligned}
\mathrm{R}_{1,m}(\vec{u}_{m-1}) = {} & u_{m-1}(x) - (1-\chi m)\left(\frac{1}{2}x^2 - \frac{1}{60}x^5\right) \\
& + \frac{1}{2}\int_0^x\int_0^x\left(\sum_{j=0}^{m-1} v_j'(x) v_{m-j-1}'(x)\right)\mathrm{d}x\mathrm{d}x \\
& - \frac{1}{2}\int_0^x\int_0^x\int_0^x\left(\sum_{j=0}^{m-1} u_j(t) u_{m-j-1}(t)\right. \\
& \left. + \sum_{j=0}^{m-1} v_j(t) v_{m-j-1}(t)\right)\mathrm{d}t\mathrm{d}x\mathrm{d}x - (1-\chi m)(a_0 x + a_1). \\
\mathrm{R}_{2,m}(\vec{u}_{m-1}) = {} & v_{m-1}(x) - (1-\chi m)\left(-\frac{1}{2}x^2 + \frac{1}{12}x^4\right) \\
& + \int_0^x\int_0^x x u_{m-1}\mathrm{d}x\mathrm{d}x - \frac{1}{4}\int_0^x\int_0^x\int_0^x\left(\sum_{j=0}^{m-1} u_j(t) u_{m-j-1}(t)\right. \\
& \left. - \sum_{j=0}^{m-1} v_j(t) v_{m-j-1}(t)\right)\mathrm{d}t\mathrm{d}x\mathrm{d}x - (1-\chi m)(b_0 x + b_1).
\end{aligned} \tag{46}
$$

Similarly, for $\mathrm{H}_1(x) = \mathrm{H}_2(x) = 1$ and $\hbar_1 = \hbar_2 = -1$, we obtain the following series solutions after substituting the values of constants a_0, a_1, b_0, b_1 which were obtained by using the initial conditions (44):

$$
\begin{aligned}
u(x) \cong U(x) = u_0(x) + \sum_{m=1}^{3} u m(x) = {} & 1 + \frac{1}{6}x^3 + 2x + \frac{13}{9979200}x^{12} + \frac{1}{24}x^4 + \frac{1}{2}x^2 \\
& + \frac{1}{120}x^5 + \frac{31}{129729600}x^{13} + \frac{1}{720}x^6 + \frac{17}{5040}x^7 \\
& + \frac{13}{8064}x^8 + \frac{1}{1890}x^9 + \frac{187}{7257600}x^{10} + \frac{53}{5702400}x^{11},
\end{aligned}
$$

$$u(x) \cong U(x) = u_0(x) + \sum_{m=1}^{3} u_m(x) = 1 + \frac{1}{6}x^3 + 2x + \frac{13}{9979200}x^{12} + \frac{1}{24}x^4 + \frac{1}{2}x^2 + \frac{1}{120}x^5 + \frac{31}{129729600}x^{13} + \frac{1}{720}x^6 + \frac{17}{5040}x^7 + \frac{13}{8064}x^8 + \frac{1}{1890}x^9 + \frac{187}{7257600}x^{10} + \frac{53}{5702400}x^{11},$$

$$v(x) \cong V(x) = v_0(x) + \sum_{m=1}^{3} v_m(x) = -1 - \frac{1}{6}x^3 + \frac{1}{1425600}x^{12} - \frac{1}{24}x^4 - \frac{1}{2}x^2 - \frac{1}{120}x^5 + \frac{1}{25945920}x^{13} - \frac{1}{720}x^6 - \frac{11}{10080}x^7 - \frac{5}{16128}x^8 - \frac{1}{17280}x^9 + \frac{121}{14515200}x^{10} + \frac{1}{483840}x^{11}. \quad (47)$$

Taking the Laplace transform of (47), we have

$$U(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{1}{s^4} + \frac{1}{s^5} + \frac{1}{s^3} + \frac{1}{s^6} + \frac{1}{s^7} + \frac{17}{s^8} + \frac{65}{s^9} + \frac{192}{s^{10}} + \frac{187}{2s^{11}} + \frac{371}{s^{12}} + \frac{624}{s^{13}} + \frac{1488}{s^{14}},$$

$$V(s) = -\frac{1}{s} - \frac{1}{s^4} + \frac{336}{s^{13}} - \frac{1}{s^5} - \frac{1}{s^3} - \frac{1}{s^6} + \frac{240}{s^{14}} - \frac{1}{s^7} - \frac{11}{2s^8} - \frac{25}{2s^9} - \frac{21}{s^{10}} + \frac{121}{4s^{11}} + \frac{165}{2s^{12}}. \quad (48)$$

Replacing $\frac{1}{s}$ with $x$in (48), we respectively obtain

$$u(x) = x + 2x^2 + x^4 + x^5 + x^3 + x^6 + x^7 + 17x^8 + 65x^9 + 192x^{10} + \tfrac{187}{2}x^{11} + 371x^{12} + 624x^{13} + 1488x^{14},$$

$$v(x) = -x - x^4 + 336 * x^{13} - x^5 - x^3 - x^6 + 240x^{14} - x^7 - \tfrac{11}{2}x^8 - \tfrac{25}{2}x^9 - 21x^{10} + \tfrac{121}{4}x^{11} + \tfrac{165}{2}x^{12}. \quad (49)$$

Application of Padé approximants [4/1]on (49) yields

$$[4/1]u(x) = \frac{-x^3+x^2+x}{1-x},$$

$$[4/1]v(x) = \frac{-x^3+x^2-x}{1-x}. \quad (50)$$

We now replace $x$in (50) with $\frac{1}{s}$to obtain

$$U(s) = \frac{-\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{s}}{1-\frac{1}{s}},$$

$$V(s) = \frac{-\frac{1}{s^3} + \frac{1}{s^2} - \frac{1}{s}}{1-\frac{1}{s}}. \quad (51)$$

Taking the inverse Laplace transform of (51), we obtain

$$u(x) = x + e^x, \quad v(x) = x - e^x$$

which are the exact solutions.

Example 4.3. Finally, we consider the nonlinear boundary value problem for integro-differential equation related to the Blasius problem [26], [33]

$$u''(x) = \alpha - \frac{1}{2}\int_0^x u(t)u''(t)\mathrm{d}t, \qquad -\infty < x < 0, \tag{52}$$

subject to the conditions

$$u(0) = 0, \quad u'(0) = 1, \qquad \lim_{x\to-\infty} u'(x) = 0. \tag{53}$$

We proceed as discussed in Examples 1 and 2 above to obtain the initial approximation

$$u_0 = a_0 x + a_1, \tag{54}$$

and the higher-order iterates of $u(x)$ are obtained from the recurrence relation

$$u_m(x) = \chi_m u_{m-1} - \Bigg(u_{m-1} + \frac{1}{2}\int_0^x\int_0^x\int_0^x \sum_{j=0}^{m-1} u_j(t)u''_{m-j-1}(t)\mathrm{d}t\mathrm{d}x\mathrm{d}x$$
$$-(1-\chi_m)\Big(a_0 x + a_1 - \frac{1}{2}\alpha^2\Big)\Bigg), \qquad -\infty < x < 0. \tag{55}$$

Thus we have the following iterates

$$u_1(x) = \frac{1}{2}\alpha x^2,$$
$$u_2(x) = -\frac{1}{48}a_0\alpha x^4 - \frac{1}{12}a_1\alpha x^3,$$
$$u_3(x) = \frac{1}{960}a_0^2\alpha x^6 + \frac{1}{160}x^5 a_1 a_0\alpha - \frac{1}{240}x^5\alpha^2 + \frac{1}{96}a_1^2\alpha x^4,$$
$$u_4(x) = -\frac{1}{21504}a_0^3\alpha x^8 + \frac{11}{20160}x^7\alpha^2 a_0 - \frac{1}{2688}x^7 a_1 a_0^2\alpha$$
$$-\frac{1}{960}x^6 a_0 a_1{}^2\alpha + \frac{1}{576}x^6\alpha^2 a_1 - \frac{1}{960}a_1^3\alpha x^5,$$
$$u_5(x) = \frac{1}{552960}a_0^4\alpha x^{10} + \frac{1}{55296}x^9 a_1 a_0^3\alpha - \frac{43}{967680}x^9\alpha^2 a_0^2$$
$$+\frac{1}{14336}x^8 a_1^2 a_0^2\alpha - \frac{1}{3840}x^8 a_1\alpha^2 a_0 + \frac{11}{161280}x^8\alpha^3$$
$$-\frac{1}{2520}x^7\alpha^2 a_1^2 + \frac{1}{8064}x^7 a_0 a_1^3\alpha + \frac{1}{11520}a_1^4\alpha x^6,$$
$$u_6(x) = -\frac{1}{1351680}x^{11} a_1 a_0^4\alpha + \frac{149}{6451200}x^{10} a_1\alpha^2 a_0^2$$
$$-\frac{1}{276480}x^{10} a_1^2 a_0^3\alpha + \frac{1}{15120}x^9 a_0\alpha^2 a_1^2$$
$$-\frac{1}{110592}x^9 a_1^3 a_0^2\alpha - \frac{1}{86016}x^8 a_0 a_1^4\alpha$$
$$-\frac{1}{16220160}a_0^5\alpha x^{12} - \frac{1}{161280}a_1^5\alpha x^7 + \frac{587}{212889600}x^{11}\alpha^2 a_0^3$$

$$- \frac{5}{387072}x^{10}\alpha^3 a_0 - \frac{13}{322560}x^9\alpha^3 a_1 + \frac{1}{15360}x^8\alpha^2 a_1^3,$$

$$u_7(x) = \frac{1}{38338560}x^{13}a_1 a_0^5\alpha - \frac{71}{46448640}x^{12}a +_1 \alpha^2 a_0^3$$
$$+\frac{1}{6488064}x^{12}a_1^2 a_0^4\alpha + \frac{1}{2027520}x^{11}a_1^3 a_0^3\alpha$$
$$+\frac{1759}{212889600}x^{11}a_0\alpha^3 a_1 - \frac{1}{158400}x^{11}a_1^2\alpha^2 a_0^2$$
$$+ \frac{1}{1105920}x^{10}a_0^2 a_1^4\alpha - \frac{229}{19353600}x^{10}a_0\alpha^2 a_1^3 + \frac{1}{1105920}x^9 a_0 a_1^5\alpha$$
$$+ \frac{1}{536739840}a_0^6\alpha x^{14} + \quad \alpha x^8 - \frac{1877}{13284311040}x^{13}\alpha^2 a_0^4$$
$$+ \frac{1}{725760}x^{12}\alpha^3 a_0^2 + \frac{7}{552960}x^{10}\alpha^3 a_1^2 - \frac{11}{1290240}x^9\alpha^2 a_1^4$$
$$- \frac{5}{4257792}x^{11}\alpha^4, \tag{56}$$

The solution of (52) in series form after finding the unknown constants a_0, a_1, by applying (53) and substituting the values into $u(x) = u_0(x) + \sum_{m=1}^{7} um(x)$ is therefore given as

$$u(x) = \frac{1}{536739840}\alpha x^{14} - \frac{1877}{13284311040}x^{13}\alpha^2 + \frac{1}{725760}x^{12}\alpha^3$$
$$- \frac{5}{387072}x^{10}\alpha^3 + \frac{587}{212889600}x^{11}\alpha^2 + \frac{1}{552960}\alpha x^{10} + \frac{11}{20160}x^7\alpha^2$$
$$- \frac{1}{16220160}\alpha x^{12} - \frac{1}{21504}\alpha x^8 - \frac{43}{967680}x^9\alpha^2 - \frac{1}{48}\alpha x^4$$
$$+ \frac{1}{960}\alpha x^6 - \frac{5}{4257792}x^{11}\alpha^4 + x - \frac{1}{240}x^5\alpha^2 + \frac{11}{161280}x^8\alpha^3 + \frac{1}{2}\alpha x^2, \tag{57}$$

from which it follows that

$$u'(x) = 1 + \alpha x - \frac{1}{12}\alpha x^3 - \frac{1}{48}x^4\alpha^2 + \frac{1}{160}\alpha x^5 + \frac{11}{2880}x^6\alpha^2$$
$$+ \left(-\frac{1}{2688}\alpha + \frac{11}{20160}\alpha^3\right)x^7 - \frac{43}{107520}x^8\alpha^2$$
$$+ \left(-\frac{25}{193536}\alpha^3 + \frac{1}{55296}\alpha\right)x^9$$
$$+ \left(\frac{587}{19353600}\alpha^2 - \frac{5}{387072}\alpha^4\right)x^{10}$$
$$+ \left(-\frac{1}{1351680}\alpha + \frac{1}{60480}\alpha^3\right)x^{11} - \frac{1877}{1021870080}x^{12}\alpha^2$$
$$+ \frac{1}{38338560}\alpha x^{13}. \tag{58}$$

If $u'(x)$ is free of singularities, it is well-known fact that the Padé approximants will converge on the entire real axis. Therefore, to determine the numerical value for the constant α, we make use of the third condition in (53) and also obtain rational function representation of (58)by the diagonal approximants [2/2], [3/3], and [4/4]. Thus, by substituting the boundary condition at $u'(-\infty) = 0$ in each Padé approximant, the approximant vanished since the coefficient of x with the highest power in the numerator vanished.

Thus, the Padé approximant [2/2] to $u'(x)$ of degree 4 from (58) is obtained as

$$[2/2]_{u'}(x) = \frac{3\left(4+3\alpha x+\left(-\alpha^2+\frac{1}{3}\right)x^2\right)}{12-3\alpha x+x^2}. \quad (59)$$

Applying the boundary condition $u'(-\infty) = 0,$ we get

$$\alpha = 0.577350269189627.$$

Similarly, for the Padé approximants $[3/3]$ and $[4/4],$ we respectively obtain

$$\alpha = 0.516397779494323$$

and

$$\alpha = 0.522703079775270.$$

The table below shows the comparison of values for α obtained using the homotopy analysis-based iterative method (HABIM) with other methods.

Table 1: Comparison of numerical values for α

Padė approximant	HABIM	VIM[26]	MADM[33]
[2/2]	0.5773502692	0.5773502691	0.5773502693
[3/3]	0.5163977795	0.5163977793	0.5163977793
[4/4]	0.5227030798	0.5227030798	0.5227030798

5. Conclusion

A new iterative technique based on homotopy analysis method in conjunction with Padé approximants for solving Volterra integro-differential equation and nonlinear non-homogeneous boundary value problem for the integro-differential equation related to the Blasius problem has been presented. The technique produced a rapidly convergent series which gave closed-form solutions of the problems considered. Also, the validity of this method in solving integro-differential equation related to Blasius problem is based on the assumption that it converges by knowing the first two conditions to determine the values of two out of three unknown constants in the series while diagonal Padé approximants were employed to determine the third unknown constant. A comparison of the present work is made with the series solutions of VIM [26], MADM [33] and it is shown that HABIM is a robust and promising tool for solving linear and nonlinear system of integro-differential equations and boundary value problems defined in semi-infinite domain.

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