Journal of Numerical Mathematics and Stochastics*, 11** (1) : 82-101, 2019 © ***JNM@S
http://www.jnmas.org/jnmas11-6.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

Bass Diffusion: A Stochastic Differential Equation Approach

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Abstract. *The need for modeling diffusion of innovation among individuals of a social system has motivated relevant research both from theoretical and applied perspective since the 1960s. Among pioneering works, a preeminent role is played by the aggregate model proposed by Bass [2]. According to this deterministic model, the diffusion dynamics is represented by an ordinary differential equation (ODE) of logistic-type. In this paper, a stochastic extension of the classical Bass model is studied. The extension is a diffusion process defined as solution of a stochastic differential equation (SDE) obtained by adding to the classical equation a suitable diffusion term. The diffusion term guarantees that the resulting stochastic process satisfies some relevant properties. In particular, the trajectories of the process are almost surely (a.s.) non-negative and bounded by the constant K representing the number of potential adopters (regularity with respect to the interval* $[0,K]$*). A known regularity result is extended here to include the case where the number of potential adopters is a deterministic (non-decreasing) function of time. This extension can be useful in situations where effects of the population dynamics need to be included into the model. We also study theoretically when the SDE Bass model admits a stationary distribution. In the case when there is no stationary distribution, stochastic stability of the steady state solutions* $\{Y_t \equiv K, t \geq 0\}$ *and* $\{Y_t \equiv 0, t \geq 0\}$ *of classical Bass model is studied. A Monte Carlo simulation procedure to approximate the transition density of the SDE process is proposed and applied here to estimate SDE Bass model parameters via approximate maximum likelihood estimation (AMLE). The results obtained are compared to those from a known method based on Gaussian approximation. An intensive simulation study shows that the proposed method outperforms the classical one.*

Key words: Diffusion Processes, Bass Model, Innovation Diffusion, Statistical Inference, Girsanov Theorem.

AMS Subject Classifications: 60F44, 90A09

1. Introduction

In marketing sciences, a growing area of research is focused on market demand for new products or services. In this field, it is of great interest to model the diffusion of *innovation* within a population of potential consumers/users in order to optimize promotion and distribution strategy. The need for modeling innovation diffusion in a statistical framework has been motivating important research from both practical and theoretical perspective for the last 30 years. Modeling diffusion of new products, services, or social behaviors can be useful in marketing strategies in the private sector, as well as in decision making processes by public authorities in social or health fields. This phenomenon has been analyzed both at microscopic level [16], [15], [7] and at aggregate level [6], [13], [2], [12], [19], [21]. In the microscopic approach, the probability for an individual of adopting the innovation depends on some characteristics of the potential adopter and on some external conditions. Typically, the probability for an individual to adopt the innovation depends on the number of people who are currently adopters: the higher the number of current adopters, the higher the probability of a new adoption. To this respect, the innovation diffusion is similar to other diffusion phenomena, such as disease diffusion, or population dynamics, and can be studied through the same methodologies. In particular, for example, birth-death Markov random processes can be used.

With the aggregated approach, the dynamics of the total number of people currently adopting among a population of potential adopters is directly modeled. The classical Bass model is one of the most popular [2] and it takes into account the main elements influencing the adopter behavior. It is commonly developed within the framework of ordinary differential equations (ODE). The rate of the current number of adopters at time t is assumed to be the sum of three terms. The first term captures the individual propensity to adopt, i.e. “self-adoption” due to external factors (e.g., advertisements). It is proportional to the current number of people who have not adopted yet the product and accounts for the individual propensity to adoption (“self-innovation"). The second one is considered the most important in many applications and it models the “imitation" process, i.e. the influence on the adoption exerted by “already adopters” via communication among population individuals, “word by mouth". It is proportional to the product of the number of people who are currently adopters times the number of the remaining ones. The third term takes into account the possibility of receding from adopting the innovation, i.e. the “disadoption". It is commonly assumed proportional to the number of current adopters, as intuition suggests.

The relation between the micro and the aggregate approaches is an interesting field of research. In particular, it is of great interest to analyze under which conditions aggregate adoption models can be derived from probabilistic micro-level models. For instance, in the limit of large population size, Shun proved that the deterministic Bass equation is fulfilled by the expected value of the total adopter number, under a micro-level dynamics governed by a pure birth-death process [16].

In this paper, the aggregated approach is followed. In particular, the classical Bass model is generalized by means of a stochastic differential equation (SDE) introducing random noise in the diffusion dynamics. Stochastic aggregated models for the diffusion of innovation have been studied by many authors. For instance, Gutierrez uses an extension of the classical

Gompertz innovation diffusion model to predict natural-gas consumption in Spain [8]. Kanniainen and Skiadas propose two stochastic extensions of the Bass model [10], [21]. In the first paper, a mean reverting Ornstein-Uhlenbeck process is added to the logarithm of the classical solution of the Bass equation. In the second one, the authors propose a SDE which is obtained from the classical Bass equation by adding an ad hoc Brownian diffusion term that allows the authors to have explicit analytic solution of the SDE. When introducing a SDE Bass model, the crucial point is the choice of the diffusion term. In fact, in order to interpret the solution of the SDE as a (continuous approximation of) a discrete stochastic process for the number of adopters, some qualitative properties have to be satisfied. For instance, it is desirable that at each time the process takes a.s. values belonging to the interval $[0, K]$, where K is the number of potential adopters (*regularity condition*). This is not the case for the SDE models in both works in [10] and [21].

Simulation and inference on discretely observed diffusion processes is a challenging area of research. In fact, apart from a few cases (e.g., solutions of linear SDE or CIR model) the transition density of the diffusion process cannot be expressed in analytic form. Hence, likelihood-based inferential approaches, such as maximum likelihood (ML) estimation and Bayesian inference, are not easy to apply because the likelihood function is not known. Unavailability of the transition density function makes also difficult to compute specific quantities, such as expectation of hitting times (e.g., extinction), or expectations at given times. In these cases, one possible approach is to simulate a large number of process realizations and to approximate the quantities of interest via sample averages. To randomly simulate approximate solutions of the considered SDE some Eulerian-type schemes are commonly used [14]. The approximate solution converges, in a suitable metrics, to the exact one as the step of the involved time discretization tends to zero. Thus, the accuracy of the estimation depends on a further parameter (the discretization step), in addition to the number of Monte Carlo (MC) replicates used to approximate expectations. Another problem with Eulerian methods is that boundedness conditions, such as regularity properties with respect to a given domain, are not ensured. For instance, in the case of the SDE Bass model, there is no guarantee that the process values remain lower than K, the number of potential adopters. A methodology for exact simulation of SDE solutions has been introduced [3]. The method consists of *retrospective sampling* based on an acceptance-rejections scheme. It allows, for a certain class of SDE, to accept (or reject) a proposal within an infinite-dimensional space, by checking only a finite number of conditions.

Inference is drawn here through MC approximation of the transition density of the process. Specifically, one starts by considering the diffusion bridge obtained by conditioning the SDE Bass process on two given consecutive observations. Then, one derives the form of the transition density in terms of the expectation of a suitable Brownian functional. This is done by exploiting the expression of the Radon-Nikodym derivative of the involved diffusion bridge with respect to (w.r.t.) the corresponding Brownian bridge [11]. This above expectation is analytically intractable but can be approximated via MC averaging. The MC approximation of the transition density relative to consecutive sampling observations provides, in turn, an MC approximation of the likelihood function. Thus, approximate maximum likelihood estimates (AMLE) of the model parameters can be obtained by maximizing the approximate likelihood. The AMLE methodology has been applied here to data simulated by the exact algorithm mentioned above to assess the performance of the proposed methodology. Moreover, comparison with a classical inference method has been performed.

2. Bass Diffusion

2.1. **The classical Bass model**

In 1969, Bass proposed the following ODE for modeling diffusion of innovation:

$$y' = a(K-y) + b\frac{y}{K}(K-y) - \mu y, \tag{1}$$

where $y(t)$ represents the number of innovation *adopters* at time t, see [2], and the symbol $'$ denotes the derivative with respect to time. The three terms on the right hand side (r.h.s.) of eq. (1) can be interpreted as follows. The first one, known as “self-innovation", models the effect of commercials on adoption and is proportional to the number of individuals $K - y(t)$ who are not adopters at time t, where the positive parameter K is the number of potential adopters. The second term is sometimes called “imitation" or “word of mouth" and captures the effect of interactions between individuals on adoption. The last term takes into account the presence of adopters who stop being so. All the parameters a, b, μ are non negative.
If $(b - a - \mu)^2 + 4ab > 0$, eq. (1) has the two steady state solutions:

$$y^{\pm} = K\ \frac{b - a - \mu \pm \sqrt{(b - a - \mu)^2 + 4ab}}{2b}. \tag{2}$$

It is easy to verify that $y^+ \in [0,K]$ and is asymptotically stable, while y^- is non positive and unstable. Eq. (1) with initial condition $y(0) = y_0 \geq 0$ admits the non-negative solution:

$$y(t) = \frac{y^+ A \exp\left\{\frac{b}{K}(y^+ - y^-)t\right\} + y^-}{1 + A\exp\left\{\frac{b}{K}(y^+ - y^-)t\right\}}, \tag{3}$$

where $A = (y_0 - y^-)/(y^+ - y_0)$. Furthermore, one can easily check that, for $t > 0$, $y'(t)$ is positive if $y_0 < y^+$, and negative if $y_0 > y^+$. It follows that for $y_0 < y^+$ ($y_0 > y^+$), the solution (3) is an increasing (decreasing) positive function converging to y^+. Thus, starting from the initial value y_0, the number of adopters tends to its asymptotic value y^+. It is worthwhile noting that, if mortality is not included in the model ($\mu = 0$), as for instance in the case when y is the number of purchases of a certain good, then $y^+ = K$, i.e., for t large enough, in practice all potential consumers become adopters.

2.2. **The SDE Bass model**

A natural way of introducing stochasticity in the model is to replace the deterministic eq. (1) by an appropriate Itô stochastic differential equation (SDE):

$$dY_t = \left[a(K - Y_t) + b\, Y_t(1 - Y_t/K) - \mu Y_t\right]dt + v(Y_t)dW_t, \tag{4}$$

where $v(\cdot)$ is the volatility function and $\{W_t, t \geq 0\}$ is a one-dimensional Brownian motion.
For instance, this is the approach of Skiadas and Giovanis (1997), who propose a particular choice for the function v, see [21]. While the choice of the drift term is naturally guided by the deterministic version of the SDE, it is not obvious how to specify the stochastic term. This is done here, by imposing some conditions ensuring that the trajectory evolves within the interval $[0,K]$. According to the terminology of Shurz (2007), we call the process $\{Y_t, t \geq 0\}$ *regular* with respect to the domain $\mathsf{D} \subset \mathbb{R}$, if $P\{Y_t \in \mathsf{D}; t > 0\} = 1$, see [20]. The following theorem shows that the solution of (4) is not regular on $[0,K]$ if v is a constant [20].

We consider the canonical set-up in which a model on the filtered space $(\Omega, \mathsf{F}, \mathsf{F}_t, P)$ is specified through the one-dimensional time homogeneous Itô SDE:

$$dY_t = d(Y_t;\theta)dt + v(Y_t;\theta)dW_t, \qquad (5)$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion and θ is a h-dimensional parameter in some subset $\Theta \subset \mathbb{R}^h$. The drift and the diffusion terms d and v are assumed to be smooth functions of their arguments $(\mathbb{R} \times \Theta) \to \mathbb{R}$. Here, it is assumed that, for each $\theta \in \Theta$, eq. (5) has a unique solution $\{Y_t, t \geq 0\}$ for each F_0-measurable initial condition Y_0. Since we are dealing with two main problems in applications, simulation and inference, and therefore we always work on a finite time interval, it is enough to assume that the functions d and v belong to $C^1(\mathbb{R})$. Moreover, it is assumed that an invariant equilibrium distribution $\mu_\theta : \mathbb{R} \to \mathbb{R}$ exists such that, if $Y_0 \sim \mu_\theta$, $\{Y_t, t \geq 0\}$ is stationary and ergodic. These assumptions are quite standard and are necessary to ensure consistency properties of various estimators.

Theorem 2.1. *Assume that $\{Y_t, t \geq 0\}$ satisfies eq. (4) with the function v constant and positive, $K > 0, a \geq 0, b \geq 0, \mu \geq 0$, and $Y_0 \in [0, K]$ is independent of the σ-algebra $v(W_t, t \geq 0)$. Then, $\{Y_t, t \geq 0\}$ is not regular with respect to $\mathsf{D} = [0, K]$.*

The previous result shows that even if the drift has the “right sign" on the boundary $\partial\mathsf{D} = \{Y = 0\} \cup \{Y = K\}$ of the region, in the classical sense that it pushes the motion towards the interior of the region, additive random noise may cause the process to leave the region in finite time. It is intuitive to limit ourselves to functions $v(\bullet)$ vanishing on $\partial\mathsf{D}$ as this is the case for the multiplicative noise choice in [20]. The next result (Shurz, 2007) shows that regularity is ensured in a class of SDEs with “multiplicative noise" [20].

Theorem 2.2. *Let $\{Y_t, t \geq 0\}$ be a solution of the SDE Bass model:*

$$dY_t = [a(K - Y_t) + bY_t(1 - Y_t/K) - \mu Y_t]\, dt + \sigma |Y_t|^\beta |1 - Y_t/K|^\gamma dW_t \qquad Y_0 \in (0, K), \qquad (6)$$

where a, b, μ, σ are non-negative constants and $\beta, \gamma \geq 1$. Then the solution is regular with respect to $\mathsf{D} = [0, K]$.

For the applied part of the work, we have adopted the particular SDE Bass model of eq. (6) with $\beta = \gamma = 1$. In the final section, we will describe some new theoretical results also regarding the general case of the SDE model of eq. (6).

3. Statistical Inference

In this Section, we describe two procedures used here to estimate model parameters of the SDE Bass model. The first one approximates the transition density via MC diffusion bridge simulation [22]. The second one is strongly related to the Euler-Maruyama scheme [14]. Each of the two approaches is applied here to solutions of SDE Bass model obtained by an “exact approach” known in the literature [3]. This allows us to deal with solutions without the numerical errors introduced by classical approximated schemes. We will shortly describe here these two approaches in general. They will be specialized in the next section to the case of the SDE Bass model.

3.1. **MC diffusion bridge simulation**

Here, we describe the first approach. From now on, we will consider time homogeneous diffusion processes specified through the SDE (5), with transition density $p(t,x,y;\theta)$ defined via:

$$P\{Y_{s+t} \in \Gamma | Y_s = x;\theta\} = \int_\Gamma p(t,x,y;\theta)dy, \quad \forall s > 0, \quad \Gamma \in \mathsf{F}_t$$
$$p(0,x,y;\theta) = \delta(y-x),$$

where $\delta(\cdot)$ denotes the Dirac-delta function, which can be obtained as limit of the zero mean Gaussian distribution density function when the variance goes to zero.
We assume that the process $\{Y_t, t \geq 0\}$ is observed at a finite collection of times $0 = t_0, t_1, \ldots, t_n = t$, the corresponding values being $\mathbf{y} = (Y_{t_0} = Y_0, \ldots, Y_{t_n} = Y_t)$, i.e., $\{Y_t, t \geq 0\}$ is discretely observed. From Markov property it follows that the log-likelihood of the n observations Y_{t_i} can be written as:

$$l(\theta|\mathbf{y}) = \sum_{i=1}^{n} \log p(\Delta_{t_i}, Y_{t_i}, Y_{t_{i+1}};\theta) = \sum_{i=1}^{n} \log p_i(\theta), \tag{7}$$

where $\Delta_{t_i} = t_{i+1} - t_i$. Under mild conditions, the estimator obtained by maximizing the function (7) has the usual nice consistency properties, but unfortunately, apart from a few cases, the transition density $p(t,x,y;\theta)$ is unknown.

To estimate the transition density function $p(t,x,y;\theta)$ in eq. (7), we make the following variable transformation containing model parameters:

$$X \doteq \eta(Y;\theta); \qquad \eta(Y;\theta) = -\int_c^Y \frac{1}{v(u,\theta)} du, \tag{8}$$

where c is an arbitrary constant, and we assume that the integral exists [5], [1]. The new process $\{X_t, t \geq 0\}$ satisfies the following SDE with unit diffusion term:

$$dX_t = \alpha(X_t;\theta)dt + dW_t, \qquad X_0 = \eta(Y_0;\theta),$$

with

$$\alpha(u;\theta) = \frac{d(\eta^{-1}(u;\theta), \theta)}{v(\eta^{-1}(u;\theta), \theta)} - \frac{1}{2} v'(\eta^{-1}(u;\theta), \theta) \tag{9}$$

where the symbol $'$ denotes the derivative with respect to y. The transition density p is related to the one $\tilde{p}$ associated to the process $\{X_t, t \geq 0\}$, via the relation:

$$p(t,x,y;\theta) = \tilde{p}(t, \eta(x;\theta), \eta(y;\theta);\theta) \, |\eta'(y;\theta)| \,. \tag{10}$$

By using eq.(10), we can write the log-likelihood w.r.t. the new variable $\mathbf{x}$ as:

$$l(\theta|\mathbf{x}) = \sum_{i=1}^{n} \log \tilde{p}_i(\theta) \,. \tag{11}$$

The estimation of the transition density of the transformed process $\{X_t, t \geq 0\}$ is based on the change of measure on the space $\mathsf{C}[0,t]$ of the continuous functions on $[0,t]$, whose typical element is denoted by W. Let $\mathbf{P}_\theta^x$ be the measure associated to the new process on $[0,t]$ (with initial value x), and $\mathbf{P}_\theta^{t,x,y}$ the measure associated to it conditioned on having its ending value y at time t. Correspondingly, $\mathbf{W}^x$ and $\mathbf{W}^{t,x,y}$ are the measures associated with the Brownian motion and Brownian bridge respectively. From the Girsanov theorem [11], we have:

$$\frac{d\mathbf{P}_\theta^x}{d\mathbf{W}^x} = Z_t^x(W),$$

where

$$Z_t^x(W) = \exp\left\{\int_0^t \alpha(W_s;\theta)dW_s - \frac{1}{2}\alpha^2(W_s;\theta)ds\right\}.$$

Using this result and the factorization of the measure associated with a given unconditioned process $\{U_s, 0 \le s \le t;\ U_0 = u\}$ as product of a bridge measure and the law of the random variable U_t, we obtain:

$$\mathsf{N}(y-x,t)Z_t^x(W) = \tilde{p}(t,x,y;\theta)\frac{dP_\theta^{t,x,y}}{dW^{t,x,y}}(W). \tag{12}$$

Letting

$$A(y;\theta) = \int_c^y \alpha(u\,;\theta)du,$$

with c arbitrary constant, and using integration by parts in order to eliminate the stochastic integral in $Z_t^x(W)$, eq. (12) becomes:

$$\frac{d\mathbf{P}_\theta^{t,x,y}}{d\mathbf{W}^{t,x,y}} = \frac{\mathsf{N}(y-x,t)}{\tilde{p}(t,x,y;\theta)} \times \exp\left\{A(y;\theta) - A(x;\theta) - \frac{1}{2}\int_0^t [\alpha^2(W_s;\theta) + \alpha'(W_s;\theta)]ds\right\}.$$

Finally, taking expectation w.r.t. the bridge measure $\mathbf{W}^{t,x,y}$, we obtain:

$$\tilde{p}(t,x,y;\theta) = \mathsf{N}(y-x,t)e^{\{A(y;\,\theta)-A(x;\,\theta)\}} \times E_{\mathbf{W}^{t,x,y}}\left[\exp\left\{-\frac{1}{2}\int_0^t \left[\alpha^2(W_S;\theta) + \alpha'(W_S;\theta)\right]ds\right\}\right]. \tag{13}$$

Eq. (13) is crucial in many inferential approaches for discretely observed diffusions. In general, the expected value in eq. (13) is intractable, but it can be estimated via numerical methods. In particular, we could use a MC approach based on repeatedly drawing from the measure $\mathbf{W}^{t,x,y}$ and averaging the values of the r.h.s. of eq. (13) over N realizations of a standard Brownian bridge. A further approximation is the discretization of the integral in eq. (13) through a partition of the time interval $[0,t]$ suitably fine to reduce bias. In order the AMLE estimator to be consistent, the transition density function has to be estimated *simultaneously* for any $\theta \in \Theta$. In other words, we need to estimate a *function* on the parameter space Θ, rather than to estimate independently its values on a grid of points in Θ.

The whole procedure consists, for $i = 1,\dots,n$, of the following steps:

MC approximation of the transition density.

1. partition the interval $[t_i, t_i + \Delta_i]$ in m_i sub-intervals $[s_k, s_{k+1}]$, $k = 0,\dots,m_i - 1$, where $s_0 = 0, s_{m_i} = \Delta_i$, and for the sake of simplicity, we have removed from the notation for times s_k the dependence of the interval index i;
2. draw N independent realizations from the m_i-dimensional distribution of the standard

Brownian Bridge from t_i to t_{i+1} corresponding to the times s_k, $k = 1, \ldots, m_i - 1$. Let $\left\{W^0\right\}_{jk}$, $(j = 1, \ldots, N;\ k = 1, \ldots, m_i)$ be the $N \times m_i$ matrix with entry at row j and column k given by the kth component of the m_i-dimensional vector corresponding to the jth realization;

3. compute the quantity

$$\tilde{p}_i^N(\theta) = \mathsf{N}(x_{i+1} - x_i, \Delta_i) \exp\{A(x_{i+1};\theta) - A(x_i;\theta)\}$$
$$\times \frac{1}{N} \sum_{j=1}^{N} \exp\left\{ -\sum_{k=1}^{m_i} \xi\left(W_{jk}^0 + \left(1 - \frac{s_k}{\Delta_i}\right) x_i + \frac{s_k}{\Delta_i} x_{i+1}; \theta \right) \right\},$$

where $\xi(x;\theta) = \frac{1}{2}(\alpha^2(x;\theta) + \alpha'(x;\theta))$ and the dependence on the parameters is made explicit.

To ensure the existence of the estimate, we need to assume that the function $\xi(x;\theta)$ is bounded from below. Based on the n estimates $\tilde{p}_i^N(\theta)$, an estimate of the likelihood function is obtained by replacing $\tilde{p}_i(\theta)$ in eq. (11) by $\tilde{p}_i^N(\theta)$.

We highlight that the basic random elements necessary to obtain the matrix W^0 are standard Gaussian variables and do not explicitly depend on the parameters to be estimated. As a result, the MC approximation of the likelihood function, converges a.s. to the true likelihood function in the $\|\cdot\|_\infty$ norm.

3.2. **Gaussian approximation**

We describe here how the Gaussian approximation method can be used to make inference on the SDE model parameters. The dynamics of the process is approximated through the Eulerian scheme:

$$\Delta Y_i \doteq Y_{i+1} - Y_i = d(Y_i;\theta)\Delta t_i + v(Y_i;\theta)\sqrt{\Delta t_i}\, Z_i, \tag{14}$$

and Z_i are standard independent Gaussian random variables ($i = 1, \ldots, n$), see [14]. Then, the corresponding approximated log-likelihood function is (up to an additive constant independent of the model parameters):

$$l^G(\theta) = -\frac{1}{2} \sum_{i=1}^{n} \log(v(Y_{t_i};\theta)\Delta t_i) - \sum_{i=1}^{n} \left\{ \frac{(\Delta Y_{t_i} - d(Y_{t_i};\theta)\Delta t_i)^2}{2v(Y_{t_i};\theta)\Delta t_i} \right\}. \tag{15}$$

3.2. **Numerical results**

We will now illustrate some results of a study to assess the performance of the proposed inferential procedure. The results will also be compared to the ones obtained from the Gaussian approximation method described before. As anticipated,we use data exactly simulated from the SDE Bass model by the method in [3].To do that, we need to limit ourselves to the case of absence of self innovation. Therefore the SDE Bass model has drift term $d(y;\theta) = d(y;b,\mu) = by(1 - y/K) - \mu y$ and volatility term $v(y;\theta) = v(y;\sigma) = \sigma|y(1 - y/K)|$. We have not made explicit dependence on K as we focus on the dynamic of the proportion, which corresponds to setting $K = 1$. The values used for the other parameters $(b = 0.32, \mu = 0.03, \sigma = 0.14)$ correspond to a model that nicely fits real data concerning number of visitors of a certain web site. We used evenly time-spaced observations with step $\Delta = t_{i+1} - t_i$.

For the proposed estimation method, samples of size $N = 100$ are used in the MC step. Corresponding to the chosen SDE Bass model, the function η and its derivative in eqs. (8) and (10) respectively, become:

$$\eta(Y;\theta) = \eta(Y;\sigma) = \frac{1}{\sigma} log(K/y - 1) \qquad \eta'(y;\theta) = \eta'(y;\sigma) = \frac{-1}{\sigma y(1 - y/K)}.$$

The expression of the function α in eq. (9) is:

$$\alpha(u;\theta) = \alpha(u;b,\mu,\sigma) = -\frac{b}{\sigma} + \frac{\mu}{\sigma} + \frac{\sigma}{2} + \frac{\mu}{\sigma} e^{-\sigma u} - \sigma \frac{e^{-\sigma u}}{1 + e^{-\sigma u}}.$$

Correspondingly, we have

$$A(y;\theta) = A(y;b,\mu,\sigma) = \left(-\frac{b}{\sigma} + \frac{\mu}{\sigma} + \frac{\sigma}{2}\right)y - \frac{\mu}{\sigma^2} e^{-\sigma y} + \ln(1 + e^{-\sigma y}), \tag{16}$$

and

$$\xi(y;\theta) = \xi(y;b,\mu,\sigma) = \frac{1}{2}\left[-\frac{b}{\sigma} + \frac{\mu}{\sigma} + \frac{\sigma}{2} + \frac{\mu}{\sigma} e^{-\sigma y} - \sigma \frac{e^{-\sigma y}}{1 + e^{-\sigma y}}\right]^2 - \frac{1}{2}\left[\mu e^{-\sigma y} - \sigma^2 \frac{e^{-\sigma y}}{(1+e^{-\sigma y})^2}\right].$$

It can be verified that for the chosen SDE model, boundedness from below of the function ξ is fulfilled. Since the explicit form of the approximated log-likelihood function is quite complex, a maximization method not requiring computation of derivatives has been used. Specifically, AMLE estimates have been obtained through bound optimization by quadratic approximation (*bobyqa*) as implemented in the R-package *minqa*. The maximization procedure is initialized by the Gaussian approximation method estimate.

We now turn to the Gaussian approximation method. Here, we maximize the approximated log-likelihood

$$l_n^G(\theta) = l_n^G(b,\mu,\sigma) = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\omega_i - b\psi_i + \mu\lambda_i)^2 - \frac{n}{2} \log \sigma^2 - \sum_{i=1}^{n} \log(1 - Y_i/K),$$

where

$$\omega_i = \frac{\Delta Y_i}{Y_i(1 - Y_i/K)\sqrt{\Delta}} \qquad \psi_i = \sqrt{\Delta} \qquad \lambda_i = \frac{\sqrt{\Delta}}{1 - Y_i/K}.$$

For a fixed value of K, as it is assumed here, maximizing the log likelihood w.r.t. the parameters b, μ, σ^2 provides the estimates:

$$\hat{\mu} = \left[-\frac{\langle \lambda,\omega \rangle}{\langle \lambda,\lambda \rangle} + \frac{\langle \psi,\omega \rangle \langle \lambda,\psi \rangle}{\langle \lambda,\lambda \rangle \langle \psi,\psi \rangle} \right] \left[1 - \frac{\langle \lambda,\psi \rangle^2}{\langle \lambda,\lambda \rangle \langle \psi,\psi \rangle} \right]^{-1}, \quad (17)$$

$$\hat{b} = \left[+\frac{\langle \psi,\omega \rangle}{\langle \psi,\psi \rangle} - \frac{\langle \lambda,\omega \rangle \langle \lambda,\psi \rangle}{\langle \lambda,\lambda \rangle \langle \psi,\psi \rangle} \right] \left[1 - \frac{\langle \lambda,\psi \rangle^2}{\langle \lambda,\lambda \rangle \langle \psi,\psi \rangle} \right]^{-1}, \quad (18)$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\omega_i - \hat{b}\psi_i + \hat{\mu}\lambda_i \right)^2, \quad (19)$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes the scalar product.

Relative errors (%) of parameters estimates vs number of observations are plotted in Figure 1 for the proposed and the Gaussian approximation methods. The sampling step is $\Delta = 10$. We notice that the performances of the proposed estimator are good while this is not the case for the classical estimator. This is confirmed by the results of the large simulation study in Table 1.

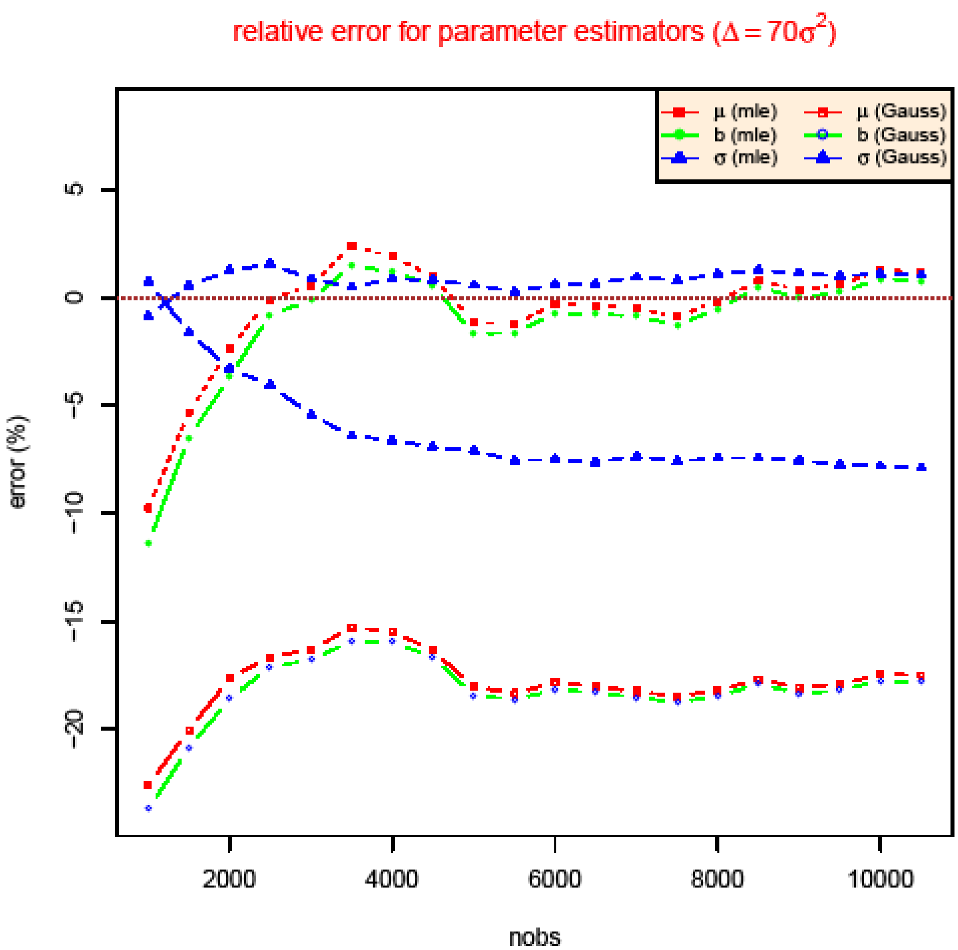

Figure 1: Relative errors of parameters estimayes for the AMLE and the Gaussian approximation methods versus the number of observations.

Here, the number of observations is kept fixed and equal to 2000. To assess the performances of the two methods, we repeated the estimation on each of 1000 independent replications of the observation sequence. Relative bias (rbias) and relative root mean square error (rrmse) for the model parameters are computed. We have also considered the estimators of mean and standard deviation of the stationary distribution (m_{staz} and std_{staz}) computed from the model parameter estimates.

Method	Error	μ	σ	b	m_{staz}	std_{staz}
AMLE	rbias	-0.02	0	-0.02	0	0.01
Gauss	rbias	-0.19	-0.16	-0.20	0	0.04
AMLE	rrmse	+0.06	+0.02	+0.06	0	0.03
Gauss	rrmse	+0.20	+0.16	+0.20	0	0.06

Table 1: Parameter estimation errors: approximate maximum likelihood method (AMLE) versus Gaussian approximation method (Gauss).

The results in Table 1 show that the proposed inferential method is successful for the estimation of the SDE Bass model parameters. Moreover, the comparison of this method with the Gaussian approximation one has been always in favor of the former. On the other side, the AMLE method requires much more computation time and memory resources.

4. Theoretical Results

In this section, we show some new theoretical results on the solution of SDE Bass model of eq.(6). The first one regards the regularity of a generalization of the model to the case when K is no more constant, but it is varying along time, in a pre-assigned way, i.e., $K \equiv K(t)$. The other results are concerning with the stationary distribution and the stochastic stability of the steady state solutions $\{Y_t \equiv K, t \geq 0\}$, and $\{Y_t \equiv 0, t \geq 0\}$ of classical Bass model.

4.1. **Solution regularity of extended SDE Bass model**

In the case when K is not constant along time, the SDE Bass model of eq. (6) becomes not homogeneous. As in the work of Shurz [20], the regularity result is here obtained using the Lyapunov function technique [9]. In fact, since the extended SDE is not homogeneous, we cannot study the regularity of solutions by means of Feller's boundary classification [4].

Theorem 4.1. *Let $\{Y_t, t \geq 0\}$ be a solution of the SDE Bass model of eq. (6), with $K = K(t)$ and initial condition $Y_0 \in (0, K(0))$, with $K(0) > 0$. Assume that $K(\bullet) \in C^1(\mathbb{R}^+ \cup \{0\})$ is a non-decreasing function and $a, b, \mu, \sigma, \alpha, \beta$ are as in theorem 2.2. Then the solution is regular with respect to $D(t) = (0, K(t))$.*

Proof. Let $T > 0$ and $K^* = K(T) = \max\{K(t),\ t \in [0,T]\}$. For each $t \in [0,T]$ and $n \in \mathbf{N}$, define a sequence of "time dependent" open intervals: $\mathsf{D}_n(t) = (e^{-n}, K(t) - e^{-n})$. Define the

random times sequence: $\tau_n = \inf\{u \in [0,T] : Y_u \notin \mathsf{D}_n(u)\}$, with the convention that $\tau_n = +\infty$, if $\{u \in [0,T] : Y_u \notin \mathsf{D}_n(u)\} = \emptyset$. Analogously, we introduce the random time $\tau = \inf\{u > 0 : Y_u \notin \mathsf{D}(u)\}$. For $t \in [0,T]$, let us introduce the Lyapunov function $V(x,t) = K^* - \log(x(K(t)-x))$. From an elementary inequality, it follows that $V(x,t) = K^* - K(t) + K(t) - x - \log(K(t)-x) + x - \log x \geq K^* - K(t) + 2 \geq 2$.
Let us introduce a second Lyapunov function W defined as

$$W(x,t) = e^{-\int_0^t c(s)ds} V(x,t); \qquad c(t) \doteq \frac{a + b + \sigma^2 K(t)^{2\alpha+2\beta-4} + \mu}{2}.$$

Now, for fixed $t \in [0,T]$, we introduce another stopping time $\tau_n^* = t \wedge \tau_n$ and we use the Dynkin formula (see [17]) to obtain an expression of the expectation of function W computed at the (random) point $(Y_{\tau_n^*}, \tau_n^*)$:

$$\left.\begin{aligned} E[W(Y_{\tau_n^*}, \tau_n^*)] &= W(Y_0, 0) + E\left[\int_0^{\tau_n^*} \left(\frac{\partial W}{\partial u}(Y_u, u) + \mathsf{L}_0 W(Y_u, u)\right) du\right]; \\ \mathsf{L}_0 &\doteq d\,\frac{\partial}{\partial y} + \frac{1}{2} v^2 \frac{\partial^2}{\partial y^2}. \end{aligned}\right\} \tag{20}$$

The integrand in eq. (20) can be written as:

$$-cW + \exp\left(-\int_0^u c(s)ds\right)\frac{\partial V}{\partial u} + \exp\left(-\int_0^u c(s)ds\right)\mathsf{L}_0 V, \tag{21}$$

where the symbol $'$ denotes the derivative with respect to time. Now, we prove that $\mathsf{L}_0 V \leq cV$. Note that, for fixed $t \in [0,T]$ and $0 < y < K(t)$, from $V(y,t) \geq 2$, it follows that

$$\begin{aligned} cV(y,t) - \mathsf{L}_0 V(y,t) &\geq a + b + \sigma^2 K^{2\alpha+2\beta-4} + \mu - \mathsf{L}_0 V(y,t) \\ &= a + b + \sigma^2 K^{2\alpha+2\beta-4} + \mu - \left((a + b\frac{y}{K})(K-y) - \mu y\right) \\ &\quad \times \left(-\frac{1}{y} + \frac{1}{K-y}\right) - \frac{\sigma^2}{2K^2} y^{2\alpha}(K-y)^{2\beta}\left(\frac{1}{y^2} + \frac{1}{(K-y)^2}\right) \\ &\geq \sigma^2 K^{2\alpha+2\beta-4} + \frac{(a+by/K)(K-y)}{y} + \frac{\mu y}{K-y} \\ &\quad - \frac{\sigma^2}{2K^2} y^{2\alpha-2}(K-y)^{2\beta-2}((K-y)^2 + y^2) \\ &\geq \sigma^2 K^{2\alpha+2\beta-4} - \frac{\sigma^2}{2K^2} y^{2\alpha-2}(K-y)^{2\beta-2}[(K-y)^2 + y^2] \\ &\geq \sigma^2 K^{2\alpha+2\beta-4}\left(1 - \frac{(K-y)^2 + y^2}{2K^2}\right) \geq 0, \end{aligned}$$

where, in order to make notation simpler, we have omitted the argument t in the functions c and K. From (20), (21) and $\mathsf{L}_0 V \leq cV$, when $Y_u \in (0, K(u))$ for $u \in (0, \tau_n^*)$, and $\alpha, \beta \geq 1$, it follows that

$$E[W(Y_{\tau_n^*}, \tau_n^*)] \leq V(Y_0, 0) + E\left[\int_0^{\tau_n^*} \exp\left(-\int_0^u c(s)ds\right)\frac{\partial V}{\partial u} du\right]$$

$$= V(Y_0,0) + E\left[\int_0^{\tau_n^*} \exp\left(-\int_0^u c(s)ds\right)\left(-\frac{K'(u)}{K(u)-Y_u}\right)du\right]$$
$$\leq V(Y_0,0),$$

where the last inequality holds because $K(\cdot)$ is not decreasing. Now, it can be seen that for $t > 0$, it holds $V(\cdot,t) > 1+n$ on $\mathsf{D}(t) \setminus \mathsf{D}_n(t)$. Thus, $V > 1+n$ on the open set

$$U_n \doteq \left\{(y,s) : \ K(s) - e^{-n} < y < K(s), 0 \leq s \leq t\right\} \cup \left\{(y,s) : \ 0 < y < e^{-n}, 0 \leq s \leq t\right\}.$$

For fixed $t \in [0,T]$ we have: $P(\tau < t) \leq P(\tau_n < t) = P(\tau_n^* < t) = E(\mathbf{1}_{\tau_n^*<t})$.
Using the inequality $E(W(Y_{\tau_n^*},\tau_n^*)) \leq V(Y_0,0)$, previously proved, and the fact that $c(t)$ is positive and not decreasing, we obtain:

$$P(\tau < t) \leq E(\mathbf{1}_{\tau_n^*<t}) \leq E\left[\frac{\exp\int_{\tau_n^*}^t c(s)ds V(Y_{\tau_n^*},\tau_n^*)}{\inf\{V(x,s) : (x,s) \in U_n\}}\mathbf{1}_{\tau_n^*<t}\right]$$
$$\leq \exp\int_0^t c(s)ds\, E\left[\frac{\exp-\int_0^{\tau_n^*} c(s)ds V(Y_{\tau_n^*},\tau_n^*)}{\inf\{V(x,s) : (x,s) \in U_n\}}\mathbf{1}_{\tau_n^*<t}\right]$$
$$\leq \exp\int_0^t c(s)ds\, E\left[\frac{W(Y_{\tau_n^*},\tau_n^*)}{\inf\{V(x,s) : (x,s) \in U_n\}}\right]$$
$$\leq \exp\int_0^t c(s)ds\, \frac{V(Y_0,0)}{1+n} \overset{n\to\infty}{\longrightarrow} 0.$$

Since times t and T are arbitrary, the thesis follows. ■

4.2. **Stationary distribution and stochastic stability**

In this section, we cope with the particular SDE Bass model with $\alpha = \beta = 1$, and K constant. We notice that the result of the previous section allows us to remove the absolute values appearing in the definition of the stochastic term in the SDE Bass model of eq. (6). Now, we are interested in analyzing conditions for the parameters a, b, μ and σ that ensure the existence of a stationary distribution. When the stationary distribution does not exist, we study the stochastic stability of the steady state solutions $\{Y_t \equiv 0, t \geq 0\}$ and $\{Y_t \equiv K, t \geq 0\}$ of classical Bass model. Furthermore, hereinafter we will assume that the parameter b is strictly positive. In fact, the condition $b = 0$ would imply the absence of the “word of mouth" effect, and the corresponding process is of no interest for modeling diffusion of innovations. Moreover, the corresponding SDE would become linear. It is useful to work with “proportion" process $\{Z_t = Y_t/K, t \geq 0\}$, which corresponds to set $K = 1$.

As done for inference, we deal with process $\{X_t, t \geq 0\}$ obtained from $\{Y_t, t \geq 0\}$ applying the same Lamperti transformation as before $x = \eta(y) = 1/\sigma \log(1/y - 1)$, see [5],[1]. To find the stationary distribution density $q(\cdot)$ of the process $\{X_t, t \geq 0\}$, we look for a solution of the homogenous Fokker-Planck equation (see [18]):

$$\frac{1}{2}q''(x) - \frac{d}{dx}(\alpha(x)q(x)) = 0, \tag{22}$$

where

$$\alpha(x) = -\frac{a}{\sigma} - \frac{b}{\sigma} + \frac{\mu}{\sigma} + \frac{\sigma}{2} - \frac{a}{\sigma}e^{-\sigma x} + \frac{\mu}{\sigma}e^{-\sigma x} - \sigma\frac{e^{-\sigma x}}{1+e^{-\sigma x}}.$$

After integrating eq. (22), we have:

$$\frac{1}{2}q'(x) - (\alpha(x)q(x)) = C, \qquad (23)$$

where C is a constant. The general solution of eq. (23) can be written as:

$$q(x) = K_1 e^{2A(x)} + 2Ce^{2A(x)}\int_d^x e^{-2A(t)}dt = e^{2A(x)}\left[K_1 + 2C\int_d^x e^{-2A(t)}dt\right],$$

where

$A(x) = \int_d^x \alpha(t)dt = \left(-\frac{a}{\sigma} - \frac{b}{\sigma} + \frac{\mu}{\sigma} + \frac{\sigma}{2}\right)x - \frac{a}{\sigma^2}e^{-\sigma x} - \frac{\mu}{\sigma^2}e^{-\sigma x} + \log(1+e^{-\sigma x}) - K_2,$
K_1 and d are constants such that $0 < q(d) = K_1$,
$K_2 = Dd - \frac{a}{\sigma^2}e^{-\sigma d} - \frac{\mu}{\sigma^2}e^{-\sigma d} + \log(1+e^{-\sigma d})$, and D is the coefficient of the linear term of the expression of function A.In order for $q(\cdot)$ to be a probability density, it must be positive everywhere on $(-\infty,+\infty)$ and tend to zero as $x \to \pm\infty$. As we will see, this implies some restrictions on the family of solutions of the SDE Bass model. Let us consider separately some different cases.

(i) $a > 0$, $\mu > 0$ (*both self-innovation and recess are allowed*)
In this case, we have:

$$\lim_{x\to\pm\infty} A(x) = \lim_{x\to\pm\infty}\left[Dx - \frac{a}{\sigma^2}e^{-\sigma x} - \frac{\mu}{\sigma^2}e^{-\sigma x} + \log(1+e^{-\sigma x}) - K_2\right] = -\infty.$$

The requirement of non negativity for $q(\cdot)$ implies that $C = 0$. In fact, for $C \neq 0$, the quantity $1 + (2C/K_1)\int_d^x e^{-2A(t)}dt$, for $|x|$ large enough, would have different sign for different sign of $x - d$, hence $q(\cdot)$ would fail to be positive on the entire real axis. In the case $C = 0$, the function $q(x) = K_1 e^{2A(x)}$ is positive and integrable. Coming back to the stationary distribution $p(\cdot)$ for the original variable Y, we have

$$\begin{aligned} p(y) &= q(\eta(y/K))\frac{1}{K}|\eta'|_{z=y/K} \\ &= \mathsf{Z}^{-1}\exp\left\{\bar{p}\log\frac{y}{K-y} - \frac{2a}{\sigma^2}\frac{K-y}{y} - \frac{2\mu}{\sigma^2}\frac{y}{K-y} + 2\log\left(\frac{K}{K-y}\right)\right\}\frac{1}{\sigma y(K-y)} \\ &= \mathsf{N}^{-1}\frac{1}{y(K-y)^3}\left(\frac{y}{K-y}\right)^{\bar{p}}\exp\left\{-\frac{2a}{\sigma^2}\frac{K-y}{y} - \frac{2\mu}{\sigma^2}\frac{y}{K-y}\right\}, \end{aligned}$$

where $\bar{p} = -2D/\sigma = \frac{2}{\sigma^2}(a+b-\mu) - 1$, Z is the integral of the factor following Z^{-1}, and

$$\mathsf{N} = \int_0^K \frac{1}{y(K-y)^3}\left(\frac{y}{K-y}\right)^{\bar{p}}\exp\left\{-\frac{2a}{\sigma^2}\frac{K-y}{y} - \frac{2\mu}{\sigma^2}\frac{y}{K-y}\right\}dy.$$

We notice that, the normalizing constant can be expressed as

$$\mathsf{N} = \frac{1}{K^3}\left(\frac{a}{\mu}\right)^{\bar{p}/2}\left\{K_{\bar{p}}(\sqrt{\delta}) + 2\left(\frac{a}{\mu}\right)^{1/2}K_{\bar{p}+1}(\sqrt{\delta}) + \left(\frac{a}{\mu}\right)K_{\bar{p}+2}(\sqrt{\delta})\right\},$$

where $\delta = 16a\mu/\sigma^4$, and $K_q(\cdot)$ denotes the modified Bessel function of second kind of index q. Similar calculations allow us to express, in terms of the modified Bessel function of second kind of index q, the first two moments m, m_2 of the random variable (r.v.) distributed accordingly to the stationary distribution:

$$m = K\frac{\sqrt{a/\mu}\,K_{\bar{p}+1}(\sqrt{\delta}) + (a/\mu)K_{\bar{p}+2}(\sqrt{\delta})}{2\left[\sqrt{a/\mu} - (\bar{p}+1)/\sqrt{\delta}\right]K_{\bar{p}+1}(\sqrt{\delta}) + [1 + (a/\mu)]K_{\bar{p}+2}(\sqrt{\delta})}$$

$$m_2 = \frac{1}{\mathsf{N}K}(\mu/a)^{-\bar{p}/2-1}K_{\bar{p}+2}(\sqrt{\delta}).$$

It is interesting to study the deterministic limit ($\sigma^2 \to 0$) of the above expressions. Direct analysis is difficult because both the indices and the argument of the K-functions depend on parameter σ^2. However, asymptotic behavior can be analyzed using Laplace's approximation method. Using this approximation for the integrals appearing in the definitions of m, and m_2, we obtain:

$$m \approx K\frac{b - a - \mu + \sqrt{(b - a - \mu)^2 + 4ab}}{2b} \qquad m_2 \approx m^2.$$

Note that the limit value coincides with the positive solution y^+ in (2). In other words, as one could expect, the stationary distribution in the limit of small noise tends to be concentrated on the (positive) steady state solution of the corresponding deterministic model.

(ii) $a = 0$, $\mu > 0$ (*no self-innovation and possibility of recessing*)
We first consider the case where: $\sigma^2 < 2(b - \mu)$. In this case, we have

$$\lim_{x\to\pm\infty} A(x) = \lim_{x\to\pm\infty}\left[\left(\frac{\mu}{\sigma} - \frac{b}{\sigma} + \frac{\sigma}{2}\right)x - \frac{\mu}{\sigma^2}e^{-\sigma x} + \log(1 + e^{-\sigma x}) - K_2\right] = -\infty.$$

As in the previous case, the positivity of $q(\cdot)$ implies $C = 0$. As before, the function q satisfies the requirements for being a probability density, which is given by:

$$q(x) = K_1 e^{2A(x)} = \mathsf{N}_0^{-1}\exp\left\{2\left(\frac{\mu}{\sigma} - \frac{b}{\sigma} + \frac{\sigma}{2}\right)x - 2\frac{\mu}{\sigma^2}e^{-\sigma x} + 2\log(1 + e^{-\sigma x})\right\},$$

where N_0 is the normalization constant. Returning to the original variable, the stationary density is:

$$p(y) = \mathsf{N}_0^{-1}\frac{1}{y(K-y)^3}\left(\frac{y}{K-y}\right)^p\exp\left\{-\frac{2\mu}{\sigma^2}\frac{y}{K-y}\right\},$$

where

$$\mathsf{N}_0 = \int_0^K \frac{1}{y(K-y)^3}\left(\frac{y}{K-y}\right)^p\exp\left\{-\frac{2\mu}{\sigma^2}\frac{y}{K-y}\right\}dy,$$

and $p = 2b/\sigma^2 - 2\mu/\sigma^2 - 1 > 0$. The normalization constant can be expressed as

$$\mathsf{N}_0 = \frac{1}{K^3}\left(\frac{\sigma^2}{2\mu}\right)^p \Gamma(p)\frac{1}{\mu^2}\left[b^2 - \frac{\sigma^2}{2}(\mu + b)\right],$$

where $\Gamma(\cdot)$ denotes the Gamma function. The expected value m of the r.v. following this distribution, is given by:

$$m = K\frac{b(b - \mu - \sigma^2/2)}{b^2 - \sigma^2(\mu + b)/2}.$$

Similar calculations lead to the following expression for the second moment:

$$m_2 = K^2(b - \mu)\left(b - \mu - \frac{\sigma^2}{2}\right)\left(b^2 - \frac{\sigma^2}{2}(\mu + b)\right)^{-1}.$$

Remark 4.1. It is interesting to study the behavior of the first two moments in the "deterministic limit", i.e. as $\sigma^2 \to 0$. It can be seen that:

$$\lim_{\sigma^2 \to 0} m = K\frac{b - \mu}{b}; \quad \lim_{\sigma^2 \to 0}(m_2 - m^2) = 0.$$

The right equation above shows that, as one would expect, the stationary distribution tends to become singular with mass on the steady state solution of the deterministic model, as the noise vanishes.

In the special case $\sigma^2 = 2(b - \mu)$, $A(x)$ tends to a constant as $x \to +\infty$, so that $q(x)$ cannot vanish at $x = +\infty$. Therefore, in this case the stationary distribution density does not exist.

Now we deal with the case: $\sigma^2 > 2(b - \mu)$. In this case, it holds

$$\lim_{x \to +\infty} A(x) = \lim_{x \to +\infty}\left[\left(\frac{\mu}{\sigma} - \frac{b}{\sigma} + \frac{\sigma}{2}\right)x - \frac{\mu}{\sigma^2}e^{-\sigma x} + \log(1 + e^{-\sigma x}) - K_2\right] = +\infty.$$

If $C = 0$, then

$$\lim_{x \to +\infty} q(x) = \lim_{x \to +\infty} K_1 e^{2A(x)} = +\infty.$$

In the opposite case, by applying the L'Hopital's rule we find:

$$\lim_{x \to +\infty} q(x) = \lim_{x \to +\infty} e^{2A(x)}\left[K_1 + 2C\int_d^x e^{-2A(t)}dt\right] = C\left(-\frac{\mu}{\sigma} + \frac{b}{\sigma} - \frac{\sigma}{2}\right)^{-1} \neq 0.$$

Therefore, also in this case the stationary distribution density does not exist.

Since in the case of no self innovation ($a = 0$), both drift and diffusion functions vanish for $Y = 0$, the process $\{Y_t \equiv 0, t \geq 0\}$ is a (trivial) solution of the SDE Bass model. Thus, it is interesting to study the stability properties of this solution when the parameter are such that an invariant distribution does not exist. In particular, in analogy with the deterministic framework, it is of interest to analyze the qualitative behavior of solutions corresponding to initial conditions y_0 that belong to a small right interval of 0. The analysis of the stochastic stability is based on the Lyapunov theory [9]. Without loss of generality, we set $K = 1$. Then the following theorem holds.

Theorem 4.2. *The solution* $\{Y_t \equiv 0, t \geq 0\}$ *of the SDE Bass model without self innovation:*

$$dY_t = [bY_t(1 - Y_t) - \mu Y_t]dt + \sigma Y_t(1 - Y_t)dW_t$$

is stochastically asymptotically stable for $\sigma^2 > 2(b - \mu)$.

Proof. The generator of this SDE is:

$$\mathsf{L} = [by(1 - y) - \mu y]\frac{\partial}{\partial y} + \frac{\sigma^2}{2}y^2(1 - y)^2\frac{\partial^2}{\partial y^2}.$$

Let us define the Lyapunov function:

$$V(y) = \left|\int_0^y g(x)dx\right|; \qquad g(y) \doteq \exp\left\{-A\log\left|\frac{y}{1 - y}\right| + \frac{2b}{\sigma^2}\frac{y}{1 - y}\right\}.$$

Assuming $A < 1$, so that V is well defined in a neighborhood of $\{0\}$,

$$\begin{aligned}\mathsf{L}V(y) &= sgn(y)g(y)[by(1 - y) - \mu y] + sgn(y)g(y)\left[-\frac{A}{y(1 - y)} + \frac{2b}{\sigma^2}\frac{1}{(1 - y)^2}\right]\frac{\sigma^2}{2}y^2(1 - y)^2 \\ &= g(y)y\, sgn(y)\left[b - \mu - \frac{\sigma^2}{2}A + \frac{\sigma^2}{2}Ay\right].\end{aligned}$$

Since $g(y)y\, sgn(y) > 0$, the thesis is proved if there is a neighborhood D of $y = \{0\}$, such that

$$b - \mu - \frac{\sigma^2}{2}A + \frac{\sigma^2}{2}Ay < 0$$

for $y \in D$. This is the case, since the last expression is a continuous function of (A, y), taking a negative value at point $(1, 0)$. ■

Remark 4.2. It is interesting to note that the presence of sufficiently strong noise can determine disadoption of all individuals with high probability, also in situations where the corresponding deterministic system goes away from zero. Specifically, this happens when $b - \mu - \sigma^2/2 \leq 0 < b - \mu$. The steady state solution $y(t) \equiv 0$ of the deterministic model is unstable, but the solution $\{Y_t \equiv 0, t \geq 0\}$ of the SDE Bass model is asymptotically stable in probability. This situation is typical of logistic models and in population dynamics is responsible for the fact that “noise can cause extinction".

(iii) $a > 0$, $\mu = 0$ (*self-innovation is present but recess is not allowed*)
In this case we have:

$$\lim_{x \to -\infty} A(x) = \lim_{x \to -\infty}\left[\left(-\frac{a}{\sigma} - \frac{b}{\sigma} + \frac{\sigma}{2}\right)x - \frac{a}{\sigma^2}e^{-\sigma x} + \log(1 + e^{-\sigma x}) - K_2\right] = +\infty.$$

The same analysis as in the case $\sigma^2 = 2(b - \mu)$ leads to the conclusion that the stationary distribution density does not exist.
Without loss of generality, let us assume that $K = 1$. We note that in the treated case of no recess ($\mu = 0$), $\{Y_t \equiv 1, t \geq 0\}$ is a solution of the SDE Bass model equation

$$dY_t = (a + bY_t)(1 - Y_t)dt + \sigma Y_t(1 - Y_t)dW_t.$$

Thus, since the process does not admit a proper stationary distribution, one could ask whether $\{Y_t \equiv 1, t \geq 0\}$ is stable in probability. Note that the transformed process defined by $\{Z_t = 1 - Y_t, t \geq 0\}$ is solution of the SDE

$$dZ_t = -[aZ_t + bZ_t(1 - Z_t)]dt - \sigma Z_t(1 - Z_t)dW_t,$$

with $Z_0 = 1 - Y_0$. Thus, stochastic stability of $\{Y_t \equiv 1, t \geq 0\}$ is equivalent to stochastic stability of $\{Z_t \equiv 0, t \geq 0\}$.
It can be shown that the Lyapunov function:

$$V(z) = \left| \int_0^z \exp\left\{ -\frac{2b}{\sigma^2} \frac{x}{1 - x} \right\} dx \right|$$

satisfies equation

$$\mathsf{L}V(z) = -(a + b)z\, sgn(z) \exp\left\{ -\frac{2b}{\sigma^2} \frac{z}{1 - z} \right\}.$$

Thus, $\mathsf{L}V(z) < 0, \forall z \in \mathbb{R}$. This shows that $\{Z_t \equiv 0, t \geq 0\}$ (correspondingly, solution $\{Y_t \equiv 1, t \geq 0\}$) is a solution of the SDE Bass model stable in probability.

5. Conclusion

In this paper, a SDE extension of the classical Bass model has been studied. The extension is obtained by adding to the classical equation a suitable diffusion term ensuring that the trajectories of the process remain a.s. in the interval $[0, K]$, where K is the number of potential adopters. We have extended a known regularity result to the case when the number of potential adopters varies along time in a pre-assigned way. This extension allows to take into account the effects of the population dynamics on the diffusion of innovation. As a second issue, we have coped with the problem of the existence of a stationary distribution. It has been proved that when all the model parameters are strictly positive, a stationary distribution exists. In case of no self-innovation ($a = 0$), some conditions on the remaining parameters for the existence of a stationary distribution have been provided. Moreover, the stochastic stability of the steady state solutions of the deterministic Bass model corresponding to “no adopter" or “all adopters" has been studied, in the case when the process has not a genuine stationary distribution. We have proposed an inferential procedure that relies on the approximation of the SDE transition density by means of MC diffusion bridge simulation. We have applied this procedure to the SDE Bass model solutions obtained by an exact algorithm. A comparison of the results with those obtained by a known approach based on Gaussian approximation has been performed. The comparison has shown that the AMLE method performs better than the competing technique, the price to pay being the need to know and to properly use not trivial stochastic simulation techniques, the very intensive computation effort and the need of large computer memory.

Acknowledgments
We thank the two anonymous reviewers for their very useful comments and suggestions.

References

[1] Y. Ait-Sahalia, Maximum likelihood estimation of discretely sampled diffusions: a closed form approximation approach, *Econometrica* **70**, (2002), 223 - 262.

[2] F. M. Bass, A new product growth model for consumer durables, *Management Sciences* **15**, (1969), 215-227.

[3] A. Beskos, O. Papaspilioupolos, and G. O. Roberts, Retrospective exact simulation of diffusion sample path with applications, *Bernoulli* **12**, (2006),1077-1098.

[4] S. N. Ethier, and T. G. Kurtz, *Markov Processes: Characterization and Convergence*, Wiley, New York,1986.

[5] D. Florens, Estimation of diffusion coefficient from crossings, *Statistical Inference for Stochastic Processes* **1**, (1999),175-195.

[6] L. A. Fourt, and J. W. Woodlock, Early prediction of early success of new grocery products, *Journal of Marketing* **25**, (1960), 31-38.

[7] R. Guseo, and M. Guidolin, Automata with network incubation in information technology diffusion, *Physica A: Statistical Mechanics & Its Applications* **389**(12), (2010), 2422-2433.

[8] R. Gutierrez, A. Nafidi, and R. G. Sanchez, Forecasting natural-gas consumption in Spain by using the stochastic Gompertz innovation diffusion model, *Applied Energy* **80**, (2005),115-124.

[9] R. Z. Has'minski, *Stochastic Stability of Differential Equations*, Sijthoff & Noordhoff, 1980.

[10] J. Kanniainen, S. J. Makinen, R. Piche and A. Chakrabarti, Forecasting the diffusion innovation: A stochastic Bass model with long-normal and mean-reverting error process, *IEEE Trans. Engineering Management* **58**(2), (2011), 228-249.

[11] I. Karatzas, and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York, 1991.

[12] V. Mahajan, and Y. Wind, *Innovation Diffusion Models of New Product Acceptance*, Ballinger, Cambridge, Massachusetts, 1986.

[13] E. Mansfield, Technical change and rate of imitation, *Econometrica* **29**, (1961), 741 - 766.

[14] G. Maruyama, Continuous Markov processes and stochastic equations, *Rend. Circ. Mat. Palermo* **4**, (1955), 48 - 90.

[15] S. Moldovan, and J. Goldenberg, Cellular automata modeling of resistance to innovations: Effects and solutions, *Technological Forecasting & Social Change* **71**, (2003), 425-442.

[16] S.C. Niu, A stochastic formulation of the Bass model of new-product diffusion, *Mathematical Problems of Engineering* **8**, (2002), 429-263.

[17] B. K. Oksendal, *Stochastic Differential Equations: An Introduction With Applications*, Springer-Verlag, Berlin, 2013.

[18] H. Risken, *The Fokker-Plank Equation: Methods of Solution & Applications*, Springer Series in Synergetics, Springer, Berlin, 1996.

[19] E. M. Rogers, *Diffusion of Innovations*, The Free Press, New York, 1995.

[20] H. Shurz, Modeling, analysis and discretization of stochastic logistic equations, *International Journal of Numerical Analysis & Modeling* **4**, (2007), 178-197.

[21] C. H. Skiadas, and A. N. Giovanis, A stochastic Bass innovation diffusion model for studying the growth of electricity consumption in Greece, *Applied Stochastic Models and Data Analysis* **13**, (1997), 95-101.

[22] S. M. Yacus, *Simulation and Inference for Stochastic Differential Equations*, Springer, Berlin, 2008.

Article history: Submitted July, 06, 2020; Revised September, 25, 2020; Accepted October, 17, 2020.