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A New Approach for Computing A Posteriori Error Estimation for the Vlasov-Maxwell-Fokker-Planck System

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Abstract. *In this paper, we propose a new splitting scheme for reformulation of the Vlasov Maxwell Fokker Planck (VMFP) system, which produces a new recursive algorithm for its solution. We demonstrate that this algorithm converges to a unique solution. Also, we obtain an a posteriori error estimation, in practical finite element analysis, for sub-problems of the recursive algorithm. The paper also contains brief comments on error estimations of solutions of the VMFP system, particularly when mixed methods are employed. Finally, a numerical example is provided to demonstrate the accuracy and efficiency of the proposed algorithm.*

Key words: Vlasov-Maxwell-Fokker-Planck, Finite Element, A Posteriori Error, Convergence.

AMS Subject Classifications: 65M12, 65M15, 65M60, 82D10, 35L80

1. Introduction

Physical laws of plasma and its kinetics are described by conservation of quantities like mass, energy and momentum (see [6, 7, 13]). Also, kinetic models arise in the statistical description of a large number of particles same as electrons, molecules, massive particles and nucleons. The main parameter studied has been the ***averaged*** behavior of this collection of particles rather than the dynamics of each individual particle. Of course, there is a wide variety of physical situations where such models can be used such as plasma physics, nuclear physics, astrophysics, fluid mechanics, rarefied gazes, lasers and semi-conductors. The list of applicable models is rather impressive and extends to the variety of physical situations : electromagnetic interactions, gravitational forces, collision effects, quantum interactions, relativistic effects and nuclear forces. Here we investigate an important one of these phenomena modelled by the relativistic Vlasov Maxwell Fokker Planck system. Incidentally, existence and uniqueness of a global solution to the VMFP system were investigated by many authors in [8, 10, 11, 17, 18, 21, 22, 26, 30, 31, 33, 34, 35, 39, 40].

In an a posteriori error estimate, the error is evaluated by some norm of the residual of the approximate solution. This residual is the difference between the left and right hand side in the equation when the exact solution is replaced by the approximant. Hence a posteriori error estimates give quantitative information about the size of the error after computation of the approximate solution. There are various approaches to computing a posteriori error estimates by several authors (see [2, 3, 4, 5, 12, 13, 15, 16, 19, 20, 23, 24, 25, 28, 29, 38, 36, 32]). This paper reports on a new technique for computing the a posteriori error for the solution of the VMFP system.

The time evolution of the distribution of particles in two spatial dimensions, in other words in the three-dimensional phase space, is described by the following relativistic Vlasov-Maxwell-Fokker-Planck system:

$$\partial_t f + \hat{v}_1 \partial_x f + (E + BM\hat{v}).\nabla_v f - \beta \nabla_v.(\hat{v} f) - \nu \Delta_v f = f, \tag{1}$$

$$\partial_t E_1 = 4\pi J_1, \tag{2}$$

$$\partial_x E_1 = 4\pi \rho, \tag{3}$$

$$\partial_t E_2 = -\partial_x B - 4\pi J_2, \tag{4}$$

$$\partial_x E_2 = -\partial_t B, \tag{5}$$

$$\rho(t,x) = \int_{\Omega_v} f(t,x,v)dv - \phi(x), \tag{6}$$

$$J(t,x) = \int_{\Omega_v} \hat{v} f(t,x,v)dv, \tag{7}$$

$$\int_{\Omega_x} \rho(0,x)dx = 0, \tag{8}$$

$$f(0,x,v) = f_0(x,v) \geq 0, \tag{9}$$

$$E_1(0,x) = 4\pi \int_0^x \left(\int_{\Omega_v} f_0(y,v)dv - \phi(y)\right)dy, \tag{10}$$

$$\frac{\partial f(t,x,v)}{\partial n_v} = 0, \qquad (t,x,v) \in [0,T] \times \Omega_x \times \partial\Omega_{\overline{v}}, \tag{11}$$

$$f(t,x,v) = 0, \qquad (t,x,v) \in [0,T] \times \{0,A\} \times \Omega_v, \tag{12}$$

$$E_2(0,x) = E_2^{(0)}(x), \qquad x \in [0,A], \tag{13}$$

$$B(0,x) = B_0(x), \qquad x \in [0,A]. \tag{14}$$

In the above system, for $\beta > 0$, we introduce the following notations:

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\hat{v} = (\hat{v}_1, \hat{v}_2) \equiv \frac{v}{\sqrt{(1+|v|^2)}},$$

$$\Omega = [0,T] \times \Omega_v \times \Omega_x,$$

$$\Omega_v \subset \mathbb{R}_v^2,$$
$$\Omega_x = [0,A] \subset \mathbb{R}_x,$$
$$\partial\Omega_v^- = \{x \in \partial\Omega_x | n_v(x).v < 0\},$$
$$E(t,x) = (E_1,E_2),$$
$$B = B(t,x) \in C([0,T] \times \Omega_x),$$
$$\rho = \rho(x,t).$$

Also, $f = f(t,x,v)$, $E = E(t,x) = (E_1,E_2)(t,x)$, $\rho = \rho(t,x)$, $J = J(t,x) = (J_1,J_2)(t,x)$ and $B = B(t,x)$ are unknown functions.

We assume that there exists $\gamma > 6$ such that for any $0 \leq p \leq 4$ and the following conditions are satisfied:

i) $\partial_x^p E_2^{(0)} \in L^\infty(\Omega_x)$,
ii) $\varphi \in C_0^4(\Omega_x)$,
iii) $\partial_x^p B_0 \in L^\infty(\Omega_x)$,
iv) $f_0 = f_0(x,v)$ has compact support and $\partial_x^p f_0 \in L_\infty(\Omega_x \times \Omega_v)$.
v) $\sup_{|\alpha| \leq 1} \{(\sqrt{(1+|v|^2)}\,)^\gamma |D_v^\alpha f_0(x,v)|: x \in \Omega_x, v \in \Omega_v\} < \infty$ where $\alpha > 0$.

Here, we recall that the evolution of a collection of particles is described in terms of a statistical quantity, namely the density of particles $f(t,x,v)$ at position $x \in \mathbb{R}$ with velocity $v \in \mathbb{R}^2$, at time $t > 0$,and $E = E(t,x)$, $B = B(t,x)$ represent a self-consistent electromagnetic field. In order to simplify the presentation, we will only consider models involving only one type of particles; otherwise, one has to introduce a density function for each species, but except for notational complexities the presented results easily adapt to those systems. Of course, f being a density, satisfies $f(t,x,v) > 0$.

It is common knowledge that all numerical approaches create error bounds, especially the a posteriori error for VMFP systems. There is a considerable number of very recently published articles on these errors, but many authors either avoid computing the error bounds or follow approaches that can be quite complicated [9, 14]. These facts assert the novelty of this article, in providing a new approach for computing a posteriori error for the VMFP system.

The outline of this paper is as follows. In section 2, we give a new splitting form for VMFP. The recursive algorithm for finding solution of the splitted form of the VMFP system is given in section 3. In section 4, we give another proof for the existence of a unique solution to the continuous problem of section 3. Also, in this section, we obtain the general a posteriori error estimates for VMFP system solutions.

2. Splitted VMFP According to the Abstract Formulation

Let (1) be in the form $T[f,E,B] = f$, where

$$T[f,E,B] := \partial_t f + \hat{v}_1 \partial_x f + (E + BM\hat{v}).\nabla_v f - \beta \nabla_v.(\hat{v}f) - \nu \triangle_v f.$$

As can be seen, the operator T has five parts. Thus we consider $T[f,E,B] = \sum_{i=1}^{5} T_i(f)$, and $f = \sum_{i=1}^{5} f_i$. so it can be written $T_i = f_i$, $i = 1,\ldots,5$. That means The system is split to five differential equations with boundary conditions via five operators. This process is expressed and proved in the following theorem.

Theorem 2.1. I*f we split (1) to the following form of operators*

$$T[f,E,B] = \sum_{i=1}^{5} T_i(f),$$

where $T[f,E,B] := \partial_t f + \hat{v}_1 \partial_x f + (E + BM\hat{v}).\nabla_v f - \beta \nabla_v.(\hat{v}f) - \nu \triangle_v f$ *and we consider the equations (1-2)-(1.14). Also we assume that*

$$f = \sum_{i=1}^{5} f_i, \tag{15}$$

where $\{f_i\}_{i=1}^{5}$ *are unknown, then we obtain the following abstract formulation instead in (2)-(14):*

$$f = \Gamma_1[f_1], \tag{16}$$

$$\left[\begin{array}{l} B_i(f_{i+1},g) = L_i(g), \\ i = 1,2,3,4 \end{array} \right. , \tag{17}$$

$$\rho = \Gamma_2[f], \tag{18}$$

$$E_1(t,x) = \Gamma_3[\rho], \tag{19}$$

$$B(x,t) = \Gamma_4[B(x,t)], \tag{20}$$

$$E_2(x,t) = \Gamma_5[B]. \tag{21}$$

Proof. Let us introduce the above notations in the proof. Then we define the following operators and we set (15) then we split the above system to the following abstract formulation. Let $V = W_0^{1,p}$ and $W = W^{1,q}$, $\frac{1}{p} + \frac{1}{q} = 1$.
First, we establish from (1), (9) and (15):

$$\begin{cases} T_1 : V \longrightarrow W, \\ T_1[f] = f_1(s,x,v), \\ T_1[\quad] = \partial_t[\quad], \\ f(0,x,v) = f_0(x,v). \end{cases}$$

So *we* have:

$$f = \underbrace{f_0(x,v) + \int_0^t f_1(s,x,v)ds}_{:=\Gamma_1[f_1]} \quad .$$

Also, we can write

$$\begin{cases} T_2 : V \longrightarrow W, \\ T_2[f] = f_2, \\ T_2[\quad] = \hat{v}_1 \partial_x[\quad]. \end{cases}$$

Thus, the weak formulation of the above boundary value problem (BVP) $\forall g \in W_0^{1,p}$ is

$$B_1(f_2, g) = L_1(g),$$

where $B_1(f_2, g) = \int_\Omega f_2 \, g \, d\Omega$ and $L_1(g) = \int_\Omega \hat{v}_1 \partial_x f \, g \, d\Omega$ and

$$W_0^{m,p}[a,b] = \{f(x) : f^{(m-1)} \textit{ is absolutely continuous} f^{(m)}(x) \in L_p[a,b], \quad f(a) = f(b) = 0\}.$$

Also, if we consider another part of (1)–(12) then we have:

$$\begin{cases} T_3[f] = f_3, \\ T_3[\quad] = (E + BM\hat{v}).\nabla_v[\quad]. \end{cases}$$

Similarly, the weak formulation of the above BVP $\forall g \in W_0^{1,p}$ is

$$B_2(f_3, g) = L_2(g),$$

where $B_2(f_3, g) = \int_\Omega f_3 g d\Omega$ and $L_2(g) = \int_\Omega (E + BM\hat{v}).\nabla_v f g d\Omega$
Moreover, from another part of (1)– (12), we have:

$$\begin{cases} T_4 : V \longrightarrow W, \\ T_4[f] = f_4, \\ T_4[\quad] = -\beta\nabla_v.(\hat{v}[\quad]), \\ f(t, A, v) = 0, \quad [0, T] \times \Omega_v, \\ f(t, 0, v) = 0, \quad [0, T] \times \Omega_v, \end{cases}$$

Also

$$B_3(f_4, g) = L_3(g),$$

where $L_3(g) = \int_\Omega -\beta(\hat{v}f).\nabla g d\Omega$ and $B_3(f_4, g) = \int_\Omega f_4 g d\Omega$.

Again, from another part of (1)– (12) we have:

$$\begin{cases} T_5 : V \longrightarrow W, \\ T_5[f] = f_5 \\ T_5[\quad] = v\nabla_v[\quad], \\ \dfrac{\partial f(t,x,v)}{\partial n} = 0, \quad [0,T] \times \Omega_x \times \partial\Omega_v, \end{cases}$$

and the weak formulation of the above BVP $\forall g \in W_0^{1,p}$ is

$$B_4(f_5, g) = L_4(g),$$

where $B_4(f_5, g) = \int_\Omega f_4 g d\Omega$ and $L_4(g) = \int_\Omega \nabla_v(g + v).\nabla_v f d\Omega$.
On the other hand, if we put $\rho(t,x) = \rho = T_6[f]$ then

$$T_6[f] = \underbrace{\int_{\Omega_v} f(t,x,v)dv - \phi(x)}_{:=\Gamma_2[f]} .$$

By integration from (2) and (10), we obtain

$$\begin{cases} \partial_t E_1 = 4\pi J_1, \\ E_1(0,x) = 4\pi \int_0^x (\int_{\Omega_v} f_0(y,v)dx - \phi(y))dy. \end{cases}$$

Therefore, we have

$$E_1(t,x) = 4\pi \int_0^t J_1(s,x)ds + E_1(0,x).$$

Moreover, if we consider (4) and (13) then we arrive at the following Volterra integral equation

$$\begin{cases} \partial_t E_2 = -\partial_x B - 4\pi J_2, \\ E_2(0,x) = E_2^{(0)}(x), \\ B(x,t) = B_0(x) - \int_0^t \partial_x E_2(s,x)ds. \end{cases}$$

Hence, we have

$$E_2(t,x) = E_2^{(0)}(x) - \int_0^t (\partial_x B + 4\pi J_2)ds.$$

Finally, the validity of the Volterra integral equation:

$$B(x,t) = B_0(x) - \int_0^t \partial_x (E_2^{(0)}(x) - \int_0^s (\partial_x B + 4\pi J_2)d\bar{s})ds,$$

completes the proof. ■

Now we can introduce the following recursive algorithm for finding the solution of VMFP. We observe that with the splitting of (1)–(14), we obtain another abstract formulation for the VMFP problem. Advantages of this splitting will be described in the next sections. Using (16)-(19), we can state the following algorithm.

3. Recursive Algorithm for the VMFP System

In this section, we present a recursive algorithm to obtain the approximate solution of VMFP. Then we prove the existence and uniqueness of the solution of each splitted operator in the previous section.
The following recursive algorithm, to find the unknowns of the VMFP system, will proceed until we reach a converging result.
•Put $n = 1$ and ϵ, $B^{(0)}$, $f_1^{(0)}$ and $\phi(x)$ are given, ($f_1^{(0)}$ is initial)
•Start

$$f^{(n)} = f_0 + \int_0^t f_1^{(n-1)}(s,x,v)ds, \tag{22}$$

$$B_1(f_2^{(n)}, g) = L_1(g), \tag{23}$$

$$B_2(f_3^{(n)}, g) = L_2(g), \tag{24}$$

$$B_3(f_4^{(n)}, g) = L_3(g), \tag{25}$$

$$B_4(f_5^{(n)}, g) = L_4(g), \tag{26}$$

$$f_1^{(n)} = f^{(n)} - \sum_{i=2}^{5} f_i^{(n)}, \tag{27}$$

$$\rho^{(n)}(t,x) = \int_{\Omega_v} f^{(n)}(t,x,v)dv - \phi(x), \tag{28}$$

$$\int_{\Omega_x} \rho^{(n)}(0,x)dx = \alpha n, \tag{29}$$

$$J^{(n)}(t,x) = \begin{pmatrix} J_1^{(n)}(t,x) = \int_{\Omega_v} \frac{v_1}{\sqrt{1+|v|^2}} f^{(n)}(t,x,v)dv, \\ J_2^{(n)}(t,x) = \int_{\Omega_v} \frac{v_2}{\sqrt{1+|v|^2}} f^{(n)}(t,x,v)dv \end{pmatrix}, \tag{30}$$

$$E_1^{(n)}(t,x) = 4\pi \int_0^t J_1^{(n)}(s,x)ds + E_1(0,x), \tag{31}$$

$$E_1(0,x) = 4\pi \int_0^x (f_0(y,v)dv - \phi(y))dy, \tag{32}$$

$$B^{(n)}(x,t) = B^0(x) - \int_0^t \partial_X(E_2^{(0)}(x) - \int_0^s (\partial_X B^{(n-1)} - 4\pi J_2^{(n)})d\bar{s})ds, \tag{33}$$

$$E_2^{(n)}(t,x) = E_2^{(0)}(x) - \int_0^t (\partial_X B^{(n)} - 4\pi J_2^{(n)})ds, \tag{34}$$

•

$$\max\left(\|T[f^{(n)},E^{(n)},B^{(n)}] - f^{(n)}\|, |\alpha n|, \|\int_0^t \partial_X J_1^{(n)}(s,x)ds - \int_{\Omega_v}(f_0(x,v) - f^{(n)}(t,x,v))dv\|\right) \leq \epsilon.$$

then stop.

• $n := n+1$ and go to start.

Existence and uniqueness of solution for equations (17) and (31) are given in the following theorem.

Theorem 3.1. Let f_0, $\phi, E_1(0,x)$ and f_1 have the following properties:
$E_1(0,x), \phi \in L_2$, $\|f_0\|_{L_2} \leq C_0 < \infty$, $\|f_1\| \leq C_1$, $\|\phi\|_{L_2} \leq C_2$, $\|E_1(0,x)\| \leq C_3$, $f_0 \in L_2$ and $f_1 \in W_0^{1,p}$, such that $C_0 < C_1 < C_2 < C_3 < \infty$ then the operator equations of $\Gamma_1[f_1] = f, \Gamma_2[f] = \rho$ (that they defined in the above theorem) and $\Gamma_3[p] := E_1(t,x)$ have unique solutions.

Proof. By (16) we can write

$$\begin{aligned}
\|f\|_{W_0^{1,p}} &= \|f_0 + \int_0^t f_1(s,x,v)ds\|_{W_0^{1,p}} \\
&\leq \|f_0\|_{W^{1,p}} + t\|f_1\|_{W^{1,p}} \\
&\leq C_0 + tC_1, \quad t \in [0,T].
\end{aligned}$$

Then , on one hand, we have

$$\|f\|_{W_0^{1,p}} \leq C_1(1+T).$$

On the other hand, we obtain $\|\Gamma_1\|_{W_0^{1,p},W_0^{1,q}} = C_1(1+T)$. So Γ_1 is bounded. Also, for $\rho = \Gamma_2[f]$ we have

$$\|\rho\|_{W_0^{1,p}} \leq |\Omega_v|(1+T)C_1 + C_2 \leq (|\Omega_v|(1+T)+1)C_2.$$

Subsequently,

$$\|\Gamma_2\|_{W_0^{1,p},W_0^{1,q}} = (|\Omega_v|(1+T)+1)C_2.$$

Hence Γ_2 is bounded. Also, we may write

$$\|E_1(t,x)\|_{W^{1,p}} \leq 4\pi T(|\Omega_v|(1+T)+1)C_2 + C_3,$$

and

$$\|E_1(t,x)\|_{W^{1,p}} \leq (4\pi T(|\Omega_x|(1+T)+1)+1)C_3.$$

Additionally, we have

$$\|\Gamma_3\|_{W^{1,p},W^{1,p}} = (4\pi T(|\Omega_v|(1+T)+1)+1)C_3.$$

Since $\{\Gamma_i\}_{i=1}^3$ are bounded linear operators,then the composite operator $\Gamma = \Gamma_3(\Gamma_2(\Gamma_1))$ is a continuous linear mapping. It can also be proved that Γ and Γ_i are one to one. All these results combine to complete the proof. ■

Theorem 3.2. $B_i(f_{i+1}, g) = L_i(g), \quad \forall g \in W^{1,p}, \quad i = 1,2,3,4$ *have a unique solution.*

Proof. We observe that $B_i(f_{i+1}, g) = L_i(g), \quad i = 1,2,3,4$ have the following properties

i) $B_i(f_{i+1}, g), \quad i = 1,2,3,4$ are bilinear forms,

ii) $B_i(u,u) = \|u\|_{B_i}^2 \quad i = 1,2,3,4,$

iii) $B_i(u,v) \leq \|u\|_{W_0^{1,p}} \|v\|_{W_0^{1,q}} < C \quad i = 1,2,3,4,$

iv) $B_i(u,v) \quad i = 1,2,3,4,$ are continuous,

v) $B_i(u,v) \quad i = 1,2,3,4$ are v-elliptic form.

$B_i(u,v) \geq a\|u\|_{W_0^{1,p}}^2,$

and $\forall u \in W_0^{1,p} \quad \exists C_1, C_2 \in R^+$ s.t.

$$C_1\|u\|_{W_0^{1,p}} \leq \|u\|_{B_i} \leq C_2\|u\|_{W_0^{1,p}}$$

then $C_1\|u\|_{W_0^{1,p}}^2 \leq \|u\|_{B_i}^2$ and finally we have $B_i(u,u) \geq \frac{1}{C_1}\|u\|_{W_0^{1,p}}^2.$)

Therefore, according to the Lax-Milgram Lemma the proof is completed (see [1]).

Theorem 3.3. *If we consider the following Volterra integral equation* $\Gamma_4 : W_0^{1,p} \to W_0^{1,p}$ *for* $t \in [0,T]$

$$B^{(n+1)}(t,x) = \Gamma_4[B^{(n)}(t,x)], \tag{35}$$

where

$$\Gamma_4[B(t,x)] \; := B_0(x) - \int_0^t \partial_x(E_2^{(0)}(x) - \int_0^s (\partial_x B(t,x) - 4\pi J_2)d\bar{s})ds.$$

We note that this operator is introduced in Theorem 1 and we assume that $\partial_{xx}^2 B(\bar{s},x)$ *is continuous and satisfies the Lipschitz condition that is*

$$|\partial_{xx}^2 B_1(\bar{s},x) - \partial_{xx}^2 B_2(\bar{s},x)| \leq M|B_1(\bar{s},x) - B_2(\bar{s},x)|$$

and $T^2M < 1$ *then*

a) Γ_4 *has a unique fixed point* B^* *and*

b) *the sequence* $B^{(n+1)} = \Gamma_4(B^{(n)})$ *convergence to* B^*.

Proof. We know that $\Gamma_4[B(t,x)] \; = B(t,s)$ is equivalent with

$$B(t,x) = B_0(x) - \int_0^t \partial_x E_2^0(x) - \int_0^s \partial_{xx}^2 B(\bar{s},x)ds + 4\pi \int_0^t \int_0^s \partial_x J_2(\bar{s},x)d\bar{s}ds,$$

therefore, we write

$$\begin{cases} B(t,x) = \kappa(x) - \int_0^t \int_0^s \mathsf{F}(B(\bar{s},x))d\bar{s}ds, \\ \kappa(x) = B_0(x) - \int_0^t \partial_x E_2^0(x) + 4\pi \int_0^t \int_0^s \partial_x J_2(\bar{s},x)d\bar{s}ds. \end{cases}$$

We know that $F(B(\bar{s},x)) = \partial_{xx}^2 B(\bar{s},x)$ is continuous and satisfies the Lipschitz condition

$$B(t,x) = \kappa(x) - \int_0^t \int_0^s \mathsf{F}(B(\bar{s},x))d\bar{s}ds, \tag{36}$$

so,

$$\|\int_0^t \int_0^s \mathsf{F}(\bar{B}_1(\bar{s},x)) - \int_0^t \int_0^s \mathsf{F}(\bar{B}(\bar{s},x))\| \leq \max|\int_0^t \int_0^s (\mathsf{F}(\bar{B}_1(\bar{s},x)) - \mathsf{F}(\bar{B}(s,x))| \leq$$
$$\leq MT^2 \|\bar{B}(\bar{s},x) - \bar{\bar{B}}(\bar{s},x)\|_\infty.$$

Since $T^2 M < 1$, this is a contraction mapping and (20) has a unique solution. But we do better:

$$\|\Gamma_4^2[\bar{B}_1(\bar{s},x)] - \Gamma_4^2[\bar{\bar{B}}(\bar{s},x)]\|_\infty \leq \frac{M^2 T^4}{4} \|\bar{B}_1(\bar{s},x) - \bar{\bar{B}}(\bar{s},x)\|_\infty$$

and, in general, for $p > 1$,

$$\|\Gamma_4^p[\bar{B}_1(\bar{s},x)] - \Gamma_4^p[\bar{\bar{B}}_2(\bar{s},x)]\|_\infty \leq \frac{M^p T^{2p}}{2p} \|\bar{B}(\bar{s},x) - \bar{\bar{B}}(\bar{s},x)\|_\infty.$$

Thus for p sufficiently large Γ_4^p has bound less than one and, by Theorem of the contraction mapping (20) has a unique solution.

In most practical situations the following iteration

$$B^{(n)}(t,x) = \Gamma_4[B^{(n-1)}(t,x)],$$

cannot be carried out exactly. There may be discretization errors involved in computing $\Gamma_4[B^{(n-1)}(t,x)]$, but even if this is not the case there will always be some roundoff error. Thus the actual iterates can be considered to be the result of the perturbed iteration

$$\hat{B}^{(n)}(t,x) = \Gamma_4[\hat{B}^{(n-1)}(t,x)] + \delta n, \tag{37}$$

where $B^{(0)}(t,x) = \hat{B}^{(0)}(t,x)$ and δ_n represents the total effect of all the errors at the nth stage. If $\|\delta_n\| \leq \delta$ and Γ_4 has a contraction factor $\theta < 1$ in

$$b(B^{(0)}(t,x),r) = \{B(t,x) | \|B(t,x) - B^{(0)}(t,x)\| \leq r\}$$

where

$$r \geq r_0 + \frac{\delta}{1-\theta}, \quad r_0 = \frac{1}{1-\theta} \|B^0(t,x) - \Gamma_4(B^0(t,x)]\|,$$

then we prove the following results:

a) $\hat{B}^{(n)}(t,x) \in b(B^{(0)}(t,x),r)$,
b) $\|\hat{B}^{(n)}(t,x) - B^{*}(t,x)\| \leq \theta^n r_0 + \frac{\delta}{1-\theta}$.

Because, we have

$$\|B^{(n)}(t,x) - \hat{B}^{(n)}(t,x)\| \leq \|\Gamma_4(B^{(n-1)}(t,x)] - \Gamma_4[\hat{B}^{(n-1)}(t,x)]\| + \|\delta_n\|.$$

Assume now that $\{\hat{B}^{(i)}(t,x) \in b(B^{(0)}(t,x),r)\}$. Then

$$\begin{aligned}\|B^{(n)}(t,x) - \hat{B}^{(n)}(t,x)\| &\leq \theta\|B^{(n-1)}(t,x) - \hat{B}^{(n-1)}(t,x)\| + \delta \\ &\leq \theta^2\|B^{(n-2)}(t,x) - \hat{B}^{(n-2)}(t,x)\| + \theta\delta + \delta \\ &\leq (\theta^{n-1} + \theta^{n-2} + \ldots + 1)\delta \leq \frac{\delta}{1-\theta}.\end{aligned}$$

Thus

$$\begin{aligned}\|\hat{B}^{(n)}(t,x) - B^{(0)}(t,x)\| &\leq \|B^{(n)}(t,x) - B^{(0)}(t,x)\| + \|B^{(n)}(t,x) - \hat{B}^{(n)}(t,x)\| \\ &\leq r_0 + \frac{\delta}{1-\theta}, \qquad (38)\end{aligned}$$

and $\hat{B}^{(n)}(t,x) \in b(B^{(0)}(t,x),r)$. Since $\hat{B}^{(1)}(t,x) \in b(B^{(0)}(t,x),r)$, therefore (a) is completed. Also, (b) is obtained from

$$\|\hat{B}^{(n)}(t,x) - B^{*}(t,x)\| \leq \|\hat{B}^{(n)}(t,x) - B^{(n)}(t,x)\| + \|B^{(n)}(t,x) - B^{(*)}(t,x)\|,$$

by using the contraction mapping theorem, we have:

$$\|\hat{B}^{(n)}(t,x) - B^{*}(t,x)\| \leq \frac{\delta}{1-\theta} + \theta^n r_0, \qquad (39)$$

which completes the proof. ■

Remark 3.1. We observe that the following results are obtained. We draw two conclusions from the above three theorems

1. According to Theorem 4, it can be concluded $B^{(n)}$,has an error bound.
2. Existence and uniqueness of solution for (1)-(14) are produced.

Remark 3.2. Based on the above theorem, we obtain two results for (33):
1. Approximation of solution for (33),
2. A posterior error for (33).

4. Discretization and Error Analysis

Based on the successive algorithm of VMFP we give the following subsection.

4.1. **Discretization and error analysis for** f, ρ, y

We discrete the following subproblems:

$$f^{(n)} = f_0 + \int_0^t f_1^{(n-1)}(t,x,v)ds = \Gamma_1[f_1^{(n-1)}], \tag{40}$$

$$B_i(f_{i+1}^{(n)}, g) = L_i(g), \quad i = 1,2,3,4,5, \tag{41}$$

and

$$\rho^{(n)}(t,x) = \int_{\Omega_v} f^{(n)}(t,x,v)dv - \phi(x). \tag{42}$$

Since, (40) and (42) have integral operators, hence we investigate according to the following lemma.

Lemma 4.1. *Let* $\varphi_1, \varphi_2, \ldots$ *be a basis for* $W_0^{1,2}(\Omega)$ *and* $V_N = span\{\varphi_1, \varphi_2, \ldots, \varphi_N\} \subset W_0^{1,2}$. *Also, we assume that* Γ_N *be a sequences of uniformly bounded linear operators* $\sup_N \|\Gamma_N\| \leq l$ *such that* Γ *is defined on* V_N *that* $\Gamma_N f = \Gamma f$ *for all* $f \in V_N$. *Then*

$$\lim_{N\to\infty} (y_N = \Gamma_N f_N) \to (y = \Gamma f), \tag{43}$$

and

$$\|y - y_N\|_{W_0^{1,p}} \leq (\|\Gamma\| + l)\epsilon_N, \tag{44}$$

where $\epsilon_N = \inf_{f_N \in V_N} \|f - f_N\|$.

Proof. The essential observation is that $\|y - y_N\| = \|(\Gamma - \Gamma_N)f\| \leq \|\Gamma(f - f_N)\| + \|\Gamma_N(f - f_N)\|$. Also, we have $\Gamma f = \Gamma_N f_N$. Thus, we have

$$\|y - y_N\| \leq (\|\Gamma\| + l)\|f - f_N\|$$

for all $f_N \in V_N$ therefore we obtain (44). On the other hand, we know that $\{\varphi_i\}_{i=0}^N$ is a set of basis for $W_0^{1,p}$ there exists a sequence of $f_N \in V_N$ such that $\lim_{N\to\infty} \|f - f_N\|_{W_0^{1,p}} = 0$, therefore we have (43).

Finally, we use the following remark for approximating of integration and error analysis.

Remark 4.1. The above Lemma gives the well know result such that the Gauss-Legendre quadrature rule converges for all continuous integrands and we have $\|\Gamma_N\| = 2$. Also, we show that N-point Gauss-Legendre rule with $f \in C^{(2N)}[-1,1]$

$$|\Gamma f - \Gamma_N f| \leq \frac{1}{2^{2N-3}(2N)!} M_{2N},$$

where $M_{2N} = \max_{-1\leq t\leq 1} |f^{(2N)}|$, $\Gamma f = \int_{-1}^{1} f(t)dt$ and $\Gamma_N f = \sum_{j=1}^{N} w_j f(t_j)$ such that $\{t_j\}_{j=0}^N$ are the roots of the N th Legendre polynomial with the w_j the appropriate weights.

4.1.1 *Discretization and error analysis for (22)*

We assume that $f_1^{(n-1)}$ and f_0 are given and $f^{(n)}$ is unknown. Suppose that M is a positive integer and $h = \frac{T}{M}$ then $\{\xi_i\}_{i=0}^{M}$ constitutes a uniform partition on $[0,T]$ and $h = \xi_{i+1} - \xi_i$. Also, $\{S_j^{(i)}\}_{j=1}^{N} \in [0,\xi_i]$ be a sequence of transformation of the roots of the N Legendre polynomial then truncation error for integration and interpolation error are as follows (based on Remark 4.1). We assume that

$$f_N^{(n)}(\xi_i,x,v) = f_0(x,v) + \sum_{j=1}^{N} w_j\, f_1(s_j^{(i)},x,v), \quad i = 0,1,2,\ldots,M, \tag{45}$$

be support points. On the other hand, we know that

$$f^{(n)}(\xi_i,x,v) = f_0(x,v) + \int_0^{\xi_i} f_1^{(n-1)}(s,x,v)ds, \quad i = 0,1,2,\ldots,M, \tag{46}$$

hence, the truncation error in ξ_i is equal

$$\varepsilon\tau(\xi_i,x,v) := f^{(n)}(\xi_i,x,v) - f_N^{(n)}(\xi_i,x,v), \quad i = 0,1,2,\ldots,M. \tag{47}$$

Now, we assume that $f_{NI}^{(n)}(t,x,v) \in S_N^3$ is the interpolation for $f_N^{(n)}(t,x,v)$. The space of S_M^3 consists of all cubic spline functions which belong to $C^2([0,T] \times [0,A] \times \Omega_v)$. For

$$f_N^{(n)}(t,x,v) = f_0(x,v) + \sum_{j=1}^{N} w_j f_1(t - \xi_i + s_j^{(i)}), \quad i = 0,1,2,\ldots,M. \tag{48}$$

Therefore, we have the optimal error bound as follows:

$$C_{m,r} = \sup_{n\geq 1} \frac{\left\| \frac{\partial^r}{\partial t^r} f_N^{(n)}(t,x,v) - \frac{\partial^r}{\partial t^r} f_{NI}^{(n)}(t,x,v) \right\|_\infty}{\left\| \frac{\partial^m}{\partial t^m} f_N^{(n)}(x) \right\|_\infty h^{m-r}}, \quad 1 \leq r \leq 4, \tag{49}$$

such that $f \in C^m([0,T] \times [0,A] \times \Omega_v), 0 \leq r \leq \min\{m,3\}$ (See [3]). For the lower smooth functions ($m < 4$) it is much more difficult. It is known that $C_{4,0} = \frac{5}{384}, C_{4,1} = \frac{1}{24}$ (see [3]). $C_{4,2} = \frac{\sqrt{13}}{12}$ (see [24]), $C_{4,1} = \frac{1+\sqrt{3}}{4}$, $C_{3,3} = \frac{5+\sqrt{3}+\sqrt{6}-\sqrt{2}}{3}$ and $C_{3,2} = \frac{\sqrt{3}+\sqrt{6}}{9}$ (see [15]), $C_{11} = 3$ (see [22]). $C_{2,2} = \frac{5+\sqrt{3}}{2}$, $C_{1,0} = \frac{5}{6}$. Therefore, the size of global error for (22) is

$$\begin{aligned}
\bar{\epsilon}^m &:= \|f^{(n)}(t,x,v) - f_{NI}^{(n)}(t,x,v)\|_{L_\infty(\Omega)} \\
&\leq \|f^{(n)}(t,x,v) - f_N^{(n)}(t,x,v)\|_{L_\infty(\Omega)} + \|f_N^{(n)}(t,x,v) - f_{NI}^{(n)}(t,x,v)\|_{L_\infty(\Omega)} \\
&\leq (\|\Gamma_1\|_{H^1} + l_1)\epsilon_N + C_{m,0}\, h^m \left\|\frac{\partial^m f_N^{(n)}(x)}{\partial t^m}\right\|_{L_\infty(\Omega)}.
\end{aligned} \tag{50}$$

We repeat the above subject for computing the size of global error for (31) and (32). Also, according to the Theorems of 3 and 4 we compute the global error for (34).

Lemma 4.2. *If we assume that $\{v_{j,k}|v_{j,k} \in \Omega_v\}_{j,k=1}^{v}$ is a set of Gaussian points in Ω_v then for discretization (28) we have the following rules*

$$\rho_N^{(n)}(t,x) = \sum_{j,k=1}^{N} w_{j,k} f_{IN}(t,x,v_{j,k}) - \phi(x)$$

where $w_{j,k}$ are the coefficients of Gaussian rule and also, we have:

$$\|\rho_N^{(n)} - \rho^{(n)}\|_{L_\infty(\Omega)} \leq (\|\Gamma_2\| + l_2) C_{m,0} h^m M_m$$

here, M_m dependents to the order of regularity of $f(t,x,v)$.

Proof. An argument similar to the one used in Lemma 4.1 and (50) show that it completes the proof. ■

4.1.2 *Discretization and error analysis for (23)-(26)*

If P is a regular partitioning of Ω and $\mathsf{K} \in \mathsf{P}$ such that $\underline{h} = \max_{\mathsf{K}\in\mathsf{P}} h_{\mathsf{K}}$ and $h_{\mathsf{K}} = \dim \mathsf{K}$. Suppose $V_{\underline{h}} = span\{\psi_k\}_{k=1}^{\bar{N}}$ be a space of $W^{1,2}(\Omega)$. The Galerkin discretization $\forall g \in V_{\underline{h}}$ is as follows:

$$\begin{cases} \text{Find} \quad f_{i+1\underline{h}}^{(n)} \in V_{\underline{h}} \\ \text{s.t.} \quad B_i(f_{i+1\underline{h}}^{(n)}, g) = L_i(g), \\ \quad i = 1,2,3,4. \end{cases} \tag{51}$$

Therefore, we have for $i = 2,3,4,5$ and $\forall g \in V_{\underline{h}}$

$$B_i(f_{i+1}^{(n)} - f_{i+1\underline{h}}^{(n)}, g) = 0. \tag{52}$$

If we put $f_{i\underline{h}}^{(n)} = \sum_{j=1}^{\bar{N}} u_j^{(i,n)} \psi_j$, $i = 2,3,4,5$ and use (45) then we obtain the following linear systems associated with (51) and given basis:

$$B^{(i)} u^{(i,n)} = L^{(i)}, \quad i = 1,2,3,4, \tag{53}$$

where $B^{(i)} = (B_{jk}^{(i)} = (\psi_j^{(i)}, \psi_k))_{k,j=1}^{\bar{N}}, L^{(i)} = (L_i(\psi_k))_{k=1}^{\bar{N}}$ and $u^{(i,n)} = (u_j^{(i,n)})_{j=1}^{\bar{N}}$.

Lemma 4.3. *If $f_i^{(n)} \in V_{\underline{h}} \subset W_0^{1,p=2}(\Omega)$ satisfies (23), (24), (25), (26) and (51) then*

$$\|f_i^{(n)} - f_{i\underline{h}}^{(n)}\|_{W_0^{1,p}} \leq \beta_i \inf_{g\in V_{\underline{h}}} \|f_i - g\|_{W_0^{1,p}}, \quad i = 2,3,4,5, \tag{54}$$

i.e. the Galerkin approximation $f_{i\underline{h}}$, $i = 2,3,4,5$ are the best approximation from the space $V_{\underline{h}}$.

Proof. We know that $|B_{i-1}(f_i^{(n)}, g)| \leq \gamma_i \|f_i^{(n)}\|_{W_0^{1,p}} \|g\|_{W_0^{1,p}}$ $i = 2,3,4,5,$ for $f_i, g \in W_0^{1,p}$ are satisfy. Also, $B_{i-1}(f_i^{(n)}, f_i^{(n)}) \geq \alpha_i \|f_i^{(n)}\|^2_{W_0^{1,p}}$. On the other hand $\forall g \in V_{\underline{h}}$ we have

$$\alpha_i \|f_i^{(n)} - f_{i\underline{h}}^{(n)}\|^2_{W_0^{1,p}} \leq B_{i-1}(f_i^{(n)} - f_{i\underline{h}}^{(n)}, f_i^{(n)} - f_{i\underline{h}}^{(n)})$$

$$= B_{i-1}(f_i^{(n)} - f_{ih}^{(n)}, f_i^{(n)} - g) \leq \gamma_i \| f_i^{(n)} - f_{i\underline{h}}^{(n)} \|_{W_0^{1,p}} \| f_i^{(n)} - g \|_{W_0^{1,p}}.$$

Moreover, we have

$$\| f_i^{(n)} - f_{i\underline{h}}^{(n)} \|_{W_0^{1,p}} \leq \frac{\gamma_i}{\alpha_i} \| f_i^{(n)} - g \|_{W_0^{1,p}} \quad i = 2,3,4,5,$$

also, we write $\forall g \in V_{\underline{h}}$

$$\| f_i^{(n)} - f_{ih}^{(n)} \|_{W_0^{1,p}} \leq \frac{\gamma_i}{\alpha_i} \inf \| f_i^{(n)} - g \|_{W_0^{1,p}}, \quad i = 2,3,4,5, \tag{55}$$

and this is a priori estimator for $B_{i-1}(f_i^{(n)}, g) = L_i(g), \quad i = 2,3,4,5.$ ■

Remark 4.2. If we put $g = \pi f_i \quad i = 2,3,4,5$ in (55) and πf_i is obtained by tensor product B-splines (based on the de Boor-Fix functional) on domain Ω, we estimate $\| f_i - \pi f_i \|_{W^{1,p}}$ (See Bernhard Möβner and Ulrich Reif [7]).

4.1.3(a) *A posteriori error estimation for (23)*

If we define the energy norm in the following form:

$$\| \bar{v} \|_{L^2(\Omega)} = \left(\int_\Omega |\bar{v}(t,x,v)|^2 ds \right)^{\frac{1}{2}},$$

then we can begin introducing the error indicators. Let $f_{2\underline{h}}^{(n)}$ be the solution of standard finite element approximation. Hence, we write $e_2^{(n)} = f_2^{(n)} - f_{2\underline{h}}^{(n)}$ then

$$\begin{aligned} \| e_2^{(n)} \|_{L_2(\Omega)}^2 &= \int_\Omega (f_2^{(n)} - f_{2\underline{h}}^{(n)}) e_2^{(n)} d\Omega \\ &= \int_\Omega f_2^{(n)} e_2^{(n)} d\Omega - \int_\Omega f_{2\underline{h}}^{(n)} e_2^{(n)} d\Omega \\ &= \int_\Omega (\hat{v}_1 \partial_x f^{(n)}) e_2^{(n)} d\Omega - \int_\Omega f_{2\underline{h}}^{(n)} e_2^{(n)} d\Omega. \end{aligned}$$

On the other hand, we write

$$\begin{aligned} \| e_2^{(n)} \|_{L_2(\Omega)}^2 = &\int_\Omega (\hat{v}_1 \partial_x f^{(n)})(e_2^{(n)} - \pi_h e_2^{(n)}) d\Omega + \int_\Omega (\hat{v}_1 \partial_x f^{(n)}) \pi_h e_2^{(n)} dx \\ &- \int_\Omega f_{2\underline{h}}^{(n)} (e_2^{(n)} - \pi_h e_2^{(n)}) d\Omega - \int_\Omega f_{2\underline{h}}^{(n)} \pi_h e_2^{(n)} d\Omega, \end{aligned}$$

since, we know that

$$\int_\Omega (\hat{v}_1 \partial_x f^{(n)}) \pi_h e_2^{(n)} d\Omega = \int_\Omega f_{2\underline{h}}^{(n)} \pi_h e_2^{(n)} d\Omega, \pi_h e_2^{(n)}, f_{2\underline{h}}^{(n)} \in V_{\underline{h}}.$$

Thus with defining $R_2(f_{2\underline{h}}^{(n)}) = \hat{v}_1 \partial_x f^{(n)} - f_{2\underline{h}}^{(n)}$, we have

$$\begin{aligned}\|e_2^{(n)}\|^2_{L_2(\Omega)} &= \int_\Omega (\hat{v}_1\partial_x f^{(n)})(e_2^{(n)} - \pi_h e_2^{(n)})d\Omega - \int_\Omega f_{2\underline{h}}^{(n)}(e_2^{(n)} - \pi_h e_2^{(n)})d\Omega \\ &= \int_\Omega (\hat{v}_1\partial_x f^{(n)} - f_{2\underline{h}}^{(n)})(e_2^{n)} - \pi_h e_2^{(n)})d\Omega \\ &= \int_\Omega R_2(f_{2\underline{h}}^{(n)})(e_2^{(n)} - \pi_h e_2^{(n)})d\Omega \\ &\leq (\int_\Omega R_2^2(f_{2\underline{h}}^{(n)})d\Omega)^{\frac{1}{2}}(\int_\Omega (e_2^{(n)} - \pi_h e_2^{(n)})^2 d\Omega)^{\frac{1}{2}}.\end{aligned}$$

On the other hand, we write

$$(\int_\Omega (e_2^{(n)} - \pi_h e_2^{(n)})^2 d\Omega)^{\frac{1}{2}} \leq C_2\|e_2^{(n)}\|_{L_2(\Omega)},$$

because, π_h is a linear piecewise interpolation. Therefore, we have:

$$\|e_2^{(n)}\|_{L_2(\Omega)} \leq C_2(\int_\Omega R_2^2(f_{2h}^{(n)})d\Omega)^{\frac{1}{2}}. \quad (56)$$

Thus the following lemma is true.

Lemma 4.4. The a posteriori error for (23) is as follows:

$$\|e_2^{(n)}\|_{L_2(\Omega)} \leq C_2(\int_\Omega R_2^2(f_{2\underline{h}}^{(n)})d\Omega)^{\frac{1}{2}} = C_2\|R_2(f_{2\underline{h}}^{(n)})\|_{L_2(\Omega)}. \quad (57)$$

Remark 4.3. We approximate $(\|R_2(f_{2\underline{h}}^{(n)})\|_{L_2(\Omega)}$:

$$\begin{aligned}R_2(f_{2\underline{h}}^{(n)}) &= \hat{v}_1\partial_x f^{(n)} - f_{2\underline{h}}^{(n)} + \hat{v}_1\partial_x f_{NI}^{(n)} - \hat{v}_1\partial_x f_{NI}^{(n)} \\ &= (\hat{v}_1\partial_x(f^{(n)} - f_{NI}^{(n)})) + (\hat{v}_1\partial_x f_{NI}^{(n)} - f_{2h}^{(n)}),\end{aligned}$$

therefore, we write

$$\|R_2(f_{2\underline{h}}^{(n)})\|_{L_2(\Omega)} \leq \|\hat{v}_1\|_{L_2(\Omega)}\|\partial_x(f^{(n)} - f_{NI}^{(n)})\|_{L_2(\Omega)} + \|\hat{v}_1\partial_x f_{NI}^{(n)} - f_{2\underline{h}}^{(n)}\|_{L_2(\Omega)}.$$

According to (19) and $f_{NI}^{(n)}, f_{2\underline{h}}^{(n)}$ that are known we have:

$$\|R_2(f_{2\underline{h}}^{(n)})\|_{L_2(\Omega)} \leq C_{m,r}\|\hat{v}_1\|_{L_2(\Omega)}h^{m-r}M + \|\hat{v}_1\|_{L_2(\Omega)}\|\partial_x f_{NI}^{(n)}\|_{L_2(\Omega)} + \|f_{2\underline{h}}^{(n)}\|_{L_2(\Omega)}.$$

4.1.3(b) *A posteriori error estimation for (24)*

According to the section 4.1.3 (a) we state the following Lemma.

Lemma 4.5. The a posteriori error for (24) is as follows:

$$\|e_3^{(n)}\|_{L_2(\Omega)} \leq C_3(\|E\|+\|B\|\|M\|\|\hat{v}\|)(C_{m,r}\bar{M}h^{m-r} + \|\nabla_x f_{NI}^{(n)}\|) + \|f_{3h}^{(n)}\|. \quad (58)$$

Proof. We assume that $e_3^{(n)} = f_3^{(n)} - f_{3\underline{h}}^{(n)}$ and we have

$$\begin{aligned}
\|e_3^{(n)}\|^2_{L_2(\Omega)} &= \int_\Omega (f_3^{(n)} - f_{3\underline{h}}^{(n)})e_3^{(n)} d\Omega \\
&= \int_\Omega f_3^{(n)} e_3^{(n)} d\Omega - \int_\Omega f_{3\underline{h}}^{(n)} e_3^{(n)} d\Omega \\
&= \int_\Omega ((E + BM\hat{v}).\nabla_v f^{(n)}) e_3^{(n)} d\Omega - \int_\Omega f_{3\underline{h}}^{(n)} e_3^{(n)} d\Omega \\
&= \int_\Omega ((E + BM\hat{v}).\nabla_v f^{(n)})(e_3^{(n)} - \pi_{\underline{h}} e_3^{(n)}) d\Omega \\
&\quad + \int_\Omega ((E + BM\hat{v}).\nabla_v f^{(n)})\pi_{\underline{h}} e_3^{(n)} d\Omega - \int_\Omega f_{3\underline{h}}^{(n)} (e_3^{(n)} - \pi_h e_3^{(n)}) d\Omega \\
&\quad - \int_\Omega f_{3h}^{(n)} \pi_{\underline{h}} e_3^{(n)} d\Omega.
\end{aligned}$$

Since, we know that for all $\pi_{\underline{h}} e_3^{(n)} f_{3\underline{h}}^{(n)} \in V_{\underline{h}}$

$$\int_\Omega ((E + BM\hat{v}).\nabla_v f^{(n)})\pi_{\underline{h}} e_3^{(n)} d\Omega = \int_\Omega f_{3\underline{h}}^{(n)} \pi_{\underline{h}} e_3^{(n)} d\Omega.$$

Thus, we have

$$\begin{aligned}
\|e_3^{(n)}\|^2_{L_2(\Omega)} &= \int_\Omega ((E + BM\hat{v}).\nabla_v f^{(n)})(e_3^{(n)} - \pi_{\underline{h}} e3^{(n)}) d\Omega \\
&\quad - \int_\Omega f_{3\underline{h}}^{(n)} (e_3^{(n)} - \pi_{\underline{h}} e_3^{(n)}) d\Omega \\
&= \int_\Omega ((E + BM\hat{v}).\nabla_v f^{(n)} - f_{3\underline{h}}^{(n)})(e_3^{(n)} - \pi_{\underline{h}} e_3^{(n)}) d\Omega \\
&\leq (\int_\Omega R_3^2(f_{3\underline{h}}^{(n)}) d\Omega)^{\frac{1}{2}} (\int_\Omega (e_3^{(n)} - \pi_{\underline{h}} e_3^{(n)})^2 d\Omega)^{\frac{1}{2}} \\
&\leq C_3 \|R_3(f_{3\underline{h}}^{(n)})\|_{L_2(\Omega)} \|e_3^{(n)}\|_{L_2(\Omega)},
\end{aligned}$$

where $R_3^2(f_{3\underline{h}}^{(n)}) = (E + BM\hat{v}).\nabla_v f^{(n)} - f_{3\underline{h}}^{(n)}$. Therefore, we write

$$\|e_3^{(n)}\|_{L_2(\Omega)} \leq C_3 \|R_3(f_{3\underline{h}}^{(n)})\|_{L_2(\Omega)}.$$

Also, we approximate $\|R_3(f_{3\underline{h}}^{(n)})\|_{L_2(\Omega)}$ as follows:

$$\begin{aligned}
\|R_3(f_{3\underline{h}}^{(n)})\|_{L_2(\Omega)} &= \|(E + BM\hat{v}).\nabla_x f^{(n)} - f_{3\underline{h}}^{(n)} + (E + BM\hat{v}).\nabla_x f_{NI}^{(n)} \\
&\quad - (E + BM\hat{v}).\nabla_x f_{NI}^{(n)}\|_{L_2(\Omega)} \\
&\leq \|(E + BM\hat{v}).\nabla_x (f^{(n)} - f_{NI})\|_{L_2(\Omega)} + \|(E + BM\hat{v}).\nabla_x f_{NI}^{(n)} - f_{3\underline{h}}^{(n)}\|_{L_2(\Omega)} \\
&\leq \|E + BM\hat{v}\|_{L_2(\Omega)} C_{m,r} \bar{M} h^{m-r} + \|(E + BM\hat{v})\|_{L_2(\Omega)} \|\nabla_x f_{NI}^{(n)}\|_{L_2(\Omega)}) + \|f_{3\underline{h}}^{(n)}\|_{L_2(\Omega)} \\
&= \|E + BM\hat{v}\|_{L_2(\Omega)} (C_{m,r} \hat{M} h^{m-r} + \|\nabla_x f_{NI}^{(n)}\|_{L_2(\Omega)}) + \|f_{3h}^{(n)}\|_{L_2(\Omega)},
\end{aligned}$$

hence, we have

$$\|R_3(f_{3\underline{h}}^{(n)})\|_{L_2(\Omega)} \le (\|E\|_{L_2(\Omega)} + \|B\|_{L_2(\Omega)}\|M\|_{L_2(\Omega)}\|\hat{v}\|_{L_2(\Omega)})(C_{m,r}\bar{M}h^{m-r} + \|\nabla_x f_{NI}^{(n)}\|_{L_2(\Omega)}) + \|f_{3\underline{h}}^{(n)}\|_{L_2(\Omega)}.$$

If we consider $E = (E_1, E_2)$ and $\|E\|_{L_2(\Omega)}$ are locally bounded in $b(B^{(0)}(t,x), r)$ then based on Theorem 3.2, we know that B is locally bounded in $b(B^{(0)}(t,x), r)$. ■

4.1.3(c) *A posteriori error estimation for (25)*

If we consider $f_{4\underline{h}}^{(n)} \in V_{\underline{h}}$ as the finite element approximation such that according to (25) we have

$$B_3(f_{4\underline{h}}^{(n)}, g_{\underline{h}}) = L_3(g_{\underline{h}}) \quad \forall g_{\underline{h}} \in V_{\underline{h}} \,. \tag{59}$$

The residual operator R_3 is defined by

$$R_3(f_4^{(n)}) = f_4^{(n)} - T_4 f^{(n)}.$$

and therefore we have:

$$(R_3(f_4^{(n)}), g) = L_3(g) - B_3(f_4^{(n)}, g) \quad g, f_4(n) \in W^{1,2}$$

we observe, in particular that

$$(R_3(f_4^{(n)}), g) = 0 \quad \forall g \in W^{1,2}, \tag{60}$$

and

$$(R_3(f_{4\underline{h}}^{(n)}), g_{\underline{h}}) = 0 \quad \forall g_{\underline{h}} \in V_{\underline{h}}.$$

The derivative of the residual operator at $k \in W^{1,2}$ is defined as follows

$$\frac{1}{\phi} \lim_{\epsilon \to 0} \frac{R_3(k+\epsilon\phi) - R_3(k)}{\epsilon} = DR_3(k) \quad 0 \neq \phi \in W^{1,2}. \tag{61}$$

Therefore, the derivative of the residual operator at $f_4 \in W^{1,2}$ is defined for $\phi^{(n)} \in W^{1,2}$

$$\begin{aligned}
(DR_3(f_4^{(n)})\phi^{(n)}, g) &= \left(\lim_{\epsilon \to 0} \frac{R_3(f_4^{(n)} + \epsilon\phi^{(n)}) - R_3(f_4^{(n)})}{\epsilon}, g \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[\Big(R_3(f_4^{(n)} + \epsilon\phi^{(n)}), g \Big) - \Big(R_3(f_4^{(n)}), g \Big) \Big] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} [L_3(g) - B_3(f_4^{(n)} + \epsilon\phi^{(n)}, g) - L_3(g) + B_3(f_4^{(n)}, g)] \\
&= \lim_{h \to 0} \frac{1}{\epsilon} [-\int_\Omega (f_4^{(n)} + \epsilon\phi)^{(n)} g d\Omega + \int_\Omega (f_4^{(n)}) g d\Omega] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\epsilon(-1) \int_\Omega \phi^{(n)} g d\Omega],
\end{aligned}$$

if $\phi^{(n)} = g$ then

$$(DR_3(f_4^{(n)})g,g) = -(g,g) = -\int_\Omega g^2 d\Omega = -\|g\|^2_{W^{1,2}}. \tag{62}$$

Since, DR_3 the bounded linear mapping of $L(W^{1,p},(W^{1,p})^*)$ and if we put $\forall\phi^{(n)} \in W^{1,2}$, $f_{4\underline{h}}^{(n)} = f_4^{(n)} + \underline{h}\phi^{(n)}$ then we have

$$\|DR_3(f_4^{(n)}) - DR_3(f_{4\underline{h}}^{(n)})\|_{W^{1,2}} \le \bar{C}\|f_4^{(n)} - f_{4\underline{h}}\|_{W^{1,2}} = \bar{C}\underline{h}\|\phi^{(n)}\|_{W^{1,2}} \le \bar{\bar{C}}\underline{h}.$$

Theorem 4.1. *Let $f_4^{(n)}$ be a unique regular solution for (25) and $f_{4\underline{h}}^{(n)}$ is the approximated solution of standard finite element such that $e_4^{(n)} = f_4^{(n)} - f_{4\underline{h}}^{(n)}$ then we have the following inequality:*

$$\frac{1}{C}\|R_3^{(n)}(f_{4\underline{h}})\|_{(W^{1,p})^*} \le \|e_4^{(n)}\|_{W^{1,2}} \le C\|R_3(f_{4\underline{h}}^{(n)}\|_{(W^{1,p})^*}.$$

Proof. We write

$$(R_3(f_{4\underline{h}}^{(n)}),g) = (R_3(f_4^{(n)}),g) + \int_0^1 (DR_3(f_4^{(n)} + se_4^{(n)})e_4^{(n)},g)ds$$

by using (60), we have:

$$(R_3(f_{4\underline{h}}^{(n)}),g) = \int_0^1 (DR_3(f_4^{(n)} + se_4^{(n)})e_4^{(n)},g)ds \quad \forall g \in W^{1,2}.$$

Hence, we write

$$(DR_3(f_4^{(n)})e_4^{(n)},g) = \int_0^1 (DR_3(f_4^{(n)})e_4^{(n)} - DR_3(f_4^{(n)} + se_4^{(n)})e_4^{(n)},g)ds + (R_3(f_{4\underline{h}}^{(n)}),g),$$

or

$$(DR_3(f_4^{(n)})e_4^{(n)},g) = (\mathsf{F}^{(n)},g). \tag{63}$$

If $f_4^{(n)} \in W^{1,2}$ be a regular solution then there is a positive constant C_0 such that from (63) we have:

$$\|e_4^{(n)}\|_{W^{1,2}} \le \bar{C}_0\|\mathsf{F}^{(n)}\|_{W^{1,2}}.$$

The quantity appearing on the right hand side may be estimated as follows:

$$|\int_0^1 (DR_3(f_4^{(n)})e_4^{(n)} - DR_3(f_4^{(n)} + se_4^{(n)})e_4^{(n)},g)ds|$$
$$\le \int_0^1 \|DR_3(f_4^{(n)})e_4^{(n)} - DR_3(f_4^{(n)} + se_4^{(n)})e_4^{(n)}\|_{L(H^1,(H^1)^*)}\|g\|_{H^1}ds$$

and

$$\|DR_3(f_4^{(n)})e_4^{(n)} - DR_3(f_4^{(n)} + se_4^{(n)})e_4^{(n)}\|_{L(H^1,(H^1)^*)}$$
$$\leq s\|DR_3(f_4^{(n)}) - DR_3(f_4^{(n)} + se_4^{(n)})\|_{L(H^1,(H^1)^*)}\|e\|_{H^1},\ s \in [0,1].$$

If the derivation of R_3 is Lipschitzian, then we have

$$|\int_0^1 (DR_3(f_4^{(n)})e_4^{(n)} - DR_3(f_4^{(n)} + se_4^{(n)})e_4^{(n)}, g)dt| \leq \bar{\bar{C}}_1\|e_4^2\|_{H^1}\int_0^1 sds\|g\|_{H^1}.$$

Therefore, we have

$$\|\mathsf{F}^{(n)}\|_{(H^1)^*} \leq \|R_3(f_{4\underline{h}}^{(n)})\|_{(H^1)^*} + \frac{1}{2}\bar{\bar{C}}_1\|e_4^{(n)}\|^2_{H^1(\Omega)},$$

consequently

$$\|e_4^{(n)}\|_{H^1} \leq \bar{C}_0\|R_3(f_{4\underline{h}}^{(n)})\|_{(H^1)^*} + \frac{1}{2}\bar{C}_0\bar{\bar{C}}_1\|e_4^{(n)}\|^2_{H^1(\Omega)},$$

and if $\|e_4^{(n)}\|_{H^1}$ is sufficiently small, say

$$\|e_4^{(n)}\|_{H^1} \leq \frac{1}{\bar{C}_0\bar{\bar{C}}_1}.$$

Thus, we write $\|e_4^{(n)}\|_{H^1} \leq 2\bar{C}_0\|R_3(f_{4\underline{h}}^{(n)}\|_{(H^1)^*}$. Using the bounds derived earlier leads to the conclusion

$$\|R_3(f_{4\underline{h}}^{(n)})\|_{(H^1)^*} \leq \|DR_3(f_{4\underline{h}}^{(n)})\|_{L(H^1,(H^1)^*)}\|e_4^{(n)}\|_{H^1} + \frac{1}{2}\bar{\bar{C}}_1\|e\|^2_{H^1}.$$

Therefore, the proof is completed. ■

4.1.3(d) *A posteriori error estimation for (26)*

Theorem 4.2. *Suppose that $f_5^{(n)}$ is a regular and unique solution of (26) such that $e_4^{(n)} = f_5^{(n)} - f_{5\underline{h}}^{(n)} \in W^{1,2}$ denotes the error in the standard finite element approximation then*

$$\frac{1}{\underline{C}}\|R_4(f_{5\underline{h}}^{(n)})\|_{(H^1)^*} \leq \|e_5^{(n)}\|_{H^1} \leq \underline{C}\|R_4(f_{5\underline{h}}^{(n)})\|_{(H^1)^*}.$$

Proof. We define DR_4, moreover, by using (61) and repeat all of result from section 4.1.4(c) and some tedious manipulation yields the above inequality. ■

4.2. **Estimation of the residual** $R_{i-1}(f_{ih}^{(n)}),\quad i = 4,5$

In the above theorems, we observe that we need to estimate the residual $R_{i-1}(f_{ih}^{(n)}),\quad i = 4,5$. We write

$$\begin{aligned}(R_3(f_{4h}^{(n)}),g) &= L_3(f_{4h}^{(n)}) - B_3(f_{4h}^{(n)},g)\\ &= \sum_{\mathsf{K}\in\mathsf{P}}(\int_{\mathsf{K}}(\beta(\hat{v}f^{(n)}).\nabla_v g)d\mathsf{K} - \int_{\mathsf{K}} f_{4h}^{(n)} g d\mathsf{K}),\end{aligned}$$

and

$$\begin{aligned}(R_4(f_{5h}^{(n)}),g) &= L_4(f_{5h}^{(n)} - B_4(f_{5h}^{(n)},g)\\ &= \sum_{\mathsf{K}\in\mathsf{P}}(\int_{\mathsf{K}}\nabla_v(g+v).\nabla_v f^{(n)}dk - \int_{\mathsf{K}} f_{5h}^{(n)} g d\mathsf{K}),\end{aligned}$$

where P is a regular partitioning of Ω and $\mathsf{K}\in\mathsf{P}$ such that $h = \max_{\mathsf{K}\in\mathsf{P}} L_{\mathsf{K}}$, $L_{\mathsf{K}} = \mathrm{diam}\mathsf{K}$. We assume the following conditions are satisfied
(i) $f_{ih}^{(n)} + f_{ih'}^{(n)} = 0 \quad i = 4,5$ on $\partial\mathsf{K}\cap\partial\mathsf{K}'$, $\forall\mathsf{K},\mathsf{K}'\in\mathsf{P}$ also we assume $(h = \mathrm{diam}\mathsf{K}, h' = \mathrm{diam}\mathsf{K}')$.
Physically, this condition express the requirement that the approximations should not be generated on the actual interface. Naturally since the true solution is known on the partition of exterior boundary $\tilde{\Gamma}_N$ where a Neumann condition is prescribed, the approximate solution is chosen to concede with the true solution on $\tilde{\Gamma}_N$,
(ii) $f_{ih}^{(n)} = f_i^{(n)}$ on $\partial\mathsf{K}\cap\tilde{\Gamma}_N, i = 4,5$.
Together, the conditions (i) and (ii) imply that for all $g\in W^{1,2} = H^1$ we have:

$$\int_{\tilde{\Gamma}_N} f_i^{(n)} g ds = \sum_{\mathsf{K}\in\mathsf{P}}\int_{\partial\mathsf{K}} f_{ih}^{(n)} g ds.$$

Therefore, the residuals may be decomposed into local conditions,

$$(R_i(f_{i+1}^{(n)},g)_{\mathsf{K}} = \sum_{\mathsf{K}\in\mathsf{P}}(R_{ik}(f_{i+1,h}^{(n)}),g)_{\mathsf{K}}, \quad i = 3,4$$

where

$$(R_{3\mathsf{K}}(f_{4h}^{(n)}),g)_{\mathsf{K}} = \int_{\mathsf{K}}\beta(\hat{v}f^{(n)}).\nabla_v g d\mathsf{K} - \int_{\mathsf{K}} f_{4h}^{(n)} g d\mathsf{K} + \int_{\partial\mathsf{K}} f_{4h}^{(n)} g ds,$$

and

$$(R_{4\mathsf{K}}(f_{5h}^{(n)}),g)_{\mathsf{K}} = \int_{\mathsf{K}}\nabla_v(g+v).\nabla_v f^{(n)} d\mathsf{K} - \int_{\mathsf{K}} f_{5h}^{(n)} g d\mathsf{K} + \int_{\partial\mathsf{K}} f_{5h}^{(n)} g ds,$$

are the residuals around the approximate solutions $\{f_{ih}^{(n)}\}_{i=4}^{5}$.
Define $\{\phi_{i_{\mathsf{K}}}^{(n)} | \phi_{i_{\mathsf{K}}}^{(n)} \in \nabla_{\mathsf{K}}, i = 4,5$ to be the solutions of the local residual problems:

$$B_{i\mathsf{K}}(\phi_{i+1\mathsf{K}}^{(n)},g) = (R_{i\mathsf{K}}(f_{i+1,h}^{(n)}),g) \quad \forall g\in V_h, \quad i = 3,4.$$

so that

$$(R_i(f_{i+1,h}^{(n)}),g) = \sum_{\mathsf{K}\in\mathsf{P}} B_{i\mathsf{K}}(\phi_{i+1,\mathsf{K}}^{(n)},g), \quad i = 3,4.$$

In particular, with the aid of Hölder's inequality we obtain

$$\begin{aligned} |B_{i\mathsf{K}}(\phi^{(n)}_{i+1,\mathsf{K}}, g)| &= |\int_{\mathsf{K}} \phi^{(n)}_{i+1,\mathsf{K}} g d\mathsf{K}| \\ &\leq \|\phi^{(n)}_{i+1,\mathsf{K}}\|_{L_{q_1}(\mathsf{K})} \|g\|_{L_{q_2}(\mathsf{K})}, \end{aligned}$$

where $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Therefore, we have

$$\begin{aligned} |R_i(f^{(n)}_{i+1,h}, g)| &\leq \textstyle\sum_{\mathsf{K}\in\mathsf{P}} |B_{i\mathsf{K}}(\phi^{(n)}_{i+1,\mathsf{K}}, g)| \\ &\leq \textstyle\sum_{\mathsf{K}\in\mathsf{P}} (\|\phi^{(n)}_{i,\mathsf{K}}\|_{L_{q_1}(\mathsf{K})} \|g\|_{L_{q_2}(\mathsf{K})}), \\ i = 4, 5. \end{aligned}$$

Hence, in Theorems 4.1 and 4.2 we have for $i = 4, 5$

$$\|e^{(n)}_i\|_{L_2(\mathsf{K})} \leq C_i h \sum_{\mathsf{K}\in\mathsf{P}} \|\phi^{(n)}_{i+1,\mathsf{K}}\|_{L_2(\mathsf{K})},$$

where $C_i > 0$.
Finally, the general a posteriori error estimate theorem is given:

Theorem 4.3. *Suppose that* $e^{(n)} = f^{(n)} - f^{(n)}_{\underline{h}} \in W^{1,2}$ *denotes the error in the standard finite element approximation and* $f^{(n)}_{\underline{h}} = \sum_{i=1}^{5} f^{(n)}_{i\underline{h}}$ *then*

$$\begin{aligned} \|e^{(n)}\|_{L_2(\Omega)} \leq\ & C_2[C_{m,r}\|\hat{v}_1\|_{L_2(\Omega)} h^{m-r} M + \|\hat{v}_1\|_{L_2(\Omega)} \|\partial_x f^{(n)}_{NI}\|_{L_2(\Omega)} + \|f^{(n)}_{2\underline{h}}\|_{L_2(\Omega)}] \\ & + C_3[(\|E\|_{L_2(\Omega)} + \|B\|_{L_2(\Omega)} \|M\|_{L_2(\Omega)} \|\hat{v}\|_{L_2(\Omega)})(C_{m,r}\bar{M} h^{m-r} + \|\nabla_x f^{(n)}_{NI}\|_{L_2(\Omega)}) \\ & + \|f^{(n)}_{3\underline{h}}\|_{L_2(\Omega)}] + \sum_{i=4}^{5} C_i h \sum_{\mathsf{K}\in\mathsf{P}} \|\phi^{(n)}_{i+1,\mathsf{K}}\|_{L_2(\mathsf{K})}. \end{aligned}$$

Proof. We have $f^{(n)} = \sum_{i=1}^{5} f^{(n)}_i$ of (15) and

$$\|e^{(n)}\|_{L_2(\Omega)} = \|f^{(n)} - f^{(n)}_{\underline{h}}\|_{L_2(\Omega)} = \|\sum_{i=1}^{5} f^{(n)}_i - \sum_{i=1}^{5} f^{(n)}_{i\underline{h}}\|_{L_2(\Omega)} = \sum_{i=1}^{5} \|f^{(n)}_i - f^{(n)}_{i\underline{h}}\|_{L_2(\Omega)}.$$

Therefore, by using the above relation and the subsections i.e. 4.1.1, 4.1.3.(a)-4.1.3.(d) and some tedious manipulation yields the above inequality. ■

5. Numerical Experimentation

To show the effectiveness of the proposed algorithm for solving the VMFP system , we examine a numerical example. Here we do some calculations to show the convergence and stability of the answer and then show the results in the graphs.
In f, ρ, β definitions, because the integral operator is used, we use the 10 points Gauss-Legendre quadrature rule to calculate the approximate value as well as their error. let

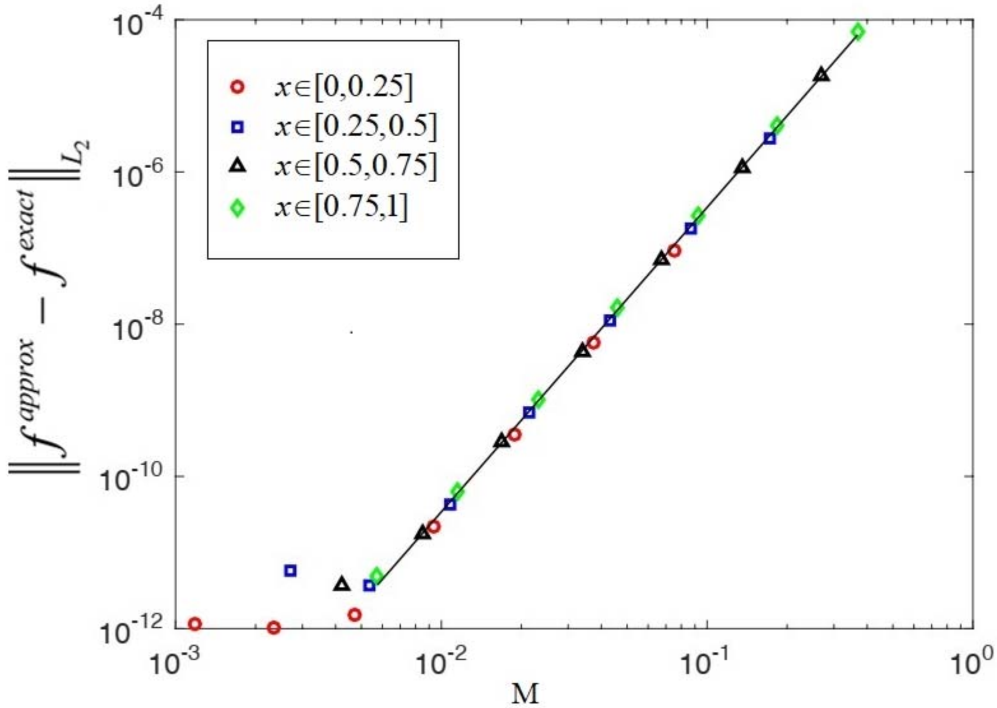

Figure 1: The L^2-norm of the difference between the exact and approximate solution of *f*.

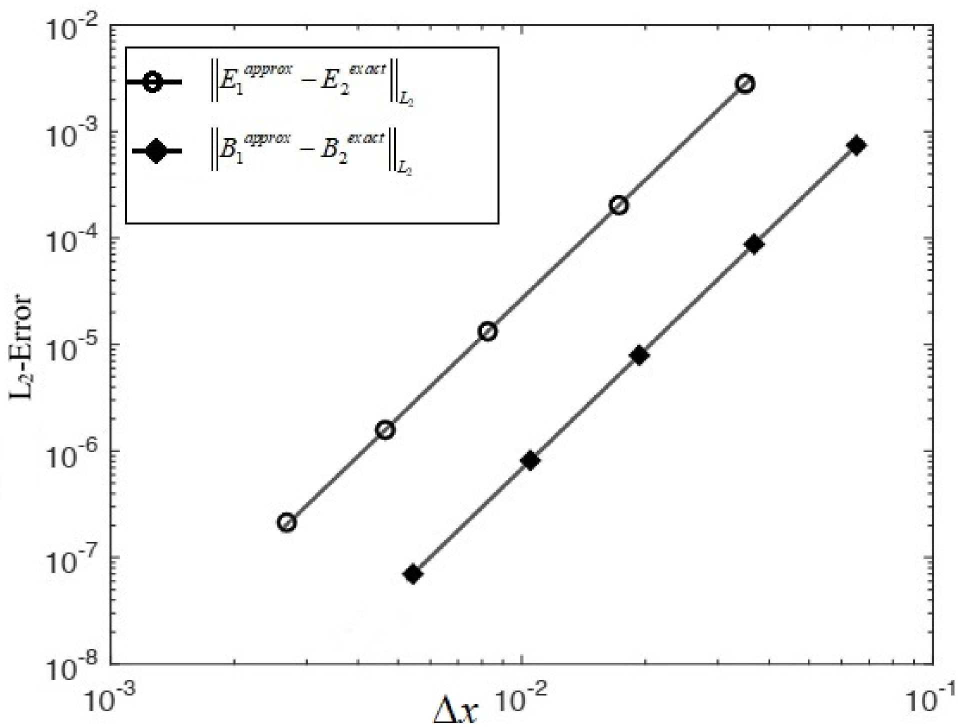

Figure 2: Comparison of L^2-norm of errors electromagnetic fields $B(x,t), E(x,t)$ as a function of $\triangle x$ at $t = 1.2$.

$$f_0(x,v) = \frac{10}{\pi}\exp{-10[(v_1 - 0.75)^2 + (v_2 - 0.5)^2]}$$

as an initial condition. To discrete the interval $[0,T]$, we set $T = 2$ and $M = 10^4$. Therefore $\{\xi_i\}_{i=0}^{10^4}$ constitutes a uniform partition on $[0,2]$.
The $L2$-norm of the difference between the exact and approximate solution of f for different values of M, is shown on figure 1. we have shown log scale plot for comparison of L_2 norm of errors electromagnetic fields $B(x,t), E(x,t)$ as a function of Δx at time $t = 1.2$ in figure2.
We also have dedicated Figure 3 to convergence of $\rho(x,t)$ as a function of Δx at time $t = 1.2$.

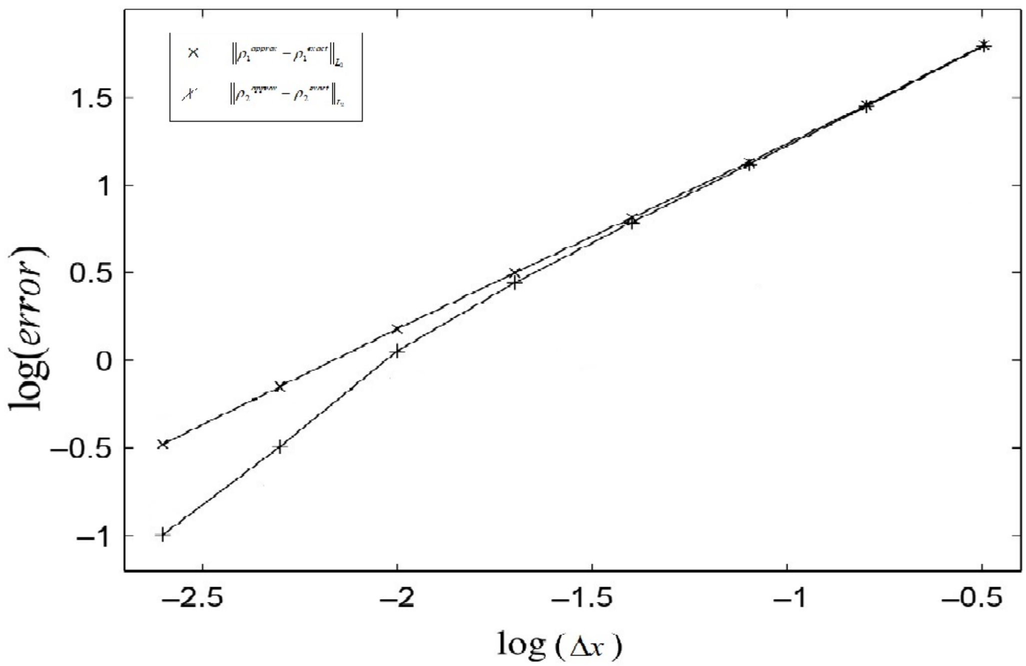

Figure 3: Convergence of $\rho(x,t), E(x,t)$ as a function of $\triangle x$ at $t = 1.2$.

6. Conclusion

In this paper, we advance a new recursive algorithm for solving the Vlasov-Maxwell-Fokker-Planck system. The algorithm employs a new splitting scheme for the VMFP system and performs an a posteriori error analysis of the resulting discretized solution.

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